ALMOST-RAINBOW EDGE-COLORINGS
OF SOME SMALL SUBGRAPHS

ELLiot Krop

Department of Mathematics, Clayton State University
2000 Clayton State Boulevard, Morrow, GA 30260 USA
e-mail: ElliotKrop@clayton.edu

AND

IRina Krop

DePaul University
1 E. Jackson, Chicago, IL 60604 USA
e-mail: irina.krop@gmail.com

Abstract

Let $f(n, p, q)$ be the minimum number of colors necessary to color the edges of $K_n$ so that every $K_p$ is at least $q$-colored. We improve current bounds on these nearly “anti-Ramsey” numbers, first studied by Erdős and Gyárfás. We show that $f(n, 5, 9) \geq \frac{7}{4}n - 3$, slightly improving the bound of Axenovich. We make small improvements on bounds of Erdős and Gyárfás by showing $\frac{5}{6}n + 1 \leq f(n, 4, 5)$ and for all even $n \not\equiv 1 \pmod{3}$, $f(n, 4, 5) \leq n - 1$. For a complete bipartite graph $G = K_{m,m}$, we show an $n$-color construction to color the edges of $G$ so that every $C_4 \subseteq G$ is colored by at least three colors. This improves the best known upper bound of Axenovich, Füredi, and Mubayi.

Keywords: Ramsey theory, generalized Ramsey theory, rainbow-coloring, edge-coloring, Erdős problem.

2010 Mathematics Subject Classification: 05A15, 05C38, 05C55.

1. Introduction

1.1. Definitions

For basic graph theoretic notation and definition see Diestel [3]. All graphs $G$ are undirected with the vertex set $V$ and edge set $E$. We use $|G|$ for $|V|$ and $|G|$ for
$|E|$. $K_n$ denotes the complete graph on $n$ vertices and $K_{n,m}$ the bipartite graph with $n$ vertices and $m$ vertices in the first and second part, respectively. For any edge $(u, v)$, let $C(u, v)$ be the color on that edge, and for any vertex $v$, let $C(v)$ be the set of colors on the edges incident to $v$. We say that an edge-coloring is proper if every pair of incident edges are of different colors. If vertices $u, v$ are adjacent, we write $u \sim v$.

1.2. Coloring edges

Given a graph $G$ of order $n$ and integers $p, q$ so that $2 \leq p \leq n$ and $1 \leq q \leq \binom{p}{2}$, call an edge-coloring $(p, q)$ if every $K_p \subseteq K_n$ receives at least $q$ colors on its edges. Let $f(n, p, q)$ be the minimum colors in a $(p, q)$ coloring of $K_n$. This generalization of classical Ramsey functions was first mentioned by Erdős in [4] and later studied by Erdős and Gyárfás in [5]. Further, define $\phi(n, p, q)$ to be the minimum colors in a proper $(p, q)$ coloring of $K_n$.

Extending the definition, for any graph $G$, call an edge coloring $(H, q)$ if every subgraph $H \subseteq G$ receives at least $q$ colors on its edges. Let $f(G, H, q)$ be the minimum colors in an $(H, q)$ coloring of the edges of $G$. We say that a coloring of $H$ is almost-rainbow if $q = \|H\| - 1$, that is, one color is repeated once.

For an extended survey regarding bounds on rainbow colorings, see [7]. Using the Local Lemma, the authors in [5] were able to produce bounds for $f(n, p, q)$, with several difficult cases unresolved. Among those were $f(n, 4, 3)$, $f(n, 4, 4)$, $f(n, 4, 5)$, and $f(n, 5, 9)$. In these cases they showed that $f(n, 4, 3) \leq c\sqrt{n}$, $c\sqrt{n} \leq f(n, 4, 4) \leq cn^2$, $\frac{5n-1}{6} \leq f(n, 4, 5) \leq n$, and $\frac{4}{3}n \leq f(n, 5, 9) \leq cn^2$. The authors further mentioned that in this branch of generalized Ramsey theory, finding the orders of magnitude of $f(n, 4, 4)$ and $f(n, 5, 9)$ are “the most interesting open problems, at least to show that the latter is non-linear”. The authors then stated the linearity of said function as Problem 1.

As for $f(n, 4, 5)$, the authors showed that $\frac{5(n-1)}{6} \leq f(n, 4, 5)$ with an upper bound of $n$ for odd $n$ and $n - 1$ for even $n$ if $n - 1$ is prime.

In [9], Mubayi showed that

$$f(n, 4, 3) \leq e^{O(\sqrt{\log n})}$$

and in [8] Kostochka and Mubayi showed that for some constant $c$,

$$f(n, 4, 3) \geq \frac{c \log n}{\log \log \log n}.$$ 

Fox and Sudakov in [6], further improved the lower bound to $\frac{\log n}{3000}$. As for the other case, in [1], Axenovich showed that for some constant $c$,

$$\frac{1 + \sqrt{5}}{2} n \leq f(n, 5, 9) \leq 2n^{1 + \frac{\sqrt{5}}{40}}.$$
In that same paper, she remarked that Tóth had communicated to her that the lower bound can be improved to $2n - 6$, however, the result has remained unpublished for over ten years.

