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DIFFERENTIAL INCLUSIONS AND MULTIVALUED INTEGRALS

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Abstract

In this paper we consider the nonlocal (nonstandard) Cauchy problem for differential inclusions in Banach spaces

$$x'(t) \in F(t, x(t)), \quad x(0) = g(x), \quad t \in [0, T] = I.$$

Investigation over some multivalued integrals allow us to prove the existence of solutions for considered problem. We concentrate on the problems for which the assumptions are expressed in terms of the weak topology in a Banach space. We recall and improve earlier papers of this type. The paper is complemented by a short survey about multivalued integration including Pettis and Henstock-Kurzweil-Pettis multivalued integrals.

Keywords: nonlocal Cauchy problem, Aumann integrals, Pettis integrals, Henstock-Kurzweil-Pettis integrals, measure of weak noncompactness.

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1. Preliminaries

Let E be a Banach space and E^* be its topological dual. The closed unit ball of E^* is denoted by $B(E^*)$. By B_r we denote a closed ball $\{x \in E : \|x\| \leq r\}$. In the whole paper I stands for a compact interval $[0, T]$. By $ck(E)$, $cb(E)$, $cc(E)$, $cwk(E)$ we denote the family of all nonempty convex compact, nonempty convex bounded, nonempty convex closed, and nonempty convex weakly compact subsets of E, respectively. For every $C \in cb(E)$ the support function of C is denoted by $s(\cdot, C)$ and defined on E^* by $s(x^*, C) = \sup_{x \in C} x^* x$, for each $x^* \in E^*$.

Definition. A multifunction $G : E \to 2^E$ with nonempty, closed values is:

(i) upper hemi-continuous (uhc) [respectively weakly upper hemi-continuous i.e., w-uhc] iff for each $x^* \in E^*$ and for each $\lambda \in \mathbb{R}$ the set $\{x \in E : s(x^*, G(x)) < \lambda\}$ is open in E [in (E, w)] i.e., $s(x^*, G(\cdot))$ is an upper semi-continuous function. Here (E, w) denotes the space E endowed with the weak topology;

(ii) (see [18]) weakly sequentially upper hemi-continuous (w-seq uhc) iff for each $x^* \in E^*$ $s(x^*, G(\cdot)) : E \to \mathbb{R}$ is sequentially upper semicontinuous from (E, w) into $\mathbb{R}$.

We recall that a function $f : I \to E$ is said to be weakly continuous if it is continuous from I to (E, w). For a bounded subset $A \subset E$, we define the measure of weak noncompactness $\omega(A)$ (in the sense of DeBlasi [23]): $\omega(A)$ is the infimum of all $\varepsilon > 0$ s.t. there exists a weakly compact set K in E with $A \subseteq K + \varepsilon \cdot B_1$. For the properties of ω, see [23], for instance.

We will say that a nonnegative real-valued function $(t, r) \longrightarrow h(t, r)$ defined on $I \times \mathbb{R}_+$ is a Kamke function if it satisfies Carathéodory conditions, $h(t, 0) = 0$ and the function identically equal to zero is the unique continuous solution of the inequality $u(t) \leq \int_0^t h(s, u(s))ds$ satisfying the condition $u(0) = 0$.

Let us introduce the following definition:

Definition ([37])**.** Let $F : [a, b] \to E$ and $A \subset [a, b]$. The function $f : A \to E$ is a pseudoderivative of F on A if for each x^* in E^* the real-valued function x^*F is differentiable almost everywhere on A and $(x^*F)' = x^*f$ a.e. on A (remark that the exceptional null-measure set depends on x^*).

If the Banach space has total dual (E^* contains a countable set T which separates points of E i.e., for every $0 \neq x \in E$ there exists $x^* \in E^*$ such that $x^x x \neq 0$), the pseudoderivative, if it exists, is unique up to a null measure set. Indeed, let f_1, f_2 be two pseudoderivatives of F and $(x_n^*)_n$ a sequence of $B(E^*)$ which separates points of E. For each n, we can find $N_n \subset [0, 1]$ of null measure such that $x_n^* f_1(t) = (x_n^* F)'(t) = x_n^* f_2(t)$, for all $t \in [0, 1] \backslash N_n$. Then $N = \cup_{n \in \mathbb{N}} N_n$

satisfies $\mu(N) = 0$ and $x_n^* f_1(t) = x_n^* f_2(t), \forall t \in [0,1] \setminus N, \forall n \in \mathbb{N}$. It follows that $f_1(t) = f_2(t), \forall t \in [0,1] \setminus N$.

It is known, that the concepts of weak differentiability and pseudo-differentiability can differ even for reflexive spaces. Nevertheless these two definitions as well as strong differentiability are equivalent when E is finite dimensional.

We leave to the reader remind the notions of absolutely continuous (AC) functions and ACG_* functions ([30]). In this book different kinds of continuity with their links to differentiability are considered.

2. Multivalued integrals

For the convenience of the reader, we recall some definitions of integrals in Banach spaces.

Definition ([37])**.** The function $f : I \to E$ is Pettis integrable ((P) integrable for short) if:

(i) for every $x^* \in E^*$, $x^* f$ is Lebesgue integrable on I,

(ii) for all measurable $A \subset I$, there exists $(P) \int_A f(s)ds \in E$ such that $x^*(P) \int_A f(s)ds = (L) \int_A x^* f(s)ds$, for every $x^* \in E^*$.

By P_E^1 we will denote the space of all Pettis integrable functions on I with the values in E. In this paper we are interested in considering this space with the so called weak topology i.e., induced by the duality $(P_E^1, L^\infty \otimes E^*)$: $< u \otimes x^*, f >= \int_I u(s) x^* f(s) ds$ (cf. [13, 29]).

