SYMMETRIC HAMILTON CYCLE DECOMPOSITIONS OF COMPLETE MULTIGRAPHS

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Abstract

Let \( n \geq 3 \) and \( \lambda \geq 1 \) be integers. Let \( \lambda K_n \) denote the complete multigraph with edge-multiplicity \( \lambda \). In this paper, we show that there exists a symmetric Hamilton cycle decomposition of \( \lambda K_{2m} \) for all even \( \lambda \geq 2 \) and \( m \geq 2 \). Also we show that there exists a symmetric Hamilton cycle decomposition of \( \lambda K_{2m} - F \) for all odd \( \lambda \geq 3 \) and \( m \geq 2 \). In fact, our results together with the earlier results (by Walecki and Brualdi and Schroeder) completely settle the existence of symmetric Hamilton cycle decomposition of \( \lambda K_n \) (respectively, \( \lambda K_n - F \), where \( F \) is a 1-factor of \( \lambda K_n \)) which exist if and only if \( \lambda (n-1) \) is even (respectively, \( \lambda (n-1) \) is odd), except the non-existence cases \( n \equiv 0 \) or \( 6 \) (mod \( 8 \)) when \( \lambda = 1 \).

Keywords: complete multigraph, 1-factor, symmetric Hamilton cycle, decomposition.

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1. Introduction

Let \( n \geq 3 \) and \( \lambda \geq 1 \) be integers. Let \( \lambda K_n \) denote the complete multigraph obtained from the complete graph \( K_n \) by replacing each edge with \( \lambda \) edges. A partition of \( \lambda G \) into edge-disjoint Hamilton cycles is called Hamilton cycle decomposition of \( \lambda G \). A Hamilton cycle decomposition \( \mathcal{H} \) of \( G \) is cyclic if \( V(G) = \mathbb{Z}_n \), and \((v_0 + 1, v_1 + 1, \ldots, v_{n-1} + 1) \in \mathcal{H} \) whenever \((v_0, v_1, \ldots, v_{n-1}) \in \mathcal{H} \). It is 1-rotational if \( V(G) = \mathbb{Z}_{n-1} \cup \{\infty\} \), and \((v_0 + 1, v_1 + 1, \ldots, v_{n-1} + 1) \in \mathcal{H} \) whenever \((v_0, v_1, \ldots, v_{n-1}) \in \mathcal{H} \), where \( \infty + 1 = \infty \) is meaningful. Let the vertex set of \( \lambda K_n \) be labeled as follows:
V(\lambda K_n) = \begin{cases} 
\{0, 1, 2, 3, \ldots, m, \overline{1}, \overline{2}, \overline{3}, \ldots, \overline{m}\}, & \text{if } n \text{ is odd, say } n = 2m + 1; \\
\{1, 2, 3, \ldots, m, \overline{1}, \overline{2}, \overline{3}, \ldots, \overline{m}\}, & \text{if } n \text{ is even, say } n = 2m.
\end{cases}

A Hamilton cycle (or a 2-factor) of \lambda K_n or \lambda K_n - F is said to be \textit{symmetric} if it is invariant under the involution \( i \to \overline{i} \), where \( \overline{i} = i \) and the vertex 0 is a fixed point of this involution. A Hamilton cycle decomposition of \( \lambda K_{2n+1} \) (respectively, \( \lambda K_{2n} \)) is symmetric if it admits an involutory automorphism fixing all its cycles and fixing exactly one vertex (respectively, fixing no vertices). Also a Hamilton cycle decomposition of \( \lambda K_{2n+1} - F \) is symmetric if it admits an involutory automorphism switching all pairs of vertices that are adjacent in \( F \). A symmetric Hamilton cycle (or a 2-factor) in \( K_{n,n} \) with bipartition \( \{1, 2, 3, \ldots, n\} \) and \( \{\overline{1}, \overline{2}, \overline{3}, \ldots, \overline{n}\} \) containing the edge \( i \overline{j} \) should also contain \( \overline{i} j \). The \textit{cartesian product}, \( G_1 \boxtimes G_2 \), of the graphs \( G_1 \) and \( G_2 \) has the vertex set \( V(G_1) \times V(G_2) \) and edge set \( E(G_1 \boxtimes G_2) = \{(u_1, v_1)(u_2, v_2) \mid u_1 = u_2 \text{ and } v_1 v_2 \in E(G_2) \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E(G_1)\} \).

Buratti and Del Fra [6] proved that a cyclic Hamilton cycle decomposition of \( K_n \) exists if and only if \( n \neq 15 \) and \( n \not\in \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\} \). Jordon and Morris [9] proved that for an even \( n \geq 4 \), there exists a cyclic Hamilton cycle decomposition of \( K_n - F \) if and only if \( n \equiv 2, 4 \pmod{8} \) and \( n \neq 2p^\alpha \) where \( p \) is an odd prime and \( \alpha \geq 1 \). Buratti et al. [5] completely solved the existence of cyclic Hamilton cycle decomposition of \( \lambda K_n \) and of \( \lambda (K_{2n} - F) \) for every \( \lambda \). In general, finding necessary and sufficient conditions for the existence of cyclic \( m \)-cycle decomposition of \( K_n \) is an interesting problem and has received much attention in recent days.

