

## THE PHYLOGENY GRAPHS OF DOUBLY PARTIAL ORDERS

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### Abstract

The competition graph of a doubly partial order is known to be an interval graph. The CCE graph and the niche graph of a doubly partial order are also known to be interval graphs if the graphs do not contain a cycle of length four and three as an induced subgraph, respectively. Phylogeny graphs are variant of competition graphs. The phylogeny graph  $P(D)$  of a digraph  $D$  is the (simple undirected) graph defined by  $V(P(D)) := V(D)$  and  $E(P(D)) := \{xy \mid N_D^+(x) \cap N_D^+(y) \neq \emptyset\} \cup \{xy \mid (x, y) \in A(D)\}$ , where  $N_D^+(x) := \{v \in V(D) \mid (x, v) \in A(D)\}$ .

In this note, we show that the phylogeny graph of a doubly partial order is an interval graph. We also show that, for any interval graph  $G$ , there exists an interval graph  $\tilde{G}$  such that  $\tilde{G}$  contains the graph  $G$  as an induced subgraph and that  $\tilde{G}$  is the phylogeny graph of a doubly partial order.

**Keywords:** competition graph, phylogeny graph, doubly partial order, interval graph.

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## 1. INTRODUCTION

Throughout this note, all graphs and all digraphs are finite and simple. The notion of competition graphs was introduced by Cohen [3] in a connection with a problem in ecology. Given a digraph  $D$ , if  $(u, v)$  is an arc of  $D$ , then we call  $v$  a *prey* of  $u$  and  $u$  a *predator* of  $v$ . The *competition graph*  $C(D)$  of a digraph  $D$  is the graph which has the same vertex set as  $D$  and has an edge between two distinct vertices  $u$  and  $v$  if and only if there exists a common prey of  $u$  and  $v$  in  $D$ . Since Cohen introduced the notion of competition graphs, several variants have been defined and studied by many authors (see the survey articles by Kim [6] and Lundgren [12]). For example, Scott [22] introduced competition-common enemy graphs (or CCE graphs), Cable, Jones, Lundgren, and Seager [1] introduced niche graphs, and Sonntag and Teichert introduced competition hypergraphs [23] (see also [20] for competition multihypergraphs and [15] for the hypercompetition numbers of hypergraphs). As another variant, Roberts and Sheng [16, 17, 18, 19] introduced phylogeny graphs. The *phylogeny graph* of a digraph  $D$  is the graph which has the same vertex set as  $D$  and has an edge between two distinct vertices  $u$  and  $v$  if and only if there exists an arc from  $u$  to  $v$  or an arc from  $v$  to  $u$  or a common prey of  $u$  and  $v$  in  $D$ .

In the study of competition graphs and their variants, one of important problems is characterizing the competition graphs of interesting classes of digraphs (see [4, 5, 10, 11, 13, 21] for some studies on this research direction). In this note, we study the phylogeny graphs of doubly partial orders. Doubly partial orders are digraphs defined as follows. For a point  $x$  in  $\mathbb{R}^2$ , we denote its first and second coordinates by  $x_1$  and  $x_2$ , respectively. For  $x, y \in \mathbb{R}^2$ , we write  $x \prec y$  if  $x_1 < y_1$  and  $x_2 < y_2$ . A digraph  $D$  is called a *doubly partial order* if there exists a finite subset  $V$  of  $\mathbb{R}^2$  such that  $V(D) = V$  and  $A(D) = \{(x, v) \mid v, x \in V, v \prec x\}$ . Note that, by definition, the out-neighborhood  $N_D^+(x) := \{v \in V(D) \mid (x, v) \in A(D)\}$  of a vertex  $x$  in a doubly partial order  $D$  is the set  $\{v \in V(D) \mid v \prec x\}$ .

We recall some results on variants of competition graphs of doubly partial orders. A graph  $G$  is called an *interval graph* if there exists an assignment  $J : V(G) \rightarrow 2^{\mathbb{R}}$  of real closed intervals  $J(v)$  to the vertices  $v$  of  $G$  such that, for any two distinct vertices  $v$  and  $w$ ,  $vw \in E(G)$  if and only if  $J(v) \cap J(w) \neq \emptyset$ . In 2005, Cho and Kim [2] showed the following:

**Theorem 1.1** [2]. *The competition graph of a doubly partial order is an interval graph.*

**Theorem 1.2** [2]. *An interval graph can be made into the competition graph of a doubly partial order by adding sufficiently many isolated vertices.*

Here, we mention that Wu and Lu [24] gave a further result on this subject. They showed that a graph is the competition graph of a doubly partial order if and

only if it is an interval graph, at least half of whose maximal cliques are isolated vertices.