Definition. A function $f : I \to E$ is called Henstock-Kurzweil (shortly, (HK)) integrable if there exists an element (HK) $\int_I f(t)dt$, satisfying: for every $\varepsilon > 0$, one can find a gauge δ_ε such that, for any δ_ε-fine partition $\mathcal{P} = (I_i, t_i)_{i=1}^n$ of I,

$$\left\| \sum_{i=1}^n f(t_i)\mu(I_i) - (\text{HK}) \int_I f(t)dt \right\| < \varepsilon.$$

In particular, when $E = \mathbb{R}$, this definition describes the usual Henstock-Kurzweil integral.

Now, we recall the definition of a vector integral which is a generalization of both Pettis and Henstock integrals.

Definition ([20, 21])**.** The function $f : I \to E$ is Henstock-Kurzweil-Pettis integrable ((HKP) integrable for short) if:

(i) for all $x^* \in E^*$, $x^* f$ is Henstock-Kurzweil integrable on I,

(ii) for every $t \in I$, there exists $(HKP)\int_0^t f(s)ds \in E$ such that

$$x^*(HKP)\int_0^t f(s)ds = (HK)\int_0^t x^* f(s)ds, \quad \text{for all} \;\; x^* \in E^*.$$

It is rather unknown, that the study of multivalued integrals was begun by Alexander Dinghas in 1956 ([25]) who adapted the definition of the Riemann integral to the multivalued context (the Riemann-Minkowski integral). The same idea of Riemann sums was rediscovered by Hukuhara in 1967 [33] and then developed by other authors. But the theory of integration for multivalued mappings is intensively studied since Aumann's work of 1965 which is based on another idea: to use the integrals of all (integrable) selections of the multifunction. The Aumann integral is well suited for applications to various mathematical fields, in particular to solve some problems for differential inclusions.

In fact, the three treatments of the multivalued integrals are still considered: Riemann, Aumann and Debreu approach. In the last concept it was defined the multivalued integral as a usual integral in the space of subsets (by using the Radström or the Hörmander theorem cf. [24]). A brief survey about such a type of integrals can be found in [16].

The problem of integrability, properties of the integrals as well as comparison results between different kind of multivalued integrals are investigated as basic problems in many papers. In this paper we prove some properties of multivalued Pettis integrals. We concentrate on the most important, from an application point of view, properties of multivalued integrals. These results are similar to that known for Bochner integrals, but not the same.

If F is a measurable multifunction, we denote by S_F^1 the set of all Bochner integrable selections of F, namely

$$S_F^1 = \{f \in L^1(E) : f(s) \in F(s) \text{ almost everywhere}\}.$$

If $S_F^1 \neq \emptyset$, then we say that F is Aumann integrable and the Aumann integral (shortly (A)-integral) of F is given by

$$(A)\int_I F(s)ds = \left\{ \int_I f(s)ds : f \in S_F^1 \right\}.$$

One of the advantages for the Aumann integrals lies in the fact that the values of "integrable" multifunctions need be neither convex nor compact sets.

2.1. Pettis set-valued integrals

If we consider multivalued nonabsolute integrals then the Pettis integral is the oldest one. For instance, the results which are the most interesting for dealing with applications of multivalued integrals for differential inclusions can be found in the book of Castaing-Valadier [13] or in the paper of Tolstonogov [44]. The (Castaing-) Pettis integrability means the Lebesgue integrability of the function $s(x^*, F(\cdot))$ for multifunctions with convex compact [or: weakly compact] values, as well as "Pettis" integrability in the appropriate space of subsets (for each measurable subset A of the domain there exists a convex compact [weakly compact] which realize the integral of $s(x^*, F(\cdot))$ over A). Cf. also [1].

A complete theory of this topic (for many classes of values of F), including more general definitions of the Pettis integral for multifunctions, can be found in [28]. This paper deals also with Pettis integrability for multifunctions with possibly unbounded values and contains interesting examples of different kind of Pettis integrable multifunctions.

Let us also recall, that the Pettis integral for multifunctions was defined also via some isometries in [11] (the Debreu-type of integral).

The last concept is based on the idea of Aumann and was introduced by Valadier in [45] (via pseudo-selections) and considered also in [28].

Definition. The Aumann-Pettis integral of a multifunction is:

$$(AP)\int_I F(s)ds = \left\{(P)\int_I f(s)ds : f \in S_F^{Pe}\right\},$$

where S_F^{Pe} denotes the set of all Pettis integrable selections of F provided that this set is not empty.

As noted above, an interesting discussion about all types of Pettis integrals can be found for instance in [28].

Definition. A subset K of $P_E^1(\mu)$ is Pettis uniformly integrable (PUI) if, for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\mu(A) \le \delta_\varepsilon \implies \left\|(P)\int_A h d\mu\right\| \le \varepsilon,\ \forall h \in K.$$

In other words, if $\{\langle x^*, h\rangle : x^* \in B(E^*), h \in K\}$ is uniformly integrable.

Let us remind the following (simplified) comparison results:

Theorem 1 ([28]). *Let $F : I \to cwk(E)$ $[ck(E)]$ be measurable and scalarly integrable multifunction (i.e., the support functions are real integrable functions). Then the following statements are equivalent:*

(a) *F is Pettis integrable in $cwk(E)$ $[ck(E)]$,*

(b) *the set $\{x^* f : x^* \in B(E^*),\ f \in S_F^{Pe}\}$ is uniformly integrable,*

(c) *every measurable selection of F is Pettis integrable,*

(d) *for every measurable subset A of I the Aumann-Pettis integral I_A belongs to $cwk(E)$ $[ck(E)]$ and, for every $x^* \in E^*$, one has*

$$s(x^*, I_A) = \int_A s(x^*, F(s))ds.$$

The properties of Pettis integrals are intensively studied. Some selected results can be found in [1, 2, 3, 12, 28, 32, 46].