Walecki [10] proved the existence of a Hamilton cycle decomposition of \( K_n \) (when \( n \) is odd) and \( K_n - F \) (when \( n \) is even), where \( F \) is a 1-factor of \( K_n \). Further, it is easy to observe that the addition by \( \frac{2n-1}{2} \) gives an involutory map fixing every cycle of the decomposition to be symmetric. Akiyama [1] et al. also constructed a new symmetric Hamilton cycle decomposition of \( K_n \) for odd \( n > 7 \), but is not isomorphic to Walecki decomposition.

Brualdi and Schroeder [4] proved that \( K_n - F \) has a decomposition into Hamilton cycles which are symmetric with respect to the 1-factor \( F \) if and only if \( n \equiv 2 \) or \( 4 \pmod{8} \), and also show that the complete bipartite graph \( K_{n,n} \) (respectively \( K_{n,n} - F \)) has a symmetric Hamilton cycle decomposition if and only if \( n \) is even (respectively \( n \) is odd). As Hamilton/ symmetric Hamilton cycle decomposition of \( K_n \) for even \( n \) does not exists, considering the existence of such decomposition in \( \lambda K_n \) gets merit (for suitable \( \lambda \) and \( n \)), since it covers a wider class of graphs.

Recently, Buratti and Merola [7] observed that every cyclic Hamilton cycle decomposition of \( \lambda K_{2n} \) or \( \lambda K_{2n} - F \) whose cycles having stabilizer of even order is, in particular symmetric: the required involutory automorphism would be in fact the addition by \( n \), and also pointed that the existence of a symmetric Hamilton
cycle decomposition of $K_n - F$ for $n \equiv 4 \pmod{8}$ (part of the main result of the paper by Brualdi and Schroeder [4]) implicitly follows from the result of Jordon and Morris [9]. Also, the result of Buratti et al. [5] gives, implicitly, the existence of a symmetric Hamilton cycle decomposition of $2K_{4m}$, $m \geq 1$.

In this paper, we show that there exists a symmetric Hamilton cycle decomposition of $\lambda K_{2m}$ for all even $\lambda \geq 2$ and $m \geq 2$. Also we show that there exists a symmetric Hamilton cycle decomposition of $\lambda K_{2m} - F$ for all odd $\lambda \geq 3$ and $m \geq 2$. In fact, our results together with the results of Walecki, Brualdi and Schroeder prove that the complete multigraph $\lambda K_n$ (respectively, $\lambda K_n - F$) has a symmetric Hamilton cycle decomposition if and only if $\lambda(n - 1)$ is even (respectively, $\lambda(n - 1)$ is odd) except the non-existence cases $n \equiv 0$ or $6 \pmod{8}$ when $\lambda = 1$, which were proved by Brualdi and Schroeder.

2. Notation and Preliminaries

Throughout this paper, we use the following notation:

- $V(\lambda K_n) = \{0, 1, 2, 3, \ldots, r, \overline{1}, \overline{2}, \overline{3}, \ldots, \overline{r}\}$, if $n$ is odd, say $n = 2r + 1$;
- $\{1, 2, 3, \ldots, r, \overline{1}, \overline{2}, \overline{3}, \ldots, \overline{r}\}$, if $n$ is even, say $n = 2r$.

- $\lambda K_r^*$ is the complete multigraph with the vertex set $\{1, 2, \ldots, r\}$.
- $\lambda K_r^*$ is the complete multigraph with the vertex set $\{1, \overline{2}, \ldots, \overline{r}\}$.
- $\lambda K_{2s, 2s}$ is the complete bipartite multigraph with bipartition $\{1, 2, \ldots, 2s\}$ and $\{1, \overline{2}, \ldots, \overline{2s}\}$.
- $(1, 2, \ldots, m, \overline{1}, \overline{2}, \ldots, \overline{m})$ denotes a symmetric cycle of length $2m$.

- For our convenience, we view $\lambda K_{2r}$, $\lambda K_{2r} - F$ as follows:
  (i) $\lambda K_{2r} = \lambda K_r^* \oplus \lambda K_{r,r} \oplus \lambda K_r^*$.

  (ii) $\lambda K_{2r} - F = \lambda K_r^* \oplus \lambda K_{r,r} - F \oplus \lambda K_r^*$, where $F = \{\overline{i} \in E(K_{r,r}) \mid 1 \leq i \leq r\}$.

- $F'$ denotes the 1-factor $\{i(s + i), (s + i)i \in E(K_{2s, 2s}) \mid 1 \leq i \leq 2s\}$ of $K_{2s, 2s}$.

- $I$ denotes the 1-factor $\{i(s + i) \in E(K_{2s}^*) \mid 1 \leq i \leq s\}$ of $K_{2s}^*$.

- $\overline{I}$ denotes the 1-factor $\{i(s + i) \in E(\overline{K}_{2s}^*) \mid 1 \leq i \leq s\}$ of $\overline{K}_{2s}^*$.

To prove our results we state the following.

**Proposition 1** [1]. Let $p \geq 7$ be a prime. There exists a Hamilton cycle decomposition $G_p$ of $K_p$ which is not isomorphic to the Walecki’s decomposition $W_p$ of $K_p$. 
Theorem 3 [4]. For each integer $m \geq 1$, there exist a symmetric Hamilton cycle decomposition of $K_{2m,2m}$, and $K_{2m+1,2m+1} - F$, where $F$ is a 1-factor.