In Section 2, we show

$$f(n, 5, 9) \geq \frac{7}{4}n - 3.$$  

In Section 3, we make minimal improvements in the work of [5], showing $\frac{5}{6}(n - 1) + 1 \leq f(n, 4, 5) \leq n - 1$ for even $n$ not congruent to one mod three.

In [2], the authors showed that $f(K_{n,n}, C_4, 3) \geq \frac{2}{3}n$, $f(K_{n,n}, C_4, 3) \leq n$ for odd $n \geq 5$, and $f(K_{n,n}, C_4, 3) \leq n + 1$ for even $n \geq 5$.

In Section 4, we show

$$f(K_{n,n}, C_4, 3) \leq n,$$  

for all $n \geq 3$.

We believe that this upper-bound is the best possible.

2. **Almost-rainbow Five-cliques**

2.1. **The main tool**

Let $f(G)$ be the minimum number of colors needed to color the edges of $G$ so that every path or cycle with four edges is at least three-colored.

Let $\phi(G)$ be defined as $f(G)$ above, except replace “color” by “properly color”. By arguments from [1] it is easy to see that $f(n, 5, 9) \leq \phi(n, 5, 9) = \phi(K_n)$.

**Lemma 1.** $\phi(K_{2,n}) = \lceil \frac{3}{2}n \rceil$.

**Proof.** Suppose the edges of $G = K_{2,n}$ are properly colored so that every path of length four receives at least three colors. Call the vertices in the first part of $G$, $u$ and $v$. Choose a color $a \in C(u) \cap C(v)$ so that for some vertices $x, y$ in the second part of $G$, $a = C(u, x) = C(v, y)$. Note that there exist colors $b, c$ so that $b = C(u, y)$, $c = C(v, x)$, and $b, c \in (C(u) \cup C(v)) \setminus (C(u) \cap C(v))$. Since there are two colors for every one in $C(u) \cap C(v)$, we can say that

$$(1) \quad |C(u) \cap C(v)| \leq \left\lfloor \frac{1}{2} |(C(u) \cup C(v)) \setminus (C(u) \cap C(v))| \right\rfloor.$$  

Applying this inequality to the principle of inclusion-exclusion, we write

$$|C(u) \cup C(v)| = |C(u)| + |C(v)| - |C(u) \cap C(v)| \geq 2n - \frac{1}{3} |C(u) \cup C(v)|.$$
Solving for the union we get

\[(2) \quad |C(u) \cup C(v)| \geq \frac{3}{2}n.\]

For the upper bound, we construct an edge-coloring of \(G = K_{2,n}\) with \(\lceil \frac{3}{2}n \rceil\) colors. Label the vertices of the first part of \(G, u, v\) and the second part \(\{v_1, v_2, \ldots, v_n\}\). Let \(r = \lceil \frac{n}{2} \rceil\). Color the edges \((v_1, u), (v_2, u), \ldots, (v_r, u)\) by the colors \(1, \ldots, r\). If \(n\) is even, color the edges \((v_n, v), (v_{n-1}, v), \ldots, (v_{n-r+1}, v)\) from the set of colors \(\{1, \ldots, r\}\). If \(n\) is odd, color the edges \((v_n, v), (v_{n-1}, v), \ldots, (v_{n-r+2}, v)\) by some of the colors from the set \(\{1, \ldots, r\}\). Color the remaining edges distinctly by all the colors not previously used. Let \(i\) and \(j\) be such that \(C(u, v_i) = C(v, u_j)\). Notice that for any \(k \in \{1, \ldots, n\}\), \(\{C(u, v_i), C(u, v_j), C(v, v_i), C(u, v_k)\}\) are pairwise distinct. Hence every 4-path receives at least three colors.

\[\blacksquare\]

### 2.2. A small improvement

**Theorem 2.** \(f(n, 5, 9) \geq \frac{7}{4}n - 3\).

**Proof.** Consider a \((5, 9)\) edge-coloring of \(G = K_n\) using \(s\) colors. Using the argument of Axenovich [1], we first assume that the coloring is not proper, so there exist incident edges \((v_1, v_2)\) and \((v_1, v_3)\) of the same color. For the coloring to remain \((5, 9)\), all edges of \(G \setminus \{(v_1, v_2), (v_1, v_3)\}\) incident to \(\{v_1, v_2, v_3\}\) must be of different colors and not \(C(v_1, v_2)\) or \(C(v_2, v_3)\). Therefore, \(s \geq 3n - 7 \geq \frac{7}{4}n - 3\) for \(n \geq 5\).

Next we assume the coloring is proper. By the pigeonhole principle there exists a color, call it \(a\), used on at least \(\binom{n}{2}/s\) edges. Let \(A\) be the set of vertices adjacent to edges colored \(a\) and choose vertices \(u, v \in A\) so that \(c(u, v) = a\).