In the sequel we will need the following characterization of the set S_F^{Pe} of all Pettis-integrable selections of the multifunction F due to Castaing (cf. [2] Proposition 3.4):

Lemma 2. *Suppose that E is a separable Banach space and $F : I \to cwk(E)$ is a Pettis integrable multifunction. Then the set S_F^{Pe} is nonempty and sequentially compact for the topology of pointwise convergence on $L^\infty \otimes E^*$.*

Let us remind an auxiliary result ([41]):

Lemma 3. *Assume that E is separable. For any sequence $(\overline{y}_n)_n$ of measurable selections of a $cwk(E)$-valued measurable multifunction Γ, there exists a sequence $z_n \in conv\{\overline{y}_m, m \geq n\}$ weakly a.e. convergent to a measurable $\overline{y}$.*

2.2. Henstock-Kurzweil-Pettis set-valued integrals

As in the single-valued case, it is possible to unify the concepts of Henstock set-valued integrability (presented in [27]) and Pettis set-valued integrability by introducing a new multivalued integral, that can be found in [26].

For multifunctions with closed convex and bounded values we can define, similarly like in previous considerations, the multivalued Henstock-Kurzweil-Pettis integral by assuming scalar HK-integrability (i.e., the HK-integrability of support functionals) together with the existence of a closed convex and bounded set I_A in E such that $s(x^*, I_A) = (HK) \int_A s(x^*, F(t))dt$ for each subinterval A of I ([26, 27]). As in [28] it is possible to consider the integrability for different classes of subsets. Some examples and dependencies can be found in [26].

Also, an Aumann-type definition using HKP-integrable selections is introduced in [26]. We omit them.

The following theorem is due to Di Piazza and Musiał (here S_F^{HKP} denotes the family of all Henstock-Kurzweil-Pettis integrable selections of the multifunction F):

Proposition 4 ([26]). *Let $F : I \to cb(E)$ be a scalarly HK-integrable multifunction. If $S_F^{HKP} \neq \emptyset$, then the following conditions are equivalent:*

(i) *F is HKP-integrable in $cb(E)$;*

(ii) *for every* $f \in S_F^{HKP}$ *the multifunction* $G(t) = F(t) - f(t)$ *is Pettis integrable in* $cb(E)$,

(iii) *there exists* $f \in S_F^{HKP}$ *such that the multifunction* $G(t) = F(t) - f(t)$ *is Pettis integrable in* $cb(E)$.
We have then

$$(HKP)\int_{[a,b]} F(t)dt = (P)\int_{[a,b]} G(t)dt + (HKP)\int_{[a,b]} f(t)dt$$

for all $[a,b] \subset I$.

Both single and multivalued integrals are really useful in the theory of differential equations and inclusions and allow to unify separately considered cases (cf. [22]). Applicability of such integrals will be clarified in next sections.

2.3. Remarks about different kind of integrable selections

Sometimes it seems to be useful to consider also another type of integrable functions as a selection of multifunctions. Nevertheless, to the best of our knowledge, there is no application of such integrals for solving differential inclusions.

For Riemann-integrable selections we have some existence and nonexistence results ([19, 38]). See also [43] for lack of full characterization of the set of all Riemann-integrable selections. For real-valued multifunctions some applications are proved in [4]. Let us recall two modified examples from [19]:

(1) Let (r_n) be the sequence of all different rational numbers from $[0,1]$ and let $F : [0,1] \to 2^{c_0}$ be defined by: $F(x)$ is the unit ball in c_0 for irrational x and $F(x) = r_n$ if $x = r_n$. Then F is upper semi-continuous almost everywhere on $[0,1]$, the values of F are closed and convex and there exists a Riemann integrable selection of F which is not continuous almost everywhere.

(2) Taking a nonmeasurable set A of $[0,1]$ we can define a (multi-)function $H : [0,1] \to l_\infty[0,1]$ via formulas $H(x) = \{\theta\}$ if $x \notin A$ and $H(x) = \{e_x\}$ if $x \in A$, where $e_x(t) = 1$ for $x = t$ and $e_x(t) = 0$ for $x \neq t$. Such a multifunction is even non-measurable but has a Riemann integrable selection which is not strongly measurable. It is obvious that $S_H^1 = \emptyset$, so H is neither Aumann nor Pettis integrable.

3. Results

For an abstract Cauchy problem

$$x'(t) = f(t, x(t)) \ , \ x(0) = x_0 \tag{1}$$

the solution are considered in a generalized sense.

Definition ([21])**.** A function $x : I \to E$ is said to be a pseudo-solution of the Cauchy problem (1) if it satisfies the following conditions:

(i) x is absolutely continuous,

(ii) $x(0) = x_0$,

(iii) for each $x^* \in E^*$ there exists a set $A(x^*)$ with a Lebesgue measure zero, and such that for each $t \notin A(x^*)$, $x^*x(t)$ is differentiable and $(x^*x(t))' = x^*(f(t, x(t)))$.

Definition ([21])**.** A function $x : I \to E$ is said to be a pseudo-K-solution of the Cauchy problem (1) if it satisfies the following conditions:

(i) x is weakly ACG_* (i.e., for each $x^* \in E^*$ x^*x is ACG_*),

(ii) $x(0) = x_0$,

(iii) for each $x^* \in E^*$ there exists a set $A(x^*)$ with a Lebesgue measure zero, such that for each $t \notin A(x^*)$, $x^*x(t)$ is differentiable and $(x^*x(t))' = x^*(f(t, x(t)))$.

By x' we denote a pseudoderivative.

Let us note that the pseudo-solution of the problem (1) is equivalent to the solution of the integral problem

$$x(t) = x_0 + \int_0^t f(s, x(s))ds, \tag{2}$$

where the integral is taken in the sense of Pettis. In the case of pseudo-K-solutions we have an equivalence with solutions of (2) with the Henstock-Kurzweil-Pettis integral ([22]).