Theorem 4 [4]. Let $n > 2$ be an integer. Then $K_n - F$ has a symmetric Hamilton cycle decomposition if and only if $n \equiv 2, 4 \pmod{8}$.

Remark 5 [4]. Consider the complete bipartite graph $K_{2m,2m}$ with $V(K_{2m,2m}) = \{1, 2, \ldots, 2m, 1, 2, \ldots, 2m\}$. Let $E_k = \{ab \in E(K_{2m,2m}) \mid a + b \equiv k \pmod{2m}\}$. Clearly, each $S_i = E_{2i} \cup E_{2i+1}$ is a symmetric Hamilton cycle of $K_{2m,2m}$ and $\{S_1, S_2, \ldots, S_m\}$ gives a symmetric Hamilton cycle decomposition of $K_{2m,2m}$. Note that each $S_i$ contain the edges $\{(i+1), (i+1), (m+i)(m+i+1), (m+i)(m+i+1), i, (m+i)(m+i)\}$, $1 \leq i \leq m$ and the additions are taken with modulo $2m$.

Remark 6. Let $V(K_{2m}^*) = \{1, 2, \ldots, 2m\}$. Then $H = (1, 2, 2m, 3, 2m - 1, 4, 2m - 2, \ldots, m + 2, m + 1, 1) = \{ab \in E(K_{2m}^*) \mid a + b \equiv 2 \text{ or } 3 \pmod{2m}\}$ is a Hamilton cycle of $K_{2m}^*$. Now we define an injective map $f_i : \{1, 2, 3, \ldots, 2m\} \to \{1, 2, 3, \ldots, 2m\}$, $1 \leq i \leq 2m - 1$ as follows:

$$f_i(1) = 1,$$

$$f_i(x) = \begin{cases} x + i - 1, & \text{if } x \in \{2, 3, \ldots, 2m - i + 1\}; \\ x - 2m + i, & \text{if } x \in \{2m - i + 2, 2m - i + 3, \ldots, 2m\}. \end{cases}$$

Let $H_i = f_i(H)$. Then $\{H_1, H_2, \ldots, H_{2m-1}\}$, $\{H_1, H_2, \ldots, H_m\}$ and $\{H_{m+1}, H_{m+2}, \ldots, H_{2m-1}\}$ respectively give a Hamilton cycle decomposition of multigraphs $2K_{2m}^* \oplus I$ and $K_{2m}^* - I$, where $I = \{i(m+i) \in E(K_{2m}^*) \mid 1 \leq i \leq m\}$. Note that each $H_i$ contain the edges $\{i(i+1), (m+i)(m+i+1)\}$, $1 \leq i \leq m$ (see Figure 1).

Also we observe that the Hamilton cycle decompositions given above will imply a 1-rotational Hamilton cycle decomposition of $2K_{2m}^*, K_{2m}^* \oplus I$ and $K_{2m}^* - I$ by just replacing the symbols $1$ by $\infty$ and $x$, $2 \leq x \leq 2m$, by $x - 1$.

3. Complete Multigraphs

In this section, we investigate the existence of a symmetric Hamilton cycle decomposition of complete multigraph $\lambda K_n$, when $\lambda(n-1)$ is even. Since the symmetric Hamilton cycle decomposition of $\lambda K_n$, when $n$ odd, exists from the well known Walecki’s construction [10], our main focus is to find a symmetric Hamilton cycle decomposition of $2K_{2m}$. 


Lemma 7. For all integers \( m \geq 1 \), there exists a symmetric Hamilton cycle decomposition of \( K_{2m} \square K_2 \). 

**Proof.** Let \( V(K_{2m}) = \{u_1, u_2, \ldots, u_{2m}\} \) and \( V(K_2) = \{v_1, v_2\} \). For our convenience, we denote \( V(K_{2m} \square K_2) = \bigcup_{s=1}^{2m} V_s \), where \( V_1 = \{i \mid i = (u_i, v_1), 1 \leq i \leq 2m\} \), \( V_2 = \{i \mid i = (u_i, v_2), 1 \leq i \leq 2m\} \) and \( E(K_{2m} \square K_2) = \{ij, i\overline{j}, \overline{i}j, i \neq j, i, j = 1, 2, \ldots, 2m\} \). For \( 1 \leq k \leq 2m, 1 \leq l \leq m \), we define

\[
E_k = \{ij \in E(K_{2m} \square K_2) \mid i \neq j, i + j \equiv k \pmod{2m}\},
\]

\[
\overline{E}_k = \{i\overline{j} \in E(K_{2m} \square K_2) \mid i \neq j, i + j \equiv k \pmod{2m}\},
\]

\[
J_l = \{i\overline{i} \in E(K_{2m} \square K_2) \mid 2i \equiv 2l \pmod{2m}\}.
\]

Note that \( \overline{E}_l \cup \overline{E}_{l+1} \) and \( E_l \cup E_{l+1} \) are Hamilton paths with end vertices \( l, m + l \) and \( \overline{l}, \overline{m + l} \) of \( K_{2m} \square K_2 \) respectively. For each \( l, 1 \leq l \leq m \), we define \( H_l = \overline{E}_l \cup \overline{E}_{l+1} \cup J_l \cup E_l \cup E_{l+1} \). Clearly, each \( H_l \) is a symmetric Hamilton cycle and \( \{H_1, H_2, \ldots, H_m\} \) gives a symmetric Hamilton cycle decomposition of \( K_{2m} \square K_2 \). \( \blacksquare \)

Lemma 8. For all integers \( m \geq 1 \), there exists a symmetric Hamilton cycle decomposition of \( 2(K_{2m+1} \square K_2) \).