We say that an edge is in \(A\) if both vertices adjacent to that edge are in \(A\). Notice that the number of colors on the edges in \(A\) adjacent to \(u \geq \frac{\binom{n}{2}}{s} - 1\), the same for \(v\), and \(c(u, v)\) is counted both times. Let \(H\) be the complete bipartite graph with vertices \(\{u, v\}\) in the first part and the vertices of \(G \setminus A\) in the second part. Let the edge coloring of \(H\) be induced by the edge coloring of \(G\). For any \(x \in A\) and \(y \in G\), \(C(u, x) \neq C(v, y)\), else we produce a two-colored four-edge path. The same reasoning holds for \(y \in A\) and \(x \in G\). This implies that the colors on the edges of \(H\) are distinct from the colors previously counted. Hence we apply Lemma 1 to \(H\) to obtain

\[(3) \quad s \geq 2 \left(\frac{n}{2}\right) - 1 + 2 \left(\frac{n}{s}\right) - 1 - 1 + \frac{3}{2} \left(n - 2 \left(\frac{n}{2}\right)\right).\]

Solving for \(s\) we obtain the result.

\[\blacksquare\]
3. Almost-rainbow Four-cliques

We obtain a marginal improvement for the lower bound on $f(n, 4, 5)$ and extend the even case of the upper bound from [5] to all complete graphs with orders not congruent to one modulo three.

**Theorem 3.** (i) $\frac{5}{6}(n - 1) + 1 \leq f(n, 4, 5)$.
(ii) $f(n, 4, 5) \leq n - 1$ for even $n \not\equiv 1 \pmod{3}$.

**Proof.** Given a $(4, 5)$ coloring of the edges of $G = K_n$, for a fixed vertex $u$, let $P_u$ denote the set of edges incident to $u$, whose colors are repeated on other edges incident to $u$. Let $S_u$ denote the set of edges with non-repeated colors, incident to $u$. Let $T_u$ denote the set of edges incident to edges from $P_u$ of the same color.

\[ S_u \cup P_u \cup T_u \]

Notice that

1. $\mathcal{C}(P_u) \cap \mathcal{C}(S_u) = \emptyset$ by definition.
2. $\mathcal{C}(P_u) \cap \mathcal{C}(T_u) = \emptyset$ else we obtain an induced four-colored $K_4$ on the edges $p \in P_u$ and $t \in T_u$ that share the same same color and the edges $p_1, p_2 \in P_u$ that share the same color and are incident to $t$ ($p$ may be equal to $p_1$, depending on the coloring).
3. $\mathcal{C}(S_u) \cap \mathcal{C}(T_u) = \emptyset$ else we obtain an induced four colored $K_4$ on the the edge $s \in S_u$ and $t \in T_u$ of the same color and the two edges of $P_u$ with the same color, which are incident to $t$.
4. For any vertex $v$ distinct from $u$, if $(u, v) \in P_u$ so that $\mathcal{C}(u, v) = \mathcal{C}(u, w)$ for some $w$, then $(u, v) \notin P_v$ and $(v, w) \notin P_v$.
5. For any vertex $v$ distinct from $u$, $T_u \cap T_v = \emptyset$.

Notice that

\[ 2 \sum_{u \in V(G)} |T_u| = \sum_{u \in V(G)} |P_u|, \]
so that
\[ \sum_u |T_u| + \sum_u |P_u| = 3 \sum_u |T_u| = 3 \frac{1}{n} \sum_u |T_u| \times n \leq \binom{n}{2} \]
by the above claim 5, and we obtain
\[ \frac{1}{n} \sum_u |T_u| \leq \frac{n - 1}{6}. \]
By the pigeonhole principle, choose a vertex \( u \) so that \( |T_u| \leq n - \frac{1}{6} \). Notice that
\[ n - 1 = \deg u = |S_u| + |P_u| \leq |S_u| + \frac{n - 1}{3}, \]
so that
\[ |S_u| \geq \frac{2}{3} (n - 1). \]
Summing up the colors of edges incident to \( u \) we get
\[ |C(u)| = |S_u| + \frac{1}{2} |P_u| \geq \frac{2}{3} (n - 1) + \frac{1}{6} (n - 1) = \frac{5}{6} (n - 1). \]
However, \( C(T_u) \) must be nonempty and distinct from the colors counted above, hence
\[ |C(u)| \geq \frac{5}{6} (n - 1) + 1. \]
For the upper bound we color the edges of \( K_n \) by a classical proper coloring (see [10] for example) and show that such a coloring is \((4, 5)\).

For odd \( n \), we \( n \)-color the edges of \( K_n \) by drawing the vertices in the form of a regular \( n \)-gon and coloring the consecutive edges around the boundary in order with colors 1 to \( n \). Next we color every edge parallel to a boundary edge by the same color as that boundary edge. Call the resulting labeled graph \( G_n \). Notice that every \( K_4 \subseteq G_n \) with a pair of parallel edges is a non-rectangular trapezoid. Hence the coloring is \((4, 5)\).

For even \( n \), choose a \( K_{n-1} \) subgraph and color it as above, obtaining \( G_{n-1} \). Next construct the graph \( w \times G_{n-1} \), joining the above graph to a vertex \( w \). Since for any vertex \( u \) of \( G_{n-1} \), there are only \( n - 2 \) incident edges, some color is missing. Apply this color to the edge \((u, w)\) and continue likewise for all vertices of \( G_{n-1} \). Call the resulting labeled graph \( G_n^* \).