Now we consider the following nonlocal Cauchy differential inclusions

$$x'(t) \in F(t, x(t)), \quad x(0) = g(x), \qquad t \in [0, T] = I. \tag{3}$$

In contrast to the single-valued case, only in a limited number of papers there were considered solutions other than Carathéodory solutions. At the beginning, the Bochner integrability was replaced by the Pettis integrability, but due to other assumptions on solutions a Carathéodory solution was more convenient (cf. Castaing, Valadier [13] Theorem VI-7). But in the paper of Tolstonogov both type of solutions were checked: Carathéodory and weak solutions ([44], cf. also Maruyama [35]). The next result (motivated by some problems arising from plasmas physics) was obtained by Arino, Gautier and Penot [5] for weakly-weakly usc multifunctions F with almost weakly relatively compact range. Note that in the last paper, in fact, the existence of pseudo-solutions was proved. However under the considered additional assumptions they are Carathéodory solutions, too. Finally, it is necessary to mention the works of Chow and Schuur [14] and [15], where some contingent differential equations (and as a particular case usual differential equations) with its relations to the weak topology has been investigated. Let us note that in all mentioned papers the considered problems were local problems i.e., with a constant function $g(x) = x_0$.

In the last years, thanks to the progress of multivalued integration, this theory was rediscovered. Let us mention the papers of Godet-Thobie and Satco [29], Satco [39, 40, 41, 42], Azzam-Laouir, Castaing and Thibault [9] or Azzam-Laouir and Boutana [8] (results for different kind of multivalued problems).

For simplicity, we restrict ourselves to the multivalued nonlocal Cauchy problem (3) and we present an improvement of previous results, including the last one in [16].

Let us present two existence results for the problem (3) in Pettis integrability setting. The first one assumes the existence of a Pettis integrable multifunction that contains all values of the right member of our inclusion, together with a sequential continuity and a growth assumption on the function g in the nonlocal condition. Then we extend some previous results by dropping the assumption of a dominant Pettis integrable multifunction and assuming instead that F satisfies Pianigiani's condition with respect to the measure of weak noncompactness, which is much more general that previously considered ones. In particular, this allows to cover the case of the sum of Lipschitz and weakly compact mappings. Moreover, this results remain true also for standard Cauchy problems, i.e., $g(x) = x_0$ as well as for classical functions g which are useful to describe some phenomena by using nonlocal (nonstandard) conditions. For instance: $g_1(x) = \sum_{n=1}^{k} c_n \cdot x(t_n)$ for some $a \leq t_1 \leq \cdots \leq t_k \leq b$ (note that here x should be defined only on some $[0, \beta]$ containing all points t_n, $n = 1, 2, \dots, k$) or $g_2(x) = \frac{1}{T} \cdot \int_a^{a+T} x(s)ds$ for $0 < T < b - a$. It is a good place to explain a nonlocal character of considered solutions. In a standard theory we are looking for solutions in a set of functions satisfying the initial condition and defined on some subinterval $J \subset I$. For a given nonlocal problem we have a function g defined on a set of functions satisfying some additional conditions. Thus the function g should be defined on a set of continuous functions at least on a subinterval $[a, \beta]$ described by the formula on g. In the above examples we have $\beta \geq t_k$ for g_1 and $\beta \geq T$ for g_2. Hence the interval of existence of our solutions cannot be arbitrarily small and should be defined at least on $[a, \beta]$. By $[0, \beta]$ we will always denote such a minimal existence interval.

Theorem 5. *Assume that E is separable. Let $F : I \times E \to 2^E$ with nonempty convex and weakly compact values satisfy:*

(a) *$F(t, \cdot)$ is weakly sequentially upper hemi-continuous for each $t \in I$,*

(b) *$F(\cdot, x)$ has a weakly measurable selection for each $x \in E$,*

(c) $F(t,x) \subset G(t)$ *a.e. for some* $cwk(E)$*-valued Pettis-integrable multifunction* G*, bounded on* $[0,\beta]$ *by a constant* M*,*

(d) *the function* $g : C([0,\beta],E) \to E$ $(\beta \leq T)$ *is:*

(d1) *convex and has sublinear growth:* $\|g(x)\| \leq a\|x\| + b$ *for* $a \in (0,1)$ *and* $b > 0$*,* $b + M \geq \beta(1-a)M$*,*

(d2) *weakly-weakly sequentially continuous.*

Then there exists at least one pseudo-solution of the Cauchy problem (3) *on subinterval* $J = [0,\beta] \subset I$*. Moreover, the set of all pseudo-solutions of this problem is weakly compact in* $C(J,E)$*.*

Proof. Since E is separable, any weakly measurable function then it is also strongly measurable. In view of the definition of the pseudo-solution it can be proved in a similar way as in earlier papers about nonlocal problems, that a function $x(\cdot)$ is a pseudo-solution of problem (3) if it is a solution of the integral inclusion

$$x(t) \in g(x) + (P)\int_0^t F(s,x(s))ds.$$

Let $U = \{x_f \in C(I,E) : x_f(t) = (P)\int_0^t f(s)ds, t \in I, f \in S_G^{Pe}\}$. By assumptions, S_G^{Pe} is Pettis uniformly integrable (cf. [29]). Moreover, for any $f \in S_G^{Pe}$ and $x^* \in E^*$ we have $x^*f \leq s(x^*,G)$ and so, for arbitrary $x_f \in U$ and $t,\tau \in I$ we get:

$$\begin{aligned}\|x_f(t) - x_f(\tau)\| &= \sup_{x^*\in B(E^*)} x^*(x_f(t) - x_f(\tau)) \\ &= \sup_{x^*\in B(E^*)} \left(\int_0^t x^*f(s)ds - \int_0^\tau x^*f(s)ds\right) \\ &= \sup_{x^*\in B(E^*)} \int_\tau^t x^*f(s)ds \leq \sup_{x^*\in B(E^*)} \int_\tau^t s(x^*,G(s))ds.\end{aligned}$$

By Pettis integrability of G it follows that U is an equicontinuous subset of $C(I,E)$. Then by Pettis uniform integrability of S_G^{Pe} for each $t \in [0,\beta]$ and $f \in S_G^{Pe}$ we have $\|(P)\int_0^t f(s)\,ds\| \leq M$.