**Proof.** Let \( V(K_{2m+1}) = \{u_1, u_2, u_3, \ldots, u_{2m+1}\} \) and \( V(K_2) = \{v_1, v_2\} \). We denote \( V(K_{2m+1} \square K_2) = \bigcup_{s=1}^{2m} V_s \) where \( V_1 = \{i \mid i = (u_i, v_1), 1 \leq i \leq 2m\} \), \( V_2 = \{i \mid i = (u_i, v_2), 1 \leq i \leq 2m\} \) and \( E(K_{2m+1} \square K_2) = \{ij, i\overline{j}, \overline{i}j, i \neq j, i, j = 1, 2, \ldots, 2m + 1\} \).
For all $k$, $1 \leq k \leq 2m + 1$, we define
\[
E_k = \{ij \in E(K_{2m+1} \Box K_2) \mid i \neq j, i + j \equiv k \pmod{2m + 1}\},
\]
\[
E_k = \{ij \in E(K_{2m+1} \Box K_2) \mid i \neq j, i + j \equiv k \pmod{2m + 1}\}.
\]
Note that $E_{2l} \cup E_{2l+1}$, $E_{2l-1} \cup E_{2l}$ and $E_1 \cup E_{2m+1}$ are Hamilton paths of $K_{2m}^*$ with end vertices $l$, $m + 1 + l$; $l$, $m + l$; and $m + 1$, $2m + l$ respectively. Similarly, $E_{2l} \cup E_{2l+1}$, $E_{2l-1} \cup E_{2l}$ and $E_1 \cup E_{2m+1}$ are Hamilton paths of $K_{2m}$ with end vertices $l$, $m + 1 + l$; $l$, $m + l$; and $m + 1$, $2m + l$ respectively.

For each $l$, $1 \leq l \leq m$, we define
\[
H_l = E_{2l} \cup E_{2l+1} \cup \{\bar{l}, (m + 1 + l)(m + l + 1)\} \cup E_{2l} \cup E_{2l+1},
\]
\[
H_l' = E_{2l-1} \cup E_{2l} \cup \{\bar{l}, (m + l)(m + l + 1)\} \cup E_{2l-1} \cup E_{2l},
\]
\[
H_{2m+1} = E_1 \cup E_{2m+1} \cup \{(2m + 1)(2m + 1), (m + 1)(m + 1)\} \cup E_1 \cup E_{2m+1}.
\]
Clearly, each $H_l$, $H_l'$ are symmetric Hamilton cycles and \{H_1, H_2, \ldots, H_m, H_1', H_2', \ldots, H_m', H_{2m+1}\} gives a symmetric Hamilton cycle decomposition of $2(K_{2m+1} \Box K_2)$.

**Remark 9.** Note that the symmetric Hamilton cycles $H_l$ and $H_l'$, $1 \leq l \leq m$ obtained in Lemma 8 contain the edges \{(l(l+1), \bar{l}l(l+1))\} and \{(2m + l +1)(2m +1 + l +1), (2m + l +1)(2m +1 + l +1)\} respectively.

**Note 10.** It is observed that for every Hamilton path decomposition of $K_{2m}$ we can find a symmetric Hamilton cycle decomposition of $K_{2m+2m}$ and $K_{2m} \Box K_2$, also to every Hamilton path decomposition of $2K_{2m+1}$ we can find a symmetric Hamilton cycle decomposition of $2(K_{2m+1} \Box K_2)$.

**Theorem 11.** For all integers $m \geq 1$, there exists a symmetric Hamilton cycle decomposition of $2K_{4m+2}$.

**Proof.** Let $V(2K_{4m+2}) = \{1, 2, \ldots, 2m + 1, \bar{1}, \bar{2}, \ldots, \bar{2m + 1}\}$. Now the complete multigraph $2K_{4m+2}$ can be viewed as follows: $2K_{4m+2} = 2(K_{2m+1} \Box K_2) \oplus 2(K_{2m+1,2m+1} - F)$, where $F = \{ii \in E(K_{2m+1,2m+1}) \mid 1 \leq i \leq 2m + 1\}$ is a 1-factor of $K_{2m+1,2m+1}$. We know that $2(K_{2m+1} \Box K_2)$ and $(K_{2m+1,2m+1} - F)$ have symmetric Hamilton cycle decompositions by Lemma 8 and Theorem 3, respectively.

We recall that Buratti and Merola [7] observed that every cyclic Hamilton cycle decomposition of $\lambda K_{2n}$ or $\lambda K_{2n} - F$ whose cycles have stabilizer of even order is, in particular symmetric: the required involution automorphism would be in fact the addition by $n$. So the result of Buratti et al. [5] deduce the existence of a symmetric Hamilton cycle decomposition of $2K_{4m}$, $m \geq 1$. 