After the study of Cho and Kim, several authors studied variants of competition graphs of doubly partial orders. In 2007, Kim, Kim, and Rho [7] showed that the CCE graph of a doubly partial order is an interval graph unless it contains  $C_4$  as an induced subgraph. In 2009, Kim, Lee, Park, Park, and Sano [8] showed a similar result for niche graphs. The niche graph of a doubly partial order is an interval graph unless it contains  $C_3$  as an induced subgraph. In 2011, Park, Lee, and Kim [14] studied the  $m$ -step competition graphs of doubly partial orders, and showed that, for any positive integer  $m$ , the  $m$ -step competition graph of a doubly partial order is an interval graph. Recently, Kim, Lee, Park, and Sano [9] studied the competition hypergraphs of doubly partial orders.

As the phylogeny graph is an important variant of the competition graph, it is natural to ask whether the phylogeny graph of a doubly partial order is an interval graph or not. In the following section, we show that the phylogeny graph of a doubly partial order is always an interval graph.

## 2. MAIN RESULTS

We use the following notations in this section. For  $x, y \in \mathbb{R}^2$ ,

$$\begin{aligned} x \searrow y &\iff x_1 \leq y_1 \text{ and } y_2 \leq x_2, \\ x \wedge y &:= (\min\{x_1, y_1\}, \min\{x_2, y_2\}) \in \mathbb{R}^2. \end{aligned}$$

The following theorem is our first main result.

**Theorem 2.1.** *The phylogeny graph of a doubly partial order is an interval graph.*

**Proof.** Let  $G$  be the phylogeny graph of a doubly partial order  $D$ . We shall give an interval assignment  $J : V(G) \rightarrow 2^{\mathbb{R}}$  such that  $G$  is the intersection graph of the family of those intervals. We may disregard the isolated vertices of  $G$  since we can assign an interval to an isolated vertex that does not overlap with any other interval.

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function defined by  $f(x) = f(x_1, x_2) := x_2 - x_1$ . For a (non-isolated) vertex  $x$  of  $D$ , we define

$$J(x) := \text{conv}\{f(a) \in \mathbb{R} \mid a \in N_D^+(x) \cup \{x\}\} \subseteq \mathbb{R},$$

where  $\text{conv}(S)$  means the convex hull of a set  $S$  in  $\mathbb{R}$ . Note that  $J(x) \neq \emptyset$  since  $f(x) \in J(x)$ . We will show that the graph obtained by deleting all the isolated vertices from  $G$  is the intersection graph of the family of these intervals.

Take two adjacent vertices  $x$  and  $y$  of  $G$ . Then it holds that  $x \prec y$ , or  $y \prec x$ , or there exists a vertex  $a$  such that  $a \prec x$  and  $a \prec y$ . If  $x \prec y$ , then

$f(x) \in J(x) \cap J(y)$ . If  $y \prec x$ , then  $f(y) \in J(x) \cap J(y)$ . If there exists a vertex  $a$  such that  $a \prec x$  and  $a \prec y$ , then  $f(a) \in J(x) \cap J(y)$ . Therefore, we have  $J(x) \cap J(y) \neq \emptyset$ .

Next, take two non-isolated vertices  $x$  and  $y$  which are not adjacent in  $G$ . Then it holds that  $x \searrow y$  or  $y \searrow x$ . Without loss of generality, we may assume that  $x \searrow y$ . We claim that

$$(1) \quad \min J(x) > \max J(y).$$

Note that  $x \wedge y = (x_1, y_2)$  since  $x \searrow y$ . Since  $N_D^+(x) \cap N_D^+(y) = \emptyset$ , if  $y_2 < x_2$  then we have

$$\begin{aligned} N_D^+(x) &\subseteq \{v \in \mathbb{R}^2 \mid v_1 < x_1, y_2 \leq v_2 < x_2\} \\ &\subseteq \{v \in \mathbb{R}^2 \mid v_2 - v_1 > y_2 - x_1\} \\ &= \{v \in \mathbb{R}^2 \mid f(v) > f(x \wedge y)\}. \end{aligned}$$

Therefore,

$$(2) \quad y_2 < x_2 \quad \Rightarrow \quad \min J(x) > f(x \wedge y).$$

Similarly, if  $x_1 < y_1$  then we have

$$\begin{aligned} N_D^+(y) &\subseteq \{v \in \mathbb{R}^2 \mid x_1 \leq v_1 < y_1, v_2 < y_2\} \\ &\subseteq \{v \in \mathbb{R}^2 \mid v_2 - v_1 < y_2 - x_1\} \\ &= \{v \in \mathbb{R}^2 \mid f(v) < f(x \wedge y)\}. \end{aligned}$$