Define a multifunction $R : C([0,\beta],E) \to 2^{C([0,\beta],E)}$ by the following formula:

$$\begin{aligned}R(x) &= g(x) \\ &\quad + \left\{x_f \in C([0,\beta],E) : x_f(t) = (P)\int_0^t f(s)ds, \forall t \in [0,\beta], f \in S^{Pe}_{F(\cdot,x(\cdot))}\right\}\end{aligned}$$

or, otherwise stated, $R(x)(t) = g(x) + (P)\int_0^t F(s,x(s))ds$ for every $t \in [0,\beta]$.

We are looking for a fixed point of a multifunction R, which by Theorem 1, is well-defined and has nonempty, convex values.

Let $\overline{U} = \{x_f \in C([0,\beta], E) : x_f(t) = (P)\int_0^t f(s)ds, t \in [0,\beta], f \in S_G^{Pe}\}$. It is nonempty, convex and bounded and similarly as U, it is equicontinuous. We show now that it is closed. Take a sequence $x_{f_n}(t) = (P)\int_0^t f_n(s)ds$ convergent uniformly to a continuous function $x(t)$ and prove that $x \in \overline{U}$. As S_G^{Pe} is sequentially compact for the topology induced by the tensor product $L^\infty \otimes E^*$ (cf. [2] Proposition 3.4), then there exists a subsequence (f_{n_k}) convergent to a selection $f \in S_G^{Pe}$. This implies that $(P)\int_0^t f_{n_k}(s)ds$ weakly converge to $(P)\int_0^t f(s)ds$, whence $x(t) = (P)\int_0^t f(s)ds$ for all $t \in [0,\beta]$ and $\overline{U}$ is closed. By Ascoli's theorem, $\overline{U}$ is weakly compact in $C([0,\beta], E)$.

On the other hand, we have the following estimation:

$$\|R(x)(t)\| \leq \|g(x)\| + \left\|(P)\int_0^t F(s,x(s))ds\right\| \leq a\|x\| + b + M, \ \forall t \in [0,\beta].$$

Therefore, for a solution of our problem we obtain an "a priori" estimation, namely $\|x\| \leq a\|x\| + b + M$. Then $\|x\| \leq \frac{b+M}{1-a}$. Denote by N the right-hand side of this inequality (recall, that $a < 1$ and $b > 0$) and by B_N the ball $\{x \in C([0,\beta], E) : \|x\| \leq N\}$. Note that by Assumption (d) solutions are defined on a whole $[0,\beta]$.

The set $V = \overline{U} \cap B_N$ is weakly compact and convex. In the next part of the proof we will consider V as a domain of R. It is clear that $R : V \to 2^V$.

Now, we are in position to show, that R has a weakly-weakly sequentially closed graph and therefore, it has closed graph when $C([0,\beta], E)$ is endowed with the weak topology, because this topology is metrizable on any weakly compact set.

Let $(x_n, y_n) \in GrR, \quad (x_n, y_n) \to (x, y)$ weakly in $C([0,\beta], E)$. From assumptions it follows that $g(x_n)$ tends weakly to $g(x)$.

Moreover, y_n is of the following form

$$y_n(t) = g(x_n) + (P)\int_0^t f_n(s)ds \ , \ f_n \in S_{F(\cdot,x_n(\cdot))}^{Pe} \ , \ t \in [0,\beta].$$

Since $f_n(t) \in F(t, x_n(t)) \subset G(t)$ a.e. and S_G^{Pe} is nonempty, convex and sequentially compact for the "weak" topology of pointwise convergence on $L^\infty \otimes E^*$ (cf. [2] Proposition 3.4), we extract a subsequence (f_{n_k}) of (f_n) such that (f_{n_k}) converges $\sigma(P_E^1, L^\infty \otimes E^*)$ to a function $f \in S_G^{Pe}$.

Fix an arbitrary $x^* \in E^*$. Since (f_{n_k}) $\sigma(P_E^1, L^\infty \otimes E^*)$-converges to $f \in S_G^{Pe}$, we have $\int_A x^* f_{n_k}(s)ds \to \int_A x^* f(s)ds$ for each measurable $A \subset [0,\beta]$ and $x^* \in E^*$. Thus for $y_{n_k}(t) = g(x_{n_k}) + (P)\int_0^t f_{n_k}(s)ds$ $x^* y_{n_k}$ tends to $x^* g(x) + \int_0^t x^* f(s)ds$, whence $y(t) = g(x) + (P)\int_0^t f(s)ds$.

It remains to show that $f \in S^{Pe}_{F(\cdot,x(\cdot))}$ (in order to have $(x, y) \in GraphR$). By Lemma 3, there exists a convex combination h_k of elements of the sequence $\{f_{n_m}, m \geq k\}$ that converges weakly to a measurable selection h of G. But this sequence is Pettis uniformly integrable (by Theorem 1) and so, as a consequence of a convergence theorem (such as Theorem 8.1 in [36]), the sequence of their integrals $((P)\int_0^t h_k(s)ds)_k$ weakly converges to $(P)\int_0^t h(s)ds$. It follows that $f = h$ a.e. By weak sequential hemi-continuity of $F(t, \cdot)$ and weak convergence of $h_k(t)$ in E we obtain $h(t) \in F(t, x(t))$ a.e. Therefore, $f \in S^{Pe}_{F(\cdot,x(\cdot))}$ and so, R has weakly-weakly sequentially closed graph.