The next construction provides an alternative proof for the existence of a symmetric Hamilton cycle decomposition of $2K_{4m}$, $m \geq 1$ which is implicitly contained in Buratti et al. ([5], Lemma 3.5).

**Theorem 12.** For all integers $m \geq 1$, there exists a symmetric Hamilton cycle decomposition of $2K_{4m}$.

**Proof.** Let $V(2K_{4m}) = \{1, 2, \ldots, 2m, \overline{1}, \overline{2}, \ldots, \overline{2m}\}$. For $m = 1$ the graph is $2K_4$. Clearly, $\{(1, 2, 2, 1), (1, 2, 1, 2), (1, 1, 2, 2)\}$ gives a symmetric Hamilton cycle decomposition of $2K_4$.

For $m \geq 2$, we write $2K_{4m} = 2K'_{2m} \oplus K_{2m,2m} \oplus K'_{2m,2m} \oplus 2\overline{K}_{2m}$. Now the idea of decomposing $2K_{4m}$ into symmetric Hamilton cycles is as follows: First we decompose $K_{2m,2m}$ and $K'_{2m,2m}$ into symmetric Hamilton cycles $S_1, S_2, \ldots, S_m$ and $S'_1, S'_2, \ldots, S'_m$, and $2K_{2m}$, $2\overline{K}_{2m}$ into Hamilton cycles $\{H_1, H_2, \ldots, H_{2m-1}\}$, $\{H'_1, H'_2, \ldots, H'_{2m-1}\}$ respectively. Then by decomposing each $H_i \oplus S_i \oplus H'_i$, $1 \leq i \leq m$ and $H_{m+j} \oplus S'_j \oplus H'_{m+j}$, $1 \leq j \leq m-1$ into symmetric Hamilton cycles $C_1, C_2$ and $D_1, D_2$ respectively, we get the symmetric Hamilton cycle decomposition $\{C_1, C_2, C'_1, C'_2, \ldots, C'_m, D_1, D_2, D'_1, D'_2, \ldots, D'_m\}$ of $2K_{4m}$.

![Symmetric Hamilton cycles](image)

We know by Remark 5 that $2K_{2m,2m}$ has a symmetric Hamilton cycle decomposition $\{S_1, S_2, \ldots, S_m, S'_1, S'_2, \ldots, S'_m\}$ such that both $S_i$ and $S'_i$ contain the edges $\{i(i+1), i(i+1), (m+i)(m+i+1), (m+i+1)(m+1)i\}$. Furthermore, by Remark 6, $2K'_{2m}$ has a Hamilton cycle decomposition $\{H_1, H_2, \ldots, H_{2m-1}\}$ such that each $H_i$ contain the edges $\{i(i+1), (m+i)(m+i+1)\}$. Similarly, let $\{\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_{2m-1}\}$ be a Hamilton cycle decomposition of $2\overline{K}_{2m}$ such that each $\overline{H}_i$ contain the edges $\{i(i+1), (m+i)(m+i+1)\}$.
Now we define $C_i^1, C_i^2$ from $H_i \oplus S_i \oplus \overline{H}_i, 1 \leq i \leq m$ as follows:

$$C_i^1 = (H_i \setminus \{i(i + 1)\}) \cup (\overline{H}_i \setminus \{i(i + 1)\}) \oplus \{i(i + 1), i(i + 1)\},$$

$$C_i^2 = (S_i \setminus \{i(i + 1), i(i + 1)\}) \oplus \{i(i + 1), i(i + 1)\}.$$

Now we define $D_j^1, D_j^2$ from $H_{m+j} \oplus S'_j \oplus \overline{H}_{m+j}, 1 \leq j \leq m - 1$ as follows:

$$D_j^1 = (H_{m+j} \setminus \{(m + j)(m + j + 1)\}) \cup (\overline{H}_{m+j} \setminus \{(m + j)(m + j + 1)\})$$

$$\oplus \{(m + j)(m + j + 1), (m + j)(m + j + 1)\},$$

$$D_j^2 = (S'_j \setminus \{(m + j)(m + j + 1), (m + j)(m + j + 1)\})$$

$$\oplus \{(m + j)(m + j + 1), (m + j)(m + j + 1)\}.$$

It is easy to check that $C_i^1, C_i^2, D_j^1$ and $D_j^2$ are edge-disjoint symmetric Hamilton cycles of $2K_{4m}$, (see Figures 2 and 3). Hence $\{C_i^1, C_i^2, D_j^1, D_j^2, S'_m \mid 1 \leq i \leq m, 1 \leq j \leq m - 1\}$ gives a symmetric Hamilton cycle decomposition of $2K_{4m}$.

**Theorem 13.** For all $\lambda \equiv 0 (\text{mod } 2)$ and $n \equiv 0 (\text{mod } 2) \geq 4$, there exists a symmetric Hamilton cycle decomposition of $\lambda K_n$.

**Proof.** Follows from Theorems 11 and 12.

4. **Complete Multigraph Minus a 1-factor**

In this section, we investigate the existence of symmetric Hamilton cycle decomposition of $\lambda K_n - F$, when $\lambda K_n$ has odd regularity.
Theorem 14. For all \( \lambda \equiv 1 \pmod{2} \) and \( n \equiv 2 \) or \( 4 \pmod{8} \), there exists a symmetric Hamilton cycle decomposition of \( \lambda K_n - F \).