For vertices \( x, y, z \in G_n^* \) with so that \((x, y)\) and \((y, z)\) are boundary edges, we say that \( y \) is opposite an edge \( e \) if the line bisecting angle \( uvw \) is the perpendicular bisector of \( e \). Notice that the edges opposite to \( y \) share the same color, which is not used on any edge incident to \( y \). By the above observation, \( G_{n-1} \subseteq G_n^* \) is \((4, 5)\)-colored, hence it is enough to show that for \( w \) as chosen above in the definition of \( G_n^* \) and any other distinct vertices \( x, y, z \) of \( G_n^* \), the induced subgraph receives
at most one repeated color. Choose any vertex \( v \in G_n \). For \( i = 1, \ldots, n-2 \) label the vertices with counterclockwise distance \( i \) from \( v \), \( u_i \), where arithmetic of label indices is performed modulo \( n-1 \). Notice that the only edges that share the color \( \mathcal{C}(w, v) \) are \((u_1, u_{-1}), (u_2, u_{-2}), \ldots, (u_{n-2}, u_{-(n-2)})\). For \( i = 1, \ldots, \frac{n-2}{2} \), if \( \mathcal{C}(u, w) = \mathcal{C}(u_{-i}, v) \), then for any edge \( e \) opposite \( u_i \), \( \mathcal{C}(e) = \mathcal{C}(u_{-i}, v) \). However, this means that

\[
\mathcal{C}(u_{i-1}, u_{i+1}) = \mathcal{C}(e) = \mathcal{C}(u_{-i}, v) \iff vu_{2k} = vu_{-k}
\]

\[
\iff 3k \equiv 0 \pmod{(n-1)} \iff n \equiv 1 \pmod{3}.
\]

4. **Almost-rainbow Four-cycles**

We show the improved upper bound for the bipartite problem, when the two parts of \( G \) are of equal size.

**Theorem 4.**

\[ f(K_{n,n}, G_4, 3) \leq n, \text{ for all } n \geq 3. \]

4.1. **The coloring**

We will explore the matrix

\[
G = \begin{pmatrix}
1 & 2 & 3 & \ldots & r & \ldots & c+1 & \ldots & n \\
3 & 1 & 2 & \ldots & r-1 & \ldots & c & \ldots & n-1 \\
v_3 & n-1 & 1 & \ldots & r-2 & \ldots & c-1 & \ldots & n-2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{n+1-r} & r+1 & r+2 & \ldots & 1 & \ldots & r+c & \ldots & r \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{n-1} & 3 & 4 & \ldots & r+1 & \ldots & c+2 & \ldots & 2 \\
 n-2 & u_2 & u_3 & \ldots & u_r & \ldots & u_{c+1} & \ldots & 1 \\
\end{pmatrix}
\]

The values of \( v_i \) and \( u_i \) will be defined shortly.

Let permutation \( \sigma \) be the \( n-1 \) cycle \((1 \ 2 \ \cdots \ n-1)\). That is, \( \sigma \) sends \( i \) to \( i + 1 \pmod{n-1} \). For a natural number \( m \) we shall write \( m \pmod{(n-1)} \) for its representative in \( \{1, 2, \ldots, n-1\} \). For each \( r \) we defined \( \sigma^{(r)} \) by the rule \( \sigma^{(r)}(c) \equiv r + c \pmod{(n-1)} \). Let us start with the matrix
The entries of $G$

\[ C = \begin{pmatrix}
    2 & 3 & \ldots & c+1 & \ldots & n \\
    \sigma^0(1) & \sigma^0(2) & \ldots & \sigma^0(c) & \ldots & \sigma^0(n-1) \\
    \sigma^{n-2}(1) & \sigma^{n-2}(2) & \ldots & \sigma^{n-2}(c) & \ldots & \sigma^{n-2}(n-1) \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    \sigma^r(1) & \sigma^r(2) & \ldots & \sigma^r(c) & \ldots & \sigma^r(n-1) \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    \sigma^2(1) & \sigma^2(2) & \ldots & \sigma^2(c) & \ldots & \sigma^2(n-1)
\end{pmatrix}. \]

We define the matrix $G$ by adding the first column $V = \{v_1, \ldots, v_{n-1}, vu\}$ and the last row $U = \{vu, u_2, \ldots, u_n\}$ to the matrix $C$.

\[ G = \begin{pmatrix}
    v_1 & 2 & 3 & \ldots & c+1 & \ldots & n \\
    v_2 & \sigma^0(1) & \sigma^0(2) & \ldots & \sigma^0(c) & \ldots & \sigma^0(n-1) \\
    v_3 & \sigma^{n-2}(1) & \sigma^{n-2}(2) & \ldots & \sigma^{n-2}(c) & \ldots & \sigma^{n-2}(n-1) \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    v_{n+1-r} & \sigma^r(1) & \sigma^r(2) & \ldots & \sigma^r(c) & \ldots & \sigma^r(n-1) \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    v_{n-1} & \sigma^2(1) & \sigma^2(2) & \ldots & \sigma^2(c) & \ldots & \sigma^2(n-1) \\
    vu & u_2 & u_3 & \ldots & u_{c+1} & \ldots & u_n
\end{pmatrix}. \]