Thus R satisfies the hypothesis of Kakutani's fixed point theorem and it has a fixed point. Moreover, the set of fixed points is compact in $C([0, \beta], E)$ with respect to the weak topology. ■

By a separability assumption we obtain the existence of strong Carathéodory solutions:

Corollary 6. *Under the conditions of Theorem* 5 *all pseudo-solutions of the problem* (3) *are in fact Carathéodory solutions.*

Proof. Since E is separable, the dual E^* contains a countable T set which separates points of E (i.e., E has total dual). Thus for any $z^* \in T$ and any pseudo-solution x of the problem (3) the functions $z^*x'(\cdot)$ and $z^*f(\cdot, x(\cdot))$ differ only on negligible sets N_{z^*}. As a consequence $x'(\cdot)$ and $f(\cdot, x(\cdot))$ are different on the countable union of sets N_{z^*}, hence x is a Carathéodory solution. ■

The key point of the proof of the previous theorem was the assumption of weak compactness of the values of the dominant function G. As a consequence, we were able to use a characterization of the family of all Pettis-integrable selections of G. If we drop this property, the situation is much more complicated. As it can be expected, we will need another compactness condition.

Theorem 7. *Assume that E is separable. Let $F : I \times E \to 2^E$ with nonempty convex and weakly compact values satisfy*

(a) *$F(t, \cdot)$ is weakly sequentially upper hemi-continuous for each $t \in I$,*

(b) *$F(\cdot, x)$ has a weakly measurable selection for each $x \in E$,*

(c) *for each $x^* \in E^*$ there exists a function $p_* \in L^2(I, \mathbb{R})$ such that for each $z \in E$ and almost all $t \in I$*

$$|s(x^*, F(t, z))| \leq p_*(t) \cdot (1 + \|z\|),$$

where $\|p_*\|_2 < L < 1$.

(d) *for each bounded subset* $B \subset E$ *and* $J \subset I$

$$\lim_{\tau \to t} \omega(F(J_{\tau,t} \times B)) \leq h(t, \omega(B)),$$

where h *is a Kamke function and* $J_{\tau,t} = [\tau, t] \cap J$,

(e) *the function* $g : C(I,E) \to E$ *satisfies:*

   (e1) *has sublinear growth:* $\|g(x)\| \leq a\|x\| + b$ *for* $a < 1 - L$ *and* $b > 0$,

   (e2) *is weakly-weakly sequentially continuous,*

   (e3) *there exists* $0 < d < 1$ *such that* $\omega(g(B)) \leq d \cdot \omega_C(B)$ *for each bounded subset* B *of* $C(I,E)$.

Then there exists at least one pseudo-solution of the Cauchy problem (3) *on* I. *Moreover, the set of all pseudo-solutions of this problem is weakly compact in* $C(I,E)$.

Proof. As in the previous theorem, define a multifunction $R : C(I,E) \to 2^{C(I,E)}$ by the following formula:

$$R(x)(t) = g(x) + (P) \int_0^t F(s, x(s))\, ds.$$

By hypothesis (c) it follows that for each $x \in C(I,E)$, $F(\cdot, x(\cdot))$ is scalarly integrable. Moreover, the family $\{s(x^*, F(\cdot, x(\cdot))), x^* \in E^*, \|x^*\| \leq 1\}$ is uniformly integrable, because for every measurable $A \subset I$

$$\int_A p_*(t)(1 + \|x(t)\|)dt \leq (1 + \|x\|) \int_0^T \chi_A(t) p_*(t) dt \leq (1 + \|x\|) \cdot L \cdot (\mu(A))^{\frac{1}{2}}.$$

Theorem 5.4 in [28] asserts that $F(\cdot, x(\cdot))$ is Pettis integrable, whence R is well-defined and its values are nonempty.

Let $x^* \in E^*$ be such that $\|x^*\| \leq 1$. We have the following estimation:

$$\begin{aligned} |s(x^*, R(x)(t))| &\leq \|g(x)\| + \int_0^t |s(x^*, F(s, x(s)))| ds \\ &\leq a \cdot \|x\| + b + \int_0^t p_*(s) \cdot (1 + \|x(s)\|) ds \\ &\leq a \cdot \|x\| + b + L \cdot (1 + \|x\|). \end{aligned}$$

Thus $\|R(x)(t)\| \leq a \cdot \|x\| + b + L \cdot (1 + \|x\|)$.

Put

$$N = \frac{b + L}{1 - a - L}$$

and remark that R maps the ball of $C(I,E)$ of radius N into itself. Define a set

$$W = \left\{ x \in C(I,E) : \|x\| \leq N, \|x(t) - x(\tau)\| \leq (1+N) \cdot L \cdot |t-\tau|^{\frac{1}{2}}, \quad \forall t, \tau \in I \right\}.$$

Obviously, it is nonempty, closed, convex and strongly equicontinuous.

Now, let $x \in W$, $t_1, t_2 \in I$ and let $x^* \in E^*$ be such that $\|x^*\| \leq 1$. Then

$$\begin{aligned}
&|s\,(x^*, R(x)(t_1) - R(x)(t_2))| \\
&\quad \leq \left| \int_0^{t_1} s(x^*, F(s,x(s)))ds - \int_0^{t_2} s(x^*, F(s,x(s)))ds \right| \\
&\quad \leq \int_{t_1}^{t_2} |s(x^*, F(s,x(s)))|ds \\
&\quad \leq \int_{t_1}^{t_2} p_*(s)(1 + \|x(s)\|)ds \\
&\quad \leq (1+N) \cdot \int_0^T \chi_{[t_1,t_2]}(s) \cdot p_*(s)ds \\
&\quad \leq (1+N) \cdot L \cdot |t_1 - t_2|^{\frac{1}{2}}.
\end{aligned}$$

By taking the supremum over all x^* such that $\|x^*\| \leq 1$, we deduce that the set $R(W)$ is a strongly equicontinuous subset of E and $R(W) \subset W$. Now, we consider R as the multifunction $R : W \to 2^W$.

We show that R has a weakly-weakly sequentially closed graph. Let $(x_n, y_n) \in GrR$, $(x_n, y_n) \to (x, y)$ weakly in $C(I,E)$.

From our assumptions it follows that $g(x_n)$ tends weakly to $g(x)$.