Proof. We can write \( \lambda K_n - F = (\lambda - 1)K_n \oplus K_n - F \). Since both \( n \) and \( \lambda - 1 \) are even, \((\lambda - 1)K_n \) and \((K_n - F)\) have a symmetric Hamilton cycle decomposition by Theorems 13 and 4 respectively.

Theorem 15. For all \( n \equiv 6 \pmod{8} \), there exists a symmetric Hamilton cycle decomposition of \( 3K_n - F \).

Proof. Let \( n = 8m + 6 \) and \( V(3K_{8m+6}) = \{1, 2, \ldots, 4m + 3, \mathbb{F}, \mathbb{F}, \ldots, 4m + 3\} \). For \( m = 0 \), the graph is \( 3K_6 - F \). Clearly \{\((1, \mathbb{F}, \mathbb{F}, 2, \mathbb{F}), (1, \mathbb{F}, \mathbb{F}, 2, \mathbb{F}), (1, \mathbb{F}, \mathbb{F}, 2, \mathbb{F}), (1, \mathbb{F}, \mathbb{F}, 2, \mathbb{F})\}\) gives a symmetric Hamilton cycle decomposition of \( 3K_6 - F \), where \( F = \{1\mathbb{F}, 2\mathbb{F}, 3\mathbb{F}\} \) is a 1-factor.

Now we construct a symmetric Hamilton cycle decomposition of \( 3K_n - F \) for \( n \geq 14 \) as follows: For \( 1 \leq k \leq 4m + 3, 1 \leq i \leq 2m + 1 \), we define

\[
H_i = F_{2i} \cup F_{2i+1} \cup \{(4m + 3)i, (4m + 3)i, (4m + 3)(2m + 1 + i), (4m + 3)(2m + 1 + i)\} \cup F'_{2i} \cup F'_{2i+1},
\]

\[
S_i = E_{2i} \cup E_{2i+1} \cup \{(4m + 3)i, (4m + 3)i, (4m + 3)(2m + 1 + i), (4m + 3)(2m + 1 + i)\},
\]

where

\[
E_k = \{\alpha \beta \in E(K_{4m+2,4m+2}) \mid a \neq b, a + b \equiv k \pmod{4m + 2}\},
\]

\[
F_k = \{ab \in E(K_{4m+2}^*) \mid a + b \equiv k \pmod{4m + 2}\},
\]

\[
F'_k = \{\alpha \beta \in E(K_{4m+2}^* \mathbb{F}) \mid a + b \equiv k \pmod{4m + 2}\}.
\]

It is easy to check that each \( H_i \) is a symmetric Hamilton cycle of \( K_{8m+6} - F \) and each \( S_i \) is a symmetric 2-factor of \( K_{8m+6} - F \) containing the edges \{\(i(i+1), i(i+1)\}\}, where \( F = \{i\i \in E(K_{4m,4m+3}) \mid 1 \leq i \leq 4m + 3\} \) is a 1-factor. So we write \( K_{8m+6} - F = (\oplus_{i=1}^{2m+1} H_i) \oplus (\oplus_{i=1}^{2m+1} S_i) \). Furthermore, by Lemma 8, \( 2(K_{4m+3}^* \mathbb{F}K_2) \) has a symmetric Hamilton cycle decomposition \{\(C_1, C_2, \ldots, C_{2m+1}, C'_1, C'_2, \ldots, C'_{2m+1}, C'_{4m+3}\)\}. Now we can write

\[
3K_{8m+6} - F = 2K_{8m+6} \oplus (K_{8m+6} - F)
\]

\[
= 2(K_{4m+3}^* \mathbb{F}K_2) \oplus 2(K_{4m+3,4m+3} - F) \oplus (K_{8m+6} - F)
\]

\[
= (\oplus_{i=1}^{2m+1} C_i) \oplus (\oplus_{i=1}^{2m+1} C'_i) \oplus C'_{4m+3} \oplus 2(K_{4m,4m+3} - F) \oplus (\oplus_{i=1}^{2m+1} H_i) \oplus (\oplus_{i=1}^{2m+1} S_i).
\]
We now construct the remaining symmetric Hamilton cycles $D_1^i, D_2^i$ from $C_i \oplus S_i$, $1 \leq i \leq 2m + 1$ as follows:

$$
D_1^i = (S_i \setminus \{i(i+1), \tilde{i}(i+1)\}) \oplus \{i(i+1), \tilde{i}(i+1)\},
$$

$$
D_2^i = (C_i \setminus \{i(i+1), \tilde{i}(i+1)\}) \oplus \{i(i+1), \tilde{i}(i+1)\}.
$$

One can check that $D_1^i, D_2^i$ are symmetric Hamilton cycles of $2(K_{4m+3} \square K_2) \oplus K_{8m+6} - F$. Hence $\{D_1^i, D_2^i, C_i, C_{4m+3}, H_i \mid 1 \leq i \leq 2m + 1\}$ together with the symmetric Hamilton cycle decomposition of $2(K_{4m+3,4m+3} - F)$ which exists by Theorem 3, gives a symmetric Hamilton cycle decomposition of $3K_{8m+6} - F$. ■

**Lemma 16.** The graph $(K_{2m}^2 \oplus I) \oplus K_{2m,2m} \oplus (\overline{K}_{2m}^* \oplus \overline{I})$, where $I = \{i(m+i) \in E(K_{2m}^2) \mid 1 \leq i \leq m\}$, $I = \{i(m+i) \in E(\overline{K}_{2m}^2) \mid 1 \leq i \leq m\}$, $I = \{i(m+i) \in E(\overline{K}_{2m}^2) \mid 1 \leq i \leq m\}$ admits a symmetric Hamilton cycle decomposition for all $m \geq 1$.