The entries of $G$ will be defined as follows: for every 4-tuple $(i, j; l, m)$ with $1 \leq i < j \leq n$ and $1 \leq l < m \leq n$ the $(2 \times 2)$ matrix

\[ G(i, j; l, m) = \begin{pmatrix}
    a_{il} & a_{im} \\
    a_{jl} & a_{jm}
\end{pmatrix}. \]

We consider the colorings for the edges $V$ and $U$ in three types of even $n \pmod{6}$.

**Type 1:** Matrix $G_1 = G$ for $n \equiv 2 \pmod{6}$: \[ n = 2 + 6k, \ k \geq 1 \]

\[ a_{i,1} = \begin{cases}
    1, & i = 1, \\
    3, & i = 2, \\
    n, & 3 \leq i \leq \frac{n}{2} + 1, \\
    2(i-1) - n, & \frac{n}{2} + 2 \leq i \leq n-1, \\
    n - 2, & i = n.
\end{cases} \]

\[ a_{n,l} = \begin{cases}
    n - 2l, & 1 \leq l \leq \frac{n}{2} - 1, \\
    n, & \frac{n}{2} \leq l \leq n - 2, \\
    n - 1, & l = n - 1, \\
    1, & l = n.
\end{cases} \]

**Type 2:** Matrix $G_2 = G$ for $n \equiv 6 \pmod{6}$: \[ n = 6 + 6k, \ k \geq 1 \]

We define the matrix $G$ by adding the first column $V = \{v_1, \ldots, v_{n-1}, vu\}$ and the last row $U = \{vu, u_2, \ldots, u_n\}$ to the matrix $C$.
We define \( Y \) as \( \frac{n}{2} - 2 \) for even \( k \), and as \( \frac{n}{2} + 1 \) for odd \( k \).

\[
a_{i,1} = \begin{cases} 
1, & i = 1, \\
3, & i = 2, \\
n, & 3 \leq i \leq \frac{n}{2} + 1, \\
Y, & i = \frac{n}{2} + 2, \\
2(i-2) - n, & \frac{n}{2} + 3 \leq i \leq n-1, \\
n-2, & i = n.
\end{cases}
\]

\[
a_{n,l} = \begin{cases} 
n-2, & l = 1, \\
n-2(l+1), & 2 \leq l \leq \frac{n}{2} - 2, \\
Y, & l = \frac{n}{2} - 1, \\
n, & \frac{n}{2} \leq l \leq n-2, \\
n-1, & l = n-1, \\
1, & l = n.
\end{cases}
\]

Exception for \( n = 6; \) \([k = 0]\) the first row \( V = \{1, 5, 6, 4\} \), the last column \( U = \{3, 6, 6, 5, 1\} \).

**Type 3:** Matrix \( G_3 = G \) for \( n \equiv 4 \pmod{6}; \) \([n = 4 + 6k, k \geq 4]\)

The regularity starts with \( n > 22 \).

\[
a_{i,1} = \begin{cases} 
1, & i = 1, \\
3, & i = 2, \\
n, & 3 \leq i \leq \frac{n}{2} + 1, \\
n-9, & i = \frac{n}{2} + 2, \\
2(i-2) - n, & \frac{n}{2} + 3 \leq i \leq \frac{5n+4}{6}, \\
2(i-1) - n, & \frac{5n+10}{6} \leq i \leq n-1, \\
n-2, & i = n.
\end{cases}
\]

\[
a_{n,l} = \begin{cases} 
n-2l, & 1 \leq l \leq \frac{n-4}{6}, \\
n-2(l+1), & \frac{n+2}{6} \leq l \leq \frac{n}{2} - 2, \\
n-9, & l = \frac{n}{2} - 1, \\
n, & \frac{n}{2} \leq l \leq n-2, \\
n-1, & l = n-1, \\
1, & l = n.
\end{cases}
\]

Exceptions:

For \( n = 10 \) we replace \( (n-9) \) with \( (n-8) \).
For \( n = 16 \) we replace \( (n-9) \) with \( (n-11) \).
For \( n = 22 \) we replace \( (n-9) \) with \( (n-5) \) and the definitions:

\[
a_{i,1} = \begin{cases} 
2(i-2) - n, & \frac{n}{2} + 3 \leq i \leq \frac{5n-2}{6}, \\
2(i-1) - n, & \frac{5n+4}{6} \leq i \leq n-1.
\end{cases}
\]

\[
a_{n,l} = \begin{cases} 
n-2l, & 1 \leq l \leq \frac{n-10}{6}, \\
n-2(l+1), & \frac{n-4}{6} \leq l \leq \frac{n}{2} - 2.
\end{cases}
\]
4.2. Sketch of proof

First, we show that every 4-cycle defined in the basic coloring (matrix entries $a_{ij}$ where $1 < i \leq n, 1 \leq j < n$) is almost rainbow. That is, given $i < j$ and $l < m$ we show that $a_{i,l}, a_{j,l}, a_{i,m}, a_{j,m}$ contains at least three distinct elements in the basic coloring.

