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LIE IDEALS IN PRIME Γ-RINGS WITH DERIVATIONS

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Abstract

Let M be a 2 and 3-torsion free prime Γ-ring, d a nonzero derivation on M and U a nonzero Lie ideal of M. In this paper it is proved that U is a central Lie ideal of M if d satisfies one of the following

(i) $d(U) \subset Z$,

(ii) $d(U) \subset U$ and $d^2(U) = 0$,

(iii) $d(U) \subset U$, $d^2(U) \subset Z$.

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1. Introduction

The concept of a Γ-ring was first introduced by Nobusawa [5], and generalized by Barnes [1] as follows: A Γ-ring is a pair (M, Γ) where M and Γ are additive abelian groups for which there exists a map from $M \times \Gamma \times M$ to M (the image of (x, α, y) was denoted by $x\alpha y$) such that

(i) $(x+y)\alpha z = x\alpha z + y\alpha z$,
$x(\alpha+\beta)y = x\alpha y + x\beta y$,
$x\alpha(y+z) = x\alpha y + x\alpha z$,
(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring.

Recall that a Γ-ring M is called prime if for any two elements $x, y \in M$, $x\Gamma M\Gamma y = 0$ implies either $x = 0$ or $y = 0$, and M is called semiprime if $x\Gamma M\Gamma x = 0$ with $x \in M$ implies $x = 0$. Note that every prime Γ-ring is obviously semiprime. An additive mapping $d\colon M \to M$ is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. An additive subgroup I of M is called a left (right) ideal of M if $M\Gamma I \subset I$ ($I\Gamma M \subset I$). If I is both left and right ideal of M, then we say I is an ideal of M. The set $Z = \{x \in M; x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma\}$ is called the center of M. An additive subgroup U of M is said to be a Lie ideal of M if $[u, x]_\alpha \in U$, for all $u \in U, x \in M$ and $\alpha \in \Gamma$. M is n-torsion free if $nx = 0$, for $x \in M$ implies $x = 0$, where n is an integer. The commutator $x\alpha y - y\alpha x$ will be denoted by $[x, y]_\alpha$. We will use for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, the basic commutator identities:

$[x\alpha y, z]_\beta = x\alpha[y, z]_\beta + [x, z]_\beta\alpha y + x[\alpha, \beta]_z y$, and
$[x, y\alpha z]_\beta = y\alpha[x, z]_\beta + [x, y]_\beta\alpha z + y[\beta, \alpha]_x z$.

Throughout this paper, We consider the following assumption $x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ and it will be represented by property $(*)$ is a central.

According to the assumption property $(*)$, the above two identities reduced to
$[x\alpha y, z]_\beta = x\alpha[y, z]_\beta + [x, z]_\beta\alpha y$, and
$[x, y\alpha z]_\beta = y\alpha[x, z]_\beta + [x, y]_\beta\alpha z$.

The relationship between the derivations and Lie ideals of a prime ring has been investigated by a number of authors (see [2, 3] and [4]). In [2], Bergen, Herstien and Kerr showed that if U is a nonzero Lie ideal of a 2-torsion free prime ring R and d a nonzero derivation of R such that $d^2(U) = 0$ or $d^2(U) \subset Z$ then U is central. Our aim in this paper is generalized the above results in prime Γ-rings with Lie ideals.

2. The results

For proving the main results, we have needed some important lemmas. So we start as follows:

Remark 1. Let M be 2-torsion free prime Γ-ring and d a derivation of M. Then for all $x, y \in M$ and $\alpha \in \Gamma$, we have the followings:

(i) If $d^2 = 0$ on M, then $d = 0$,
(ii) $d([x, y]_\alpha) = [d(x), y]_\alpha + [x, d(y)]_\alpha$,
(iii) $d^2(x\alpha y) = d^2(x)\alpha y + 2d(x)\alpha d(y) + x\alpha d^2(y)$,
(iv) $d^3(x\alpha y) = d^3(x)\alpha y + 3d^2(x)\alpha d(y) + 3d(x)\alpha d^2(y) + x\alpha d^3(y)$.