**Proof.** We know by Remark 5 that $K_{2m,2m}$ has a symmetric Hamilton cycle decomposition $\{S_1, S_2, \ldots, S_m\}$ such that each $S_i$ contain the edges $\{i(i+1), \tilde{i}(i+1), (m+i)(m+i+1), (m+i)(m+i+1), \tilde{i}, (m+i)(m+i+1)\}$. Further by Remark 6, $K_{2m}^2 \oplus I$ has a Hamilton cycle decomposition $\{H_1, H_2, \ldots, H_m\}$ such that each $H_i$ contain the edges $\{i(i+1), (m+i)(m+i+1)\}$. Similarly, let $\{\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_m\}$ be a Hamilton cycle decomposition of $\overline{K}_{2m}^* \oplus \overline{I}$ such that each $\overline{H}_i$ contain the edges $\{\tilde{i}(i+1), (m+i)(m+i+1)\}$.

For each integer $i, 1 \leq i \leq m$, we construct $C_1^i, C_2^i$ as follows:

$$
C_1^i = (H_i \setminus \{i(i+1)\}) \cup (\overline{H}_i \setminus \{\tilde{i}(i+1)\}) \oplus \{i(i+1), \tilde{i}(i+1)\},
$$

$$
C_2^i = (S_i \setminus \{i(i+1), \tilde{i}(i+1)\}) \cup \{i(i+1), \tilde{i}(i+1)\}.
$$

Clearly, $\{C_1^i, C_2^i \mid 1 \leq i \leq m\}$ gives a symmetric Hamilton cycle decomposition of $(K_{2m}^2 \oplus I) \oplus K_{2m,2m} \oplus (\overline{K}_{2m}^* \oplus \overline{I})$. ■

**Lemma 17.** The graph $K_{2m}^2 \oplus F^* \overline{K}_{2m}^*$, where $F^* = \{i(m+i), \tilde{i}(m+i) \in E(K_{2m,2m}) \mid 1 \leq i \leq m\}$, $F^* = \{i(m+i), \tilde{i}(m+i) \in E(\overline{K}_{2m}^2) \mid 1 \leq i \leq m\}$ admits a symmetric Hamilton cycle decomposition for all $m \geq 1$.

**Proof.** For $1 \leq l \leq m$, we define $H_l = E_{2l} \cup E_{2l+1} \cup \{(m+l), \tilde{l}(m+l)\} \cup E_{2l} \cup \overline{E}_{2l+1}$, where

$$
E_k = \{i \in E(K_{2m}^*) \mid i \neq j, i + j \equiv k \ (\text{mod} \ 2m)\},
$$

$$
\overline{E}_k = \{\tilde{i} \in E(\overline{K}_{2m}^*) \mid i \neq j, i + j \equiv k \ (\text{mod} \ 2m)\}.
$$

Clearly, each $H_l$ is a symmetric Hamilton cycle and $\{H_1, H_2, \ldots, H_m\}$ gives a symmetric Hamilton cycle decomposition of $K_{2m}^2 \oplus F^* \overline{K}_{2m}^*$. ■
Lemma 18. The graph $K_{2m,2m} - \{F, F'\}$, where $F = \{i \in \Gamma(K_{2m,2m}) \mid 1 \leq i \leq 2m\}$, $F' = \{i(m+i), i(m+i) \in \Gamma(K_{2m,2m}) \mid 1 \leq i \leq m\}$ admits a $C_{2m}$-factorization for all $m \geq 2$.

Proof. Let $V(K_{2m,2m}) = \{1, 2, \ldots, 2m, \bar{1}, \bar{2}, \ldots, \bar{2m}\}$. By Remark 6, let $\mathcal{H} = \{H_{m+1}, H_{m+2}, \ldots, H_{2m-1}\}$ be a Hamilton cycle decomposition of $K_{2m}^* - I$, where $I = \{i(m+i) \in \Gamma(K_{2m}) \mid 1 \leq i \leq m\}$. Let $H \in \mathcal{H}$ and if $H = (1, 2, \ldots, 2m)$ in $K_{2m}^* - F$, then we define a 2-factor $C$ as $C = \{1, 2, 3, \ldots, 2m\}(\bar{1}, \bar{2}, \bar{3}, \ldots, \bar{2m})$ in $K_{2m,2m} - \{F, F'\}$. So corresponding to each $H_{m+i} \in \mathcal{H}$ we can define a $C_i$ as above. Hence $\{C_i \mid 1 \leq i \leq m-1\}$ gives a $C_{2m}$-factorization of $K_{2m,2m} - \{F, F'\}$.