Step 1. We start with the matrix $C$ and look at two occurrences, which are identical for each of the types of even $n (\text{mod} \ 6)$ specified above.

Case 1. We take the submatrix of $G(i, j; l, m)$ with $2 \leq l < m \leq n, 2 \leq i < j < n$, and let $s = (n + 1) - i, t = (n + 1) - j$. A typical $(2 \times 2)$ submatrix has the form:

$$
\begin{pmatrix}
\sigma^*(l - 1) & \sigma^*(m - 1) \\
\sigma'(l - 1) & \sigma'(m - 1)
\end{pmatrix}.
$$

We wish to show there are three distinct elements: $\sigma^*(l - 1) \neq \sigma'(l - 1), \sigma'(l - 1) \neq \sigma'(m - 1), \sigma^*(l - 1) \neq \sigma'(m - 1)$.

Suppose $\sigma^*(l - 1) \equiv \sigma'(l - 1) \Rightarrow s \equiv t (\text{mod } n - 1)$, which is a contradiction.

Suppose $\sigma^*(l - 1) \equiv \sigma'(m - 1) \Rightarrow l \equiv m (\text{mod } n - 1)$, which is a contradiction.

Suppose $\sigma'(l - 1) \equiv \sigma'(m - 1) \Rightarrow s + l \equiv t + m (\text{mod } n - 1)$, and assume there are three distinct elements: $\sigma^*(l - 1) \neq \sigma'(m - 1), \sigma^*(m - 1) \neq \sigma'(l - 1), \sigma^*(m - 1) \neq \sigma'(m - 1)$. Follow the argument above the first two inequalities are correct. Suppose $\sigma^*(m - 1) \equiv \sigma'(l - 1) \Rightarrow s + m \equiv t + l (\text{mod } n - 1)$. Subtracting equations $s + l \equiv t + m$ and $s + m \equiv t + l \Rightarrow l \equiv m (\text{mod } n - 1)$, which is a contradiction. One of the following two sets has three distinct elements: \{ $\sigma^*(l - 1), \sigma^*(m - 1), \sigma'(m - 1)$ \} or \{ $\sigma'(l - 1), \sigma^*(m - 1), \sigma^*(m - 1)$ \}.

Case 2. We take the submatrix of $G(i, j; l, m)$ with $2 \leq l < m \leq n, i = 1, 1 < j < n$, and let $r = (n + 1) - j$. A typical $(2 \times 2)$ submatrix has the form:

$$
\begin{pmatrix}
l & m \\
\sigma^*(l - 1) & \sigma^*(m - 1)
\end{pmatrix}.
$$

We wish to show there are three distinct elements: $l \neq m, m \neq \sigma^*(m - 1), l \neq \sigma^*(m - 1)$. Suppose $m \equiv \sigma/(m - 1) \Rightarrow r \equiv 1 (\text{mod } n - 1)$, which is a contradiction. Suppose $l \equiv \sigma/(m - 1) \Rightarrow l \equiv r + m - 1 (\text{mod } n - 1)$, and assume there are three distinct elements: $\sigma^*(l - 1) \neq \sigma^*(m - 1), m \neq \sigma^*(m - 1), m \neq \sigma^*(l - 1)$. The first two inequalities are correct. Suppose $m \equiv \sigma^*(l - 1) \Rightarrow m \equiv r + l - 1 (\text{mod } n - 1)$. Subtracting equations $l \equiv r + m - 1$ and $m \equiv r + l - 1 \Rightarrow r \equiv 1 (\text{mod } n - 1)$, which is a contradiction. One of the following two sets has three distinct elements: \{ $l, m, \sigma^*(m - 1)$ \} or \{ $\sigma^*(l - 1), \sigma^*(m - 1), m$ \}.

Step 2. For matrix $G(i, j; l, m)$ with $i = 1, j = n$ and $2 \leq l < m \leq n$ we look at five cases and consider every matrix type defined above of even $n (\text{mod} \ 6)$. 
Case 1. We take $G(i, j; l, m)$ with $2 \leq l < m \leq \frac{n}{2} - 1$, $i = 1$, $j = n$.

Subcase 1.1. Consider $G_1$,

$$\begin{pmatrix} l & m \\ n - 2l & n - 2m \end{pmatrix}.$$

We wish to show there are three distinct entries: $l \neq n - 2l$, $n - 2l \neq n - 2m$, $l \neq n - 2m$. Suppose $l = n - 2l \Rightarrow 3l = n$ and since $n = 2 + 6k$ this is a contradiction. Suppose $n - 2l = n - 2m \Rightarrow l = m$, which is a contradiction. Suppose $l = n - 2m$ and we wish to show there are three distinct elements: $l \neq m$, $l \neq n - 2l$, $m \neq n - 2l$. As shown above the first inequality is correct. Suppose $m = n - 2l$ and since $l = n - 2m \Rightarrow l = m$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n - 2l, n - 2m\}$ or $\{l, m, n - 2l\}$.