Therefore,

$$(3) \quad x_1 < y_1 \quad \Rightarrow \quad \max J(y) < f(x \wedge y).$$

Suppose that  $x_1 = y_1$ . Then  $x \wedge y = y$ . Since  $N_D^+(y) = N_D^+(x \wedge y) \subseteq N_D^+(x)$  and  $N_D^+(x) \cap N_D^+(y) = \emptyset$ , it holds that  $N_D^+(y) = \emptyset$ . Therefore  $J(y) = \{f(y)\}$ . Thus we have

$$(4) \quad x_1 = y_1 \quad \Rightarrow \quad \max J(y) = f(x \wedge y).$$

Suppose that  $x_2 = y_2$ . Then  $x \wedge y = x$ . Since  $N_D^+(x) = N_D^+(x \wedge y) \subseteq N_D^+(y)$  and  $N_D^+(x) \cap N_D^+(y) = \emptyset$ , it holds that  $N_D^+(x) = \emptyset$ . Therefore  $J(x) = \{f(x)\}$ . Thus we have

$$(5) \quad x_2 = y_2 \quad \Rightarrow \quad \min J(x) = f(x \wedge y).$$

Now, we consider the three possible cases (see Figure 1). If  $x_1 < y_1$  and  $y_2 < x_2$ , then (1) follows from (2) and (3). If  $x_1 = y_1$  and  $y_2 < x_2$ , then (1) follows from (2) and (4). If  $x_1 < y_1$  and  $x_2 = y_2$ , then (1) follows from (3) and (5). Thus, we have  $J(x) \cap J(y) = \emptyset$ .

Hence the theorem holds. ■

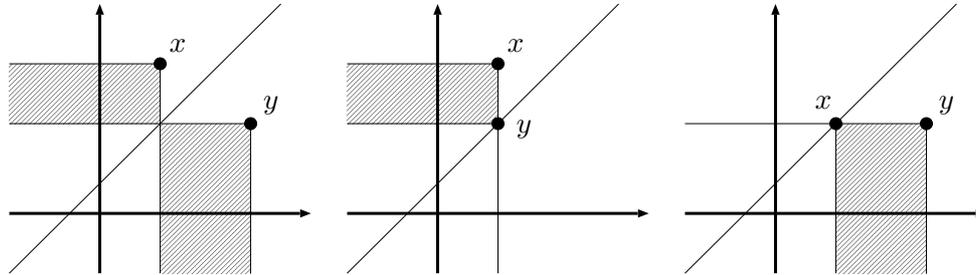


Figure 1. Pictures for Proof of Theorem 2.1.

Theorem 1.1 and Theorem 1.2 are complementary to each other. In the same fashion, after establishing Theorem 2.1, a natural question is if every interval graph is the phylogeny graph of some doubly partial order. However, the answer for this question is NO. The following theorem shows that not every interval graph  $G$  has a doubly partial order whose phylogeny graph is equal to  $G$ .

**Theorem 2.2.** *Let  $G$  be an interval graph. If  $G$  has two adjacent vertices  $u$  and  $v$  of degree at least two such that the edge  $uv$  is not contained in any triangle in  $G$ , then  $G$  cannot be the phylogeny graph of a doubly partial order.*

**Proof.** Let  $G$  be an interval graph and suppose that  $G$  has two adjacent vertices  $u$  and  $v$  of degree at least two such that the edge  $uv$  is not contained in any triangle in  $G$ . Let  $a$  (resp.  $b$ ) be a vertex of  $G$  other than  $v$  (resp.  $u$ ) which is adjacent to  $u$  (resp.  $v$ ). Then  $P = auvb$  is a path of length 3. Suppose that there exists a doubly partial order  $D$  such that the phylogeny graph of  $D$  is equal to  $G$ . Since the edge  $uv$  is not contained in any triangle in  $G$ , it holds that  $N_G(u) \cap N_G(v) = \emptyset$ . Since  $N_G(u) \cap N_G(v) = \emptyset$ , there cannot exist a vertex  $z$  in  $D$  such that  $z \prec u$  and  $z \prec v$ . However, since  $uv \in E(G)$ , either  $u \prec v$  or  $v \prec u$ . Without loss of generality, we may assume that  $v \prec u$ . Since  $vb \in E(G)$ , it holds that  $v$  and  $b$  have a common prey or  $v \prec b$  or  $b \prec v$ . If there exists a vertex  $z$  in  $D$  such that  $z \prec b$  and  $z \prec v$ , then  $z \prec u$  and  $b, u, v, z$  form a clique of size 4 in  $G$ , which is a contradiction to the fact that  $N_G(u) \cap N_G(v) = \emptyset$ . If  $v \prec b$ , then  $u$  and  $b$  have  $v$  as a common prey and so  $u$  and  $b$  are adjacent, which is also a contradiction to the fact that  $N_G(u) \cap N_G(v) = \emptyset$ . If  $b \prec v$ , then  $b \prec u$  and so  $u$  and  $b$  are adjacent, which is a contradiction. Therefore, in each case, we reach a contradiction. Hence,  $G$  cannot be the phylogeny graph of a doubly partial order. ■

The above result is contrary to Theorem 1.2 stating that we can make any interval graph into the competition graph of a doubly partial order by adding sufficiently many isolated vertices.