Since by Remark 6, each $H_{m+i} \in \mathcal{H}$ contain the edges $\{i(i+1), (m+i)(m+i+1)\}$, $C_i$ also contain the edges $\{i(i+1), i(m+i+1), (m+i)(m+i+1)\}$.

Lemma 19. The graph $(K_{2m}^* - I) \oplus K_{2m,2m} - \{F, F'\} \oplus (\overline{K}_{2m}^* - \overline{I})$, where $I = \{i(m+i) \in \Gamma(K_{2m}^*) \mid 1 \leq i \leq m\}$, $\overline{I} = \{i(m+i) \in \Gamma(K_{2m}^*) \mid 1 \leq i \leq m\}$, $F = \{i \in \Gamma(K_{2m,2m}) \mid 1 \leq i \leq 2m\}$, $F' = \{i(m+i), i(m+i) \in \Gamma(K_{2m,2m}) \mid 1 \leq i \leq m\}$ admits a symmetric Hamilton cycle decomposition for all $m \geq 1$.

Proof. We know by Remark 6, $K_{2m}^* - I$ has a Hamilton cycle decomposition $\{H_{m+1}, H_{m+2}, \ldots, H_{2m-1}\}$ such that each $H_{m+i}$ contain the edges $\{i(i+1), (m+i)(m+i+1)\}$. Similarly, $\overline{K}_{2m}^* - \overline{I}$ has a Hamilton cycle decomposition $\{\overline{H}_{m+1}, \overline{H}_{m+2}, \ldots, \overline{H}_{2m-1}\}$ such that each $\overline{H}_{m+j}$ contain the edges $\{j(j+1), (m+j)(m+j+1)\}$. Let $\{C_1, C_2, \ldots, C_{m-1}\}$ be a $C_{2m}$-factorization of $K_{2m,2m} - \{F, F'\}$ as obtained in Lemma 18. Note that each factor $C_j$ contain the edges $\{j(j+1), (m+j)(m+j+1)\}$.

For each integer $j$, $1 \leq j \leq m-1$, we construct symmetric Hamilton cycles $D_1^j, D_2^j$ as follows:

$$D_1^j = (H_j \setminus \{j(j+1)\}) \cup (\overline{H}_j \setminus \{j(j+1)\}) \cup \{j(j+1), j(j+1)\},$$

$$D_2^j = (C_j \setminus \{j(j+1), j(j+1)\}) \cup \{j(j+1), j(j+1)\}.$$ 

Then $\{D_1^j, D_2^j \mid 1 \leq j \leq m-1\}$ gives a symmetric Hamilton cycle decomposition of $(K_{2m}^* - I) \oplus K_{2m,2m} - \{F, F'\} \oplus (\overline{K}_{2m}^* - \overline{I})$.

Theorem 20. For all $n \equiv 0 \pmod{8}$, there exists a symmetric Hamilton cycle decomposition of $3K_n - F$.

Proof. Let $n = 8m$ and $V(3K_{8m}) = \{1, 2, \ldots, 4m, \bar{1}, \bar{2}, \ldots, \bar{4m}\}$. Now we write $3K_{8m} - F$, where $F = \{i \in \Gamma(K_{4m,4m}) \mid 1 \leq i \leq 4m\}$ as follows:

$$3K_{8m} - F = ((K_{4m}^* \oplus I) \oplus K_{4m,4m} \oplus (\overline{K}_{4m}^* \oplus \overline{I})) \oplus (K_{4m}^* \oplus F' \oplus \overline{K}_{4m}^*) \oplus ((K_{4m}^* - I) \oplus K_{4m,4m} - \{F, F'\} \oplus (\overline{K}_{4m}^* - \overline{I})) \oplus K_{4m,4m}.$$
Where \( I = \{i(2m+i) \in E(K_{4m}^*) \mid 1 \leq i \leq 2m\}, \ \bar{I} = \{i(2m+i) \in E(K_{4m}^*) \mid 1 \leq i \leq 2m\}, \ \bar{F}' = \{i(2m+i), \bar{i}(2m+i) \in E(K_{4m,4m}) \mid 1 \leq i \leq 2m\}. \) The remaining proof follows from Lemmas 16, 17, 19 and Remark 5.

**Theorem 21.** For all \( \lambda \equiv 1 \pmod{2} \geq 3 \) and \( n \equiv 0 \pmod{2} \geq 4 \), there exists a symmetric Hamilton cycle decomposition of \( \lambda K_n - F \).

**Proof.** Follows from Theorems 14, 15 and 20.

5. Conclusion

From the results of Sections 3 and 4 together with the known results of Section 2, we have the following:

**Theorem 22.** For \( n \geq 3 \), there exists a symmetric Hamilton cycle decomposition of \( \lambda K_n \) if and only if

(i) \( \lambda \) is even and \( n \) is odd, (or)

(ii) \( \lambda \) is odd and \( n \) is odd, (or)

(iii) \( \lambda \) is even and \( n \) is even.

**Theorem 23.** For \( n \geq 3 \), there exists a symmetric Hamilton cycle decomposition of \( \lambda K_n - F \) with respect to the 1-factor \( F \) if and only if \( \lambda \) is odd and \( n \) is even except the non-existence cases \( n \equiv 0 \) or 6 \( \pmod{8} \) when \( \lambda = 1 \).

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