Subcase 1.2. Consider $G_2$.

1. $G_2$ with $2 \leq l < m \leq \frac{n}{2} - 2$, $i = 1$, $j = n$,

$$\begin{pmatrix} l & m \\ n - 2(l + 1) & n - 2(m + 1) \end{pmatrix}.$$

We wish to show there are three distinct entries: $l \neq n - 2(m + 1)$, $l \neq n - 2(l + 1)$, $n - 2(l + 1) \neq n - 2(m + 1)$. Suppose $l = n - 2l - 2 \Rightarrow 3l = n - 2$ and since $n = 6 + 6k$ this is a contradiction. Suppose $n - 2l = n - 2m \Rightarrow l = m$, which is a contradiction. Suppose $l = n - 2(m + 1)$ and we wish to show there are three distinct elements: $l \neq m$, $l \neq n - 2(l + 1)$, $m \neq n - 2(l + 1)$. As shown above the first inequality is correct. Suppose $m = n - 2(l + 1)$ and since $l = n - 2(m + 1) \Rightarrow l = m$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n - 2(m + 1), n - 2(l + 1)\}$ or $\{l, m, n - 2(l + 1)\}$.

2. $G_2$ with $2 \leq l \leq \frac{n}{2} - 2$, $m = \frac{n}{2} - 1$, $i = 1$, $j = n$,

$$\begin{pmatrix} l & m \\ n - 2(l + 1) & Y \end{pmatrix}.$$

If $K$ is even $\Rightarrow m = \frac{n}{2} - 1$, $Y = \frac{n}{2} - 2 \Rightarrow Y = m - 1$.

We wish to show there are three distinct entries: $l \neq m$, $m \neq m - 1$, $l \neq m - 1$. Assume $l = m - 1$ and we wish to show there are three distinct entries: $m \neq m - 1$, $m \neq n - 2(l + 1)$, $m - 1 \neq n - 2(l + 1)$. Suppose $m = n - 2(l + 1)$ and since $l = m - 1$ and $m = \frac{n}{2} - 1 \Rightarrow n = 6$, which is a contradiction. Suppose $m - 1 = n - 2(l + 1)$ and since $m = \frac{n}{2} - 1$ and $l = m - 1 \Rightarrow n = 8$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, m, Y\}$ or $\{m, Y, n - 2(l + 1)\}$.

If $K$ is odd $\Rightarrow m = \frac{n}{2} - 1$, $Y = \frac{n}{2} + 1 \Rightarrow Y = m + 2$. Three distinct elements are $\{l, m, Y\}$. 
Suppose \( l \) one of the following two sets has three distinct elements: \( \{ l, n-2l, n-2m \} \) or \( \{ l, m, n-2l \} \).

1. \( G_3 \) with \( i = 1, j = n \) and \( 2 \leq l < m \leq \frac{n-4}{6} \) or \( \frac{n+2}{6} \leq l < m \leq \frac{n}{2} - 2 \). The argument is similar to above one with \( G_1 \). One of the following two sets has three distinct elements: \( \{ l, n-2l, n-2m \} \) or \( \{ l, m, n-2l \} \).

2. \( G_3 \) with \( 2 \leq l \leq \frac{n-4}{6}, \frac{n+2}{6} \leq m \leq \frac{n}{2} - 2, i = 1, j = n, \)
\[
\binom{l}{n-2l} \binom{m}{n-2(m+1)}.
\]

We wish to show there are three distinct entries: \( l \neq n-2l, n-2l \neq n-2(m+1), l \neq n-2(m+1) \). Suppose \( l = n-2l \Rightarrow 3l = n \) and since \( n = 4 + 6k \) this is a contradiction. Suppose \( n-2l = n-2(m+1) \Rightarrow l = m+1, \) which is a contradiction. Suppose \( l = n-2(m+1) \) and we wish to show there are three distinct elements: \( l \neq m, l \neq n-2l, m \neq n-2l \). The first two inequalities are correct. Suppose \( m = n-2l \) and since \( l = n-2(m+1) \Rightarrow m = l-2, \) which is a contradiction. One of the following two sets has three distinct elements: \( \{ l, n-2l, n-2(m+1) \} \) or \( \{ l, m, n-2l \} \).

3. \( G_3 \) with \( 2 \leq l \leq \frac{n-4}{6}, m = \frac{n}{2} - 1, i = 1, j = n, \)
\[
\binom{l}{n-2l} \binom{m}{n-9}.
\]

We wish to show there are three distinct entries: \( l \neq m, m \neq n-9, l \neq n-9 \). Suppose \( l = n-9 \) and since \( l < \frac{n-4}{6} \Rightarrow n-9 < \frac{n-4}{6} \Rightarrow n < 10, \) which is a contradiction. Suppose \( m = n-9 \Rightarrow \frac{n}{2} - 1 = n-9 \Rightarrow n = 16, \) which is a contradiction. There are three distinct elements \( \{ l, m, n-9 \} \).