Lemma 2 ([6], Lemma 1)**.** *Let M be 2-torsion free prime Γ-ring and Z the center of M. Then the following are satisfied:*

(i) *If $x \in Z$, and $x\Gamma y = 0$, then either $x = 0$ or $y = 0$.*
(ii) *If $x \in Z$, and $x\Gamma y \subset Z$, then either $x = 0$ or $y \in Z$.*

Lemma 3 ([3], Lemma 2)**.** *Let $0 \neq U$ be a Lie ideal of a 2-torsion free prime Γ-ring M and $U \nsubseteq Z$. If for $a, b \in M$ such that $a\Gamma U\Gamma b = 0$, then $a = 0$ or $b = 0$.*

Lemma 4. *Let U be a nonzero Lie ideal of prime Γ-ring M. If $[M, U]_\alpha \subset Z$, then $U \subset Z$.*

Proof. For all $x \in M, u \in U$ and $\alpha \in \Gamma$, we have $[x, u]_\alpha \in [M, U]_\alpha$.
Replacing x by $x\beta u$, we get

$$[x\beta u, u]_\alpha = [x, u]_\alpha \beta u \in Z, \text{ for all } x \in M,\ u \in U \text{ and } \alpha, \beta \in \Gamma.$$

Since $[x, u]_\alpha \in Z$, then by Lemma 2(ii) we obtain $[x, u]_\alpha = 0$ or $u \in Z$, then the result required. ∎

Lemma 5. *Let $0 \neq U$ be a Lie ideal of 2-torsion free prime Γ-ring M satisfying property $(*)$. If $[U, U]_\Gamma = 0$, then $U \subset Z$ (If U is a commutative Lie ideal, then U is central).*

Proof. For all $x \in M, u \in U$ and $\alpha \in \Gamma$, we have $[u, x]_\alpha \in U$. Hence by hypothesis we have

$$[u, [u, x]_\alpha]_\beta = 0, \text{ for all } x \in M,\ u \in U \text{ and } \alpha, \beta \in \Gamma.$$

Equivalently

$$u\beta[u, x]_\alpha = [u, x]_\alpha\beta u, \text{ for all } x \in M, u \in U \text{ and } \alpha, \beta \in \Gamma. \tag{1}$$

Replacing x by $x\alpha y$, for $y \in M$ and $\alpha \in \Gamma$, we get

$$u\beta x\alpha[u, y]_\alpha + u\beta[u, x]_\alpha\alpha y = x\alpha[u, y]_\alpha\beta u + [u, x]_\alpha\alpha y\beta u. \tag{2}$$

Using (1) for $u\beta[u,x]_\alpha = [u,x]_\alpha\beta u$ and $[u,y]_\alpha\beta u = u\beta[u,y]_\alpha$ in (2) we obtain

$$u\beta x\alpha[u,y]_\alpha + [u,x]_\alpha\beta u\alpha y = x\alpha u\beta[u,y]_\alpha + [u,x]_\alpha\alpha y\beta u.$$

Using property $(*)$ we get $2[u,x]_\alpha\beta[u,y]_\alpha = 0$. Since M is 2-torsion free, this leads to

$$[u,x]_\alpha\beta[u,y]_\alpha = 0, \text{ for all } x,y \in M, u \in U \text{ and } \alpha,\beta \in \Gamma.$$

Replacing y by $y\gamma x$, we find that

$$[u,x]_\alpha\beta y\gamma[u,x]_\alpha = 0, \text{ for all } x,y \in M, u \in U \text{ and } \alpha,\beta,\gamma \in \Gamma.$$

Thus $[u,x]_\Gamma\Gamma M\Gamma[u,x]_\Gamma = 0$, for all $x, \in M, u \in U$. By primeness of M, we conclude $[u,x]_\Gamma = 0$, yields $U \subset Z$. ■

Lemma 6. *Let U be a nonzero Lie ideal of 2-torsion free prime Γ-ring M and d a nonzero derivation of M. If $a \in U$ such that $[a,d(x)]_\alpha = 0$, for all $x \in M$ and $\alpha \in \Gamma$, then $a \in Z$.*

Proof. By hypothesis we have $[a,d(x)]_\alpha = 0$, for all $x \in M$ and $\alpha \in \Gamma$. Replacing x by $x\beta y$, we get

$$\begin{aligned} 0 &= [a,d(x\beta y)]_\alpha \\ &= [a,d(x)]_\alpha\beta y + d(x)\beta[a,y]_\alpha + x\beta[a,d(y)]_\alpha + [a,x]_\alpha\beta d(y) \\ &= d(x)\beta[a,y]_\alpha a + [a,x]_\alpha\beta d(y). \end{aligned}$$

Replacing x by $d(x)$, we obtain

$$d^2(x)\beta[a,y]_\alpha = 0, \text{ for all } x,y \in M \text{ and } \alpha,\beta \in \Gamma.$$