Though not every interval graph is the phylogeny graph of a doubly partial order even if we allow us to add isolated vertices, we can show that, given an interval graph  $G$ , there exists an “extension” of  $G$  such that it is the phylogeny graph of a doubly partial order.

**Theorem 2.3.** *For any interval graph  $G$ , there exists an interval graph  $\tilde{G}$  such that  $\tilde{G}$  contains the graph  $G$  as an induced subgraph and that  $\tilde{G}$  is the phylogeny graph of a doubly partial order.*

**Proof.** We may assume that  $G$  has no isolated vertices. By Theorem 1.2, there exists a doubly partial order  $D$  such that the competition graph of  $D$  is equal to the graph obtained from  $G$  by adding sufficiently many isolated vertices. Let  $\tilde{G}$  be the phylogeny graph of  $D$ . Then, by Theorem 2.1,  $\tilde{G}$  is an interval graph. Let  $S \subseteq V(D)$  be the set of vertices having a prey in  $D$ , that is,

$$S := \{v \in V(D) \mid N_D^+(v) \neq \emptyset\}.$$

Let  $H$  be the subgraph of  $\tilde{G}$  induced by  $S$ . We will show that  $E(H) = E(G)$ .

Take an edge  $uv$  of  $G$ . Since the competition graph of  $D$  contains the graph  $G$ , there exists a common prey  $z$  of  $u$  and  $v$  in  $D$ . Therefore,  $u$  and  $v$  are adjacent in  $\tilde{G}$ . Also, since  $u$  and  $v$  have a prey in  $D$ , it means that  $u, v \in S = V(H)$ . Since  $H$  is an induced subgraph of  $\tilde{G}$ , we have  $uv \in E(H)$ . Therefore,  $E(G) \subseteq E(H)$ .

Take an edge  $uv$  of  $H$ . Since  $u, v \in V(H) = S$ , we have  $N_D^+(u) \neq \emptyset$  and  $N_D^+(v) \neq \emptyset$ . To show that  $uv$  is also an edge of  $G$ , it is sufficient to show that  $u$  and  $v$  have a common prey in  $D$ . Suppose that  $u$  and  $v$  do not have a common prey in  $D$ . Since  $N_D^+(u) \neq \emptyset$  and  $N_D^+(v) \neq \emptyset$ , there exist two vertices  $a$  and  $b$  in  $D$  such that  $a \prec u$  and  $b \prec v$ . Since  $u$  and  $v$  are adjacent in  $H$  but have no common prey in  $D$ , either  $u \prec v$  or  $v \prec u$ . If  $u \prec v$  then  $a \prec v$  since  $a \prec u$ , and so  $a$  is a common prey of  $u$  and  $v$  in  $D$ , which is a contradiction. Similarly, if  $v \prec u$  then  $b \prec u$  since  $b \prec v$ , and so  $b$  is a common prey of  $u$  and  $v$  in  $D$ , which is a contradiction. Therefore,  $u$  and  $v$  have a common prey in  $D$ . Thus,  $uv$  is an edge of  $G$  and so  $E(H) \subseteq E(G)$ . Hence,  $E(H) = E(G)$ .

Let  $H_0$  be the subgraph of  $\tilde{G}$  obtained by deleting all the isolated vertices from  $H$ . Then  $H_0$  is an induced subgraph of  $\tilde{G}$  which is equal to  $G$ . We complete the proof.  $\blacksquare$

### 3. CONCLUDING REMARKS

In this note, we showed that the phylogeny graph of a doubly partial order is an interval graph (Theorem 2.1) and that any interval graph has a graph extension which is the phylogeny graph of a doubly partial order (Theorem 2.3).

By Theorem 2.3, for an interval graph  $G$ , we can define the *doubly partial order phylogeny number*  $p_{\text{dpo}}(G)$  of  $G$  to be the smallest nonnegative integer  $r$

such that  $r := |V(\tilde{G}) \setminus V(G)|$  where  $\tilde{G}$  is an interval graph containing the graph  $G$  as an induced subgraph and  $\tilde{G}$  is the phylogeny graph of a doubly partial order. The doubly partial order phylogeny number of an interval graph  $G$  may be different from the phylogeny number of  $G$ . It can be shown by using the path  $P$  of length three. By [18, Theorem 7], if  $G$  is a chordal graph, then the phylogeny number  $p(G)$  of  $G$  is equal to 0. Therefore  $p(P) = 0$ . On the other hand, it follows from Theorem 2.2 that  $p_{\text{dpo}}(P) > 0$ . It would be interesting to find the doubly partial order phylogeny numbers of various interval graphs.

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