4. \( G_3 \) with \( \frac{n+2}{6} \leq l \leq \frac{n}{2} - 2, m = \frac{n}{2} - 1, i = 1, j = n, \)
\[
\binom{l}{n-2(l+1)} \binom{m}{n-9}.
\]

We wish to show there are three distinct entries: \( l \neq m, m \neq n-9, l \neq n-9 \). Suppose \( l = n-9 \) and since \( l < \frac{n}{2} - 2 \Rightarrow n-9 < \frac{n}{2} - 2 \Rightarrow n < 14, \) which is a contradiction. Suppose \( m = n-9 \Rightarrow \frac{n}{2} - 1 = n-9 \Rightarrow n = 16, \) which is a contradiction. There are three distinct elements \( \{ l, m, n-9 \} \).

Case 2. For the submatrix \( G(i, j; l, m) \) with \( i = 1, j = n \) and \( \frac{n}{2} \leq l < m \leq n - 2 \) or \( 2 \leq l \leq \frac{n}{2} - 1, \frac{n}{2} \leq m \leq n - 2 \) three distinct elements are \( \{ l, m, n \} \).

Case 3. We take the submatrix \( G(i, j; l, m) \) with \( 2 \leq l \leq \frac{n}{2} - 1, m = n - 1, i = 1, j = n. \)

Subcase 3.1. Consider \( G_1 \),
\[
\begin{pmatrix}
l & m \\
n - 2l & n - 1
\end{pmatrix}.
\]

We wish to show there are three distinct entries: \( l \neq n - 1, n - 2l \neq n - 1, l \neq n - 2l \). Suppose \( n - 2l = n - 1 \Rightarrow l = \frac{n}{2} \), which is a contradiction. Suppose \( l = n - 2l \Rightarrow 3l = n \) and since \( n = 2 + 6k \) this is a contradiction. There are three distinct elements \( \{l, n - 1, n - 2l\} \).

Subcase 3.2. Consider \( G_2 \).

1. \( G_2 \) with \( 2 \leq l \leq \frac{n}{2} - 2, m = n - 1, i = 1, j = n, \)

\[
\begin{pmatrix}
l & m \\
n - 2(l + 1) & n - 1
\end{pmatrix}.
\]

We wish to show there are three distinct entries: \( l \neq n - 1, n - 2(l + 1) \neq n - 1, l \neq n - 2(l + 1) \). Suppose \( n - 2(l + 1) = n - 1 \Rightarrow l = -\frac{1}{2} \), which is a contradiction. Suppose \( l = n - 2(l + 1) \Rightarrow 3l = n - 2 \) and since \( n = 6 + 6k \) this is a contradiction. There are three distinct elements \( \{l, n - 1, n - 2(l + 1)\} \).

2. \( G_2 \) with \( l = \frac{n}{2} - 1, m = n - 1, i = 1, j = n, \)

\[
\begin{pmatrix}
l & m \\
y & n - 1
\end{pmatrix}.
\]

If \( K \) is even \( \Rightarrow Y = \frac{n}{2} - 2 \). If \( n - 1 = Y \Rightarrow n - 1 = \frac{n}{2} - 2 \Rightarrow n = -2 \), which is a contradiction. Three distinct entries are \( \{l, Y, n - 1\} \).

If \( K \) is odd \( \Rightarrow Y = \frac{n}{2} + 1 \), and three distinct entries are \( \{l, Y, n - 1\} \).

Subcase 3.3. Consider \( G_3 \).

1. For \( G_3 \) with \( l \leq \frac{n - 4}{6}, m = n - 1, i = 1, j = n \) three distinct elements are \( \{l, n - 1, n - 2l\} \) (similar to \( G_1 \)).

2. \( G_3 \) with \( \frac{n + 2}{6} \leq l < n - 1, m = n - 1, i = 1, j = n, \)

\[
\begin{pmatrix}
l & m \\
n - 2(l + 1) & n - 1
\end{pmatrix}.
\]

We wish to show there are three distinct entries: \( l \neq n - 1, n - 2(l + 1) \neq n - 1, l \neq n - 2(l + 1) \). Suppose \( n - 2(l + 1) = n - 1 \Rightarrow l = -\frac{1}{2} \), which is a contradiction. Suppose \( l = n - 2(l + 1) \Rightarrow 3l = n - 2 \) and since \( n = 4 + 6k \) this is a contradiction. Three distinct elements are \( \{l, n - 1, n - 2(l + 1)\} \).

3. For \( G_3 \) with \( l = \frac{n}{2} - 1, m = n - 1, i = 1, j = n \) three distinct entries are \( \{l, n - 9, n - 1\} \).

Case 4. For the submatrix \( G(i, j; l, m) \) with \( \frac{n}{2} \leq l \leq n - 2, m = n - 1, i = 1, j = n \) three distinct elements are \( \{l, n, n - 1\} \).
Case 5. For the submatrix $G(i, j; l, m)$ with $l = n - 1$, $m = n$, $i = 1$, $j = n$ three distinct elements are $\{n-1, n, 1\}$.

The argument for other steps is similar. To see the details please view the appendix to this article on ArXiv at http://arxiv.org/ or contact the first author.

References