Replacing y by $z\gamma y$, we get

$$d^2(x)\beta z\gamma[a,y]_\alpha = 0, \text{ for all } x,y,z \in M \text{ and } \alpha,\beta,\gamma \in \Gamma.$$

By primeness we get $d^2(x) = 0$ or $[a,y]_\alpha = 0$, since $d \neq 0$, therefore $a \in Z$. ■

Theorem 7. *Let U be a nonzero Lie ideals of a 2-torsion free prime Γ-ring M and d a nonzero derivation of M. If $d(U) \subset Z$, then $U \subset Z$.*

Proof. suppose that $U \nsubseteq Z$, then by Lemma 5 we have $V = [U, U] \nsubseteq Z$. Let $u, w \in U$, hence from

$$d([u, w]_\alpha) = [d(u), w]_\alpha + [u, d(w)]_\alpha = 0.$$

Since $d(u), d(w) \in Z$. It follows that $d(V) = 0$.

Let $v \in V, m \in M$ and $\alpha \in \Gamma$, since $d(v) = 0$ and $d([v, m]_\alpha) = 0$, we get

$$[v, d(m)]_\alpha = 0, \text{ for all } v \in V, m \in M \text{ and } \alpha \in \Gamma.$$

Therefore by Lemma 6 we get $v \in Z$, contradiction. Accordingly, $U \subset Z$. ■

Lemma 8. *Let $U \nsubseteq Z$ be a Lie ideal of 2-torsion free prime Γ-ring M and d a nonzero derivation of M. If $a \in M$ and $a\Gamma d(U) = 0$ $(d(U)\Gamma a = 0)$, then $a = 0$.*

Proof. For all $u \in U$, $x \in M$ and $\alpha \in \Gamma$ we have $[u, x]_\beta \gamma u \in U$. By hypothesis we have

$$\begin{aligned} 0 &= a\alpha d([u, x]_\beta \gamma u) \\ &= a\alpha [u, x]_\beta \gamma d(u), for all x \in M, u \in U and \alpha, \beta, \gamma \in \Gamma. \end{aligned}$$

Replacing x by $d(v)\lambda x$, we get

$$a\alpha u\beta d(v)\lambda x\gamma d(u) = 0, \text{ for all } u, v \in U,\ x \in M \text{ and } \alpha, \beta, \gamma, \lambda \in \Gamma.$$

By primeness we obtain $a\alpha u\beta d(v) = 0$ or $d(u) = 0$.

Now let $K = \{u \in U a\alpha u\beta d(v) = 0\}$ and $L = \{u \in U | d(u) = 0\}$. Since K and L are additive subgroups of U and $U = K \cup L$, but a group can't be union of its two proper subgroups and hence $U = K$ or $U = L$.

According to Theorem 7, $d(U) \neq 0$, which proves that $U = K$. Hence we get $a\Gamma U\Gamma d(v) = 0$, for all $v \in U$. By Lemma 3 we get $a = 0$ or $d(v) = 0$, again by Theorem 7 $d(U) \neq 0$, therefore $a = 0$. ■

Theorem 9. *Let M be a 2-torsion free prime Γ-ring, U be a nonzero Lie ideal of M and d be a nonzero derivation of M. If $d^2(U) = 0$ and $d(U) \subset U$, then $U \subset Z$.*

Proof. Suppose that $U \nsubseteq Z$, for all $x \in M$, $u \in U$ and $\alpha \in \Gamma$ we have $[x, u]_\alpha \in U$. Since $d^2(U) = 0$, then by using Remark 1 we get

$$\begin{aligned} 0 &= d^2([x\beta u, u]_\alpha) \\ &= d^2([x, u]_\alpha)\beta u + 2d([x, u]_\alpha)\beta d(u) + [x, u]_\alpha \beta d^2(u). \end{aligned}$$

Since M is 2-torsion free and $d^2(U) = 0$, then we get

$$d([x, u]_\alpha)\beta d(u) = 0, \text{ for all } x \in M,\ u \in U \text{ and } \alpha, \beta \in \Gamma.$$

Replacing u by $u + d(u)$, we get $d([x, d(u)]_\alpha \beta d(u) = 0$, so that

$$[d(x), d(u)]_\alpha \beta d(u) = 0, \text{ for all } x \in M,\ u \in U \text{ and } \alpha, \beta \in \Gamma.$$

According to Lemma 8 we get $[d(x), d(u)]_\alpha = 0$ for all $x \in M, w \in U$ and $\alpha \in \Gamma$, therefor by Lemma 6 we conclude that $d(U) \subset Z$, which is contradicts Theorem 7, this prove the theorem. ■