Moreover, Theorem 5.4 in [28] implies that the Pettis and the Aumann-Pettis integrals coincide for Pettis-integrable weakly compact convex-valued multifunctions and so, y_n is of the following form:

$$y_n(t) = g(x_n) + (P)\int_0^t f_n(s)ds\ , \forall t \in I\ , \text{ where } f_n \in S^{Pe}_{F(\cdot, x_n(\cdot))}.$$

Since for arbitrary τ we have $\{F(t, x_n(t)) : n \geq 1\} \subset \{F(J_{t,\tau} \times \{x_n(t)\}) : n \geq 1\}$, then using properties of the measure of weak noncompactness we get

$$\begin{aligned}
\omega(\{f_n(t) : n \geq 1\}) &\leq \omega(\{F(t, x_n(t)) : n \geq 1\} \\
&\leq \lim_{\tau \to t^+} \omega(\{F(J_{t,\tau} \times \{x_n(t)\}) : n \geq 1\} \\
&\leq h(t, \omega\{x_n(t) : n \geq 1\}) \quad a.e.
\end{aligned}$$

Since x_n is weakly convergent in $C(I,E)$, then $\{x_n(t) : n \geq 1\}$ is relatively weakly compact in E. Hence $\omega(\{x_n(t) : n \geq 1\}) = 0$ and since $h(t,0) = 0$, then $\omega(\{f_n(t) : n \geq 1\}) = 0$ a.e. on I. Thus the set $\{f_n(t) : n \geq 1\}$ is weakly compact for almost every t.

By redefining (if necessary) a new multifunction G, on the set of measure zero: $G(t) = \overline{\text{conv}}\{f_n(t) : n \geq 1\}$ we can say that $G(t)$ are nonempty, closed, convex and weakly compact for all t. We are able to construct such a multifunction G due to assumption (d).

Now it is clear that we are able to repeat a suitable part of the proof of Theorem 5 and obtain the desired continuity property.

It is known, that there exists a weakly compact subset K_∞ of W such that $R : K_\infty \to 2^{K_\infty}$ (cf. [18]).

Indeed, if we define a sequence of sets: $K_0 = W$, $K_{n+1} = \overline{\text{conv}} R(K_n)$, then $K_\infty = \bigcap_{n=0}^{\infty} K_n$. All the sets K_n are nonempty closed and convex. Moreover, it can be proved that they form a decreasing sequence of sets. In fact, $K_1 = \overline{\text{conv}} R(K_0) = \overline{\text{conv}} R(V) \subset V = K_0$.

As a subset of an equicontinuous set W the set K_1 is equicontinuous too.

Set $a_n(t) = \omega(R(K_n)(t))$. By the equicontinuity of K_0 and properties of the measure of weak noncompactness ω we obtain that the set $\{a_n : n \in \mathbb{N}\}$ is equicontinuous. Then a_n are absolutely continuous functions and $a_n(0) = 0$.

For arbitrary Pettis-integrable function Y we have

$$(P)\int_0^{t-\tau} Y(s)ds + (P)\int_{t-\tau}^{t} Y(s)ds = (P)\int_0^t Y(s)ds.$$

Put $E = (P)\int_0^t Y(s)ds$, $B = (P)\int_0^{t-\tau} Y(s)ds$ and $C = (P)\int_{t-\tau}^t Y(s)ds$. The integrals are considered in the Aumann sense, via Pettis-integrable selections.

As $E = \{(P)\int_0^{t-\tau} w(s)ds + (P)\int_{t-\tau}^t w(s)ds : w \in W\}$, by the definition of the Minkowski sum we get

$$E \subset \left\{(P)\int_0^{t-\tau} v(s)ds + (P)\int_{t-\tau}^{t} u(s)ds : v \in W, u \in W\right\} = B + C.$$

Then $\omega(E) \leq \omega(B+C) \leq \omega(B) + \omega(C)$ and $\omega(E) - \omega(B) \leq \omega(C)$. Similarly, for arbitrary bounded subset K we get $K + E \subset K + B + C$ and $\omega(K+E) \leq \omega(K+B) + \omega(C)$. Finally we get $\omega(K+E) - \omega(K+B) \leq \omega(C)$.

Taking $Y = F(\cdot, K_n(\cdot))$ for $0 < t - \tau < t \le \beta$, we obtain for each $n \ge 1$

$$\begin{aligned} a_n(t) - a_n(t-\tau) &= \omega(R(K_n)(t)) - \omega(R(K_n)(t-\tau)) \\ &\le \omega\left((P)\int_0^t F(s, K_n(s))ds\right) \\ &\quad - \omega\left((P)\int_0^{t-\tau} F(s, K_n(s))ds\right) \\ &\le \omega\left((P)\int_{t-\tau}^t F(s, K_n(s))\, ds\right). \end{aligned}$$

By the mean value theorem

$$(P)\int_{t-\tau}^t F(s, K_n(s))\, ds \in \tau \cdot \overline{\mathrm{conv}}\{F(s, K_n(s)) : s \in [t-\tau, t]\}.$$

Hence

$$a_n(t) - a_n(t-\tau) \le \tau \cdot \omega\left(\{F(s, K_n(s)) : s \in J_{t,\tau}\}\right).$$

Since $a_n(\cdot)$ is a real-valued absolutely continuous function, a_n is differentiable a.e.

Thus by properties of ω (cf. [23]) we have

$$\begin{aligned} a_n'(t) &\le \lim_{\tau \to 0^+} \omega\left(\{F(s, K_n(s)) : s \in J_{t,\tau}\}\right) \\ &\le \lim_{\tau \to 0^+} \omega(F(J_{t,\tau} \times K_n(J_{t,\tau}))) \quad \text{a.e.} \end{aligned}$$

Recall that K_n is strongly equicontinuous, then by Ambrosetti's Lemma for the measure of weak noncompactness we ensure that $\omega(K_n(J_{t,\tau})) = \sup_{s \in J_{t,\tau}} \omega(K_n(s))$ and $\lim_{\tau \to 0}\left(\sup_{s \in J_{t,\tau}} \omega(K_n(s))\right) = \omega(K_n(t))$.

As h is continuous we obtain

$$\begin{aligned} \lim_{\tau \to 0^+} \omega(F(J_{t,\tau} \times K_n(J_{t,\tau}))) &\le \lim_{\tau \to 0^+} \omega(F(J_{t,\tau} \times K_n(t))) \\ &\le h(t, \omega(K_n(t))). \end{aligned}$$

But $\omega(K_n(t)) = \omega(\overline{\mathrm{conv}} R(K_{n-1}(t))) = \omega(R(K_{n-1}(t))) = a_{n-1}(t)$. Therefore,

$$a_n'(t) \le h(t, \omega(K_n(t))) = h(t, a_{n-1}(t)) \quad \text{a.e.}$$

and

$$a_n(t) \leq \int_0^t h(s, a_{n-1}(s))\, ds.$$

The sequence $(a_n(t))_n$ being decreasing and bounded by 0, is pointwisely convergent (for each t) to the function $a(t)$.

By passing to the limit we get

$$a(t) \leq \int_0^t h(s, a(s))\, ds.$$

By this and the definition of the Kamke function the sequence $(a_n(t))$ converges uniformly to 0.

Using the Cantor-Kuratowski intersection lemma for the measure of weak noncompactness we deduce that K_∞ is nonempty, convex and weakly compact.

By the properties of multivalued mappings we obtain that R is weakly-weakly upper semi-continuous on weakly compact subset K_∞. Let us recapitulate: we have weakly-weakly upper semi-continuous multifunction $R : K_\infty \to cwk(K_\infty)$ $(R(K_\infty) \subset K_\infty$, cf. also Lemma 2 in [18]), so the fixed point theorem of Kakutani type for weak topology ([5]) applies to the map R and we get a fixed point $\overline{z}$ of R. Of course, $\overline{z}$ is a pseudo-solution of the problem (3). ∎

Remark 8. In the proof of the above theorem we exploit only some properties of the DeBlasi measure of weak noncompactness. Thus in the Assumption (d) we are able to put an arbitrary measure μ of weak noncompactness (from an axiomatic point of view cf. [10]) and then by the conclusion that $\mu(K_\infty) = 0$ more information about the set of solutions is allowed.

A little bit different situation is when the multifunction on the right hand side is not Pettis integrable. In such a case we are able to present an existence result for pseudo-K-solutions, using a characterization of HKP-integrable set-valued functions proved by Di Piazza and Musiał (Theorem 1 in [26]), that we recall in the sequel:

Theorem 9. *Let $\Gamma : [0,T] \to \mathcal{P}_{wkc}(E)$ be a scalarly HK-integrable multifunction. Then the following conditions are equivalent:*

(i) Γ *is HKP-integrable;*

(ii) Γ *has at least one HKP-integrable selection and for every HKP-integrable selection f there exists $G : [0,T] \to \mathcal{P}_{wkc}(E)$ Pettis integrable, such that $\Gamma(t) = f(t) + G(t), \forall t \in [0,T]$;*

(iii) *each measurable selection of Γ is HKP-integrable.*

Now, let us present a new result for differential inclusions, concerning pseudo-K-solutions (for other recent results see [41] or [42]):

Theorem 10. *Assume that E is separable. Let $F : I \times E \to 2^E$ with nonempty convex and weakly compact values satisfy hypothesis* (a), (b), (d), (e) *in Theorem* 5 *and*
(c') $F(t,x) \subset \Gamma(t)$ *a.e. for some* $cwk(E)$*-valued HKP-integrable multifunction* Γ*. Then there exists at least one pseudo-K-solution of the Cauchy problem* (3) *on* $[0,\beta] \subset I$*.*

Proof. Theorem 9 yields that there exist a HKP-integrable function f and a $\mathcal{P}_{wkc}(E)$-valued Pettis integrable multifunction G satisfying $\Gamma(t) = f(t) + G(t)$ for every $t \in I$, f is scalarly measurable and, f is measurable, because the Banach space is separable.

Then the set-valued function $\widetilde{F}(t,x) = -f(t) + F\big(t, x + (\mathrm{HKP})\int_0^t f(\tau)d\tau\big)$ satisfies conditions (a), (b), (d), (e) in Theorem 5. We prove that it also satisfies hypothesis (c). Indeed, $\widetilde{F}(t,x) \subset G(t),\ \forall t \in I,\ \forall x$.

Applying Theorem 5 we deduce the integral inclusion

$$\widetilde{x}\,(t) \in g(\widetilde{x}) + (\mathrm{P}) \int_0^t \widetilde{F}\,(s, \widetilde{x}\,(s))\, ds$$

has a solution in $C([0,\beta],E)$ for a subinterval $[0,\beta] \subset I$.

Therefore, one can find $\widetilde{x} \in C([0,\beta],E)$ such that

$$\widetilde{x}\,(t) \in g(\widetilde{x}) + (\mathrm{P}) \int_0^t -f(s) + F\left(s, \widetilde{x}(s) + (\mathrm{HKP}) \int_0^s f(\tau)d\tau\right) ds.$$

In other words

$$\widetilde{x}\,(t) + (\mathrm{HKP}) \int_0^t f(s)ds \in g(\widetilde{x}) + (\mathrm{HKP}) \int_0^t F\left(s, \widetilde{x}(s) + (\mathrm{HKP}) \int_0^s f(\tau)d\tau\right) ds.$$

Thus $x(\cdot) = \widetilde{x}(\cdot) + (\mathrm{HKP}) \int_0^{\cdot} f(\tau)d\tau$ is a weakly continuous function on $[0,\beta]$ with values in E and clearly it is a solution of our integral inclusion. ∎

In this case we should remark, that if the obtained pseudo-K-solution is (strongly) absolutely continuous, then it is also a Carathéodory solution.

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