Lemma 10. *Let M be a 2 and 3-torsion free prime Γ-ring, U be a nonzero Lie ideal of M and d be a nonzero derivation of M. If $d(U) \subset U$, $d^2(U) \subset Z$ and $d^3(U) = 0$ then $U \subset Z$.*

Proof. For all $x \in M$, $u \in U$ and $\alpha \in \Gamma$ we have $[x, u]_\alpha \in U$. Since $d^3(U) = 0$, then we obtain $d^3([x, u]_\alpha) = 0$. Replacing x by $x\beta u$ and using Remark 1(iv) we get

$$\begin{aligned} 0 &= d^3([x\beta u, u]_\alpha) \\ &= 3d^2([x, u]_\alpha)\beta d(u) + 3d([x, u]_\alpha)\beta d^2(u). \end{aligned}$$

Since M is 3-torsion free, then we get

$$d^2([x, u]_\alpha)\beta d(u) + d([x, u]_\alpha)\beta d^3(u) = 0, \text{ for all } x \in M,\ u \in U \text{ and } \alpha, \beta \in \Gamma.$$

Replacing u by $d(u)$ and using $d^2(U) = 0$ we obtain

$$d^2([x, d(u)]_\alpha)\beta d^2(u) = 0.$$

Since $d^2(U) \subset Z$, then by Lemma 2(i) we get

$$d^2([x, d(u)]_\alpha) = 0 \text{ or } d^2(u) = 0. \tag{3}$$

If $d^2([x, d(u)]_\alpha) = 0$, then replacing x by $x\beta d(u)$ we obtain

$$\begin{aligned} 0 &= d^2([x\beta d(u), d(u)]_\alpha) \\ &= d^2([x, d(u)]_\alpha \beta d(u)) \\ &= d^2([x, d(u)]_\alpha \beta d(u) + 2d([x, d(u)]_\alpha \beta d^2(u) + [x, d(u)]_\alpha \beta d^3(u)). \end{aligned}$$

Since $d^3(U) = 0$, M is a 2-torsion free and by relation (3), then the last equation reduced to

$$d([x, d(u)]_\alpha \beta d^2(u) = 0, \text{ for all } x \in M,\ u \in U \text{ and } \alpha, \beta \in \Gamma.$$

Since $d^2(U) \subset Z$, then by Lemma 2(i) we get $d([x, d(u)]_\alpha = 0$ or $d^2(u) = 0$.

If $d([x, d(u)]_\alpha = 0$, then replacing x by $x\gamma d(u)$, we obtain

$$\begin{aligned} 0 &= d([x\gamma d(u), d(u)]_\alpha) \\ &= d([x, d(u)]_\alpha \gamma d(u)) \\ &= d([x, d(u)]_\alpha)\gamma d(u) + [x, d(u)]_\alpha \gamma d^2(u) \\ &= [x, d(u)]_\alpha \beta d^2(u). \end{aligned}$$

Since $d^2(U) \subset Z$ and $d(U) \subset U$, then by Lemma 2(i) we get $[x, d(u)]_\alpha = 0$ or $d^2(u) = 0$. If $[x, d(u)]_\alpha = 0$, then we have $d(u) \subset Z$. Hence from relation (3) we have either $d(u) \subset Z$ or $d^2(u) = 0$.

Now let $K = \{u \in U | d(u) \subset Z\}$ and $L = \{u \in U | d^2(u) = 0\}$. Since K and L are additive subgroups of U and $U = K \cup L$, but a group can't be union of its two proper subgroups and hence $U = K$ or $U = L$. If $U = K$, that is $d(u) \subset Z$, then by Theorem 7 we get $U \subset Z$, or $U = L$, that is $d^2(u) = 0$, hence by Theorem 9 we get $U \subset Z$. ■

Theorem 11. *Let M be a 2 and 3-torsion free prime Γ-ring, U be a nonzero Lie ideal of M and d be a nonzero derivation of M. If $d(U) \subset U$ and $d^2(U) \subset Z$, then $U \subset Z$.*

Proof. For all $x \in M$, $u \in U$ and $\alpha \in \Gamma$ we have

$$d^2([x, u]_\alpha) \in Z. \tag{4}$$

Replacing x by $x\beta d^2(v)$, where $v \in U$ and $\beta \in \Gamma$, and using $d^2(U) \subset Z$, we get

$$2d([x, u]_\alpha)\beta d^3(v) + [x, u]_\alpha \beta d^4(v) \subset Z, forall u, v \in U, x \in M and \alpha, \beta \in \Gamma. \tag{5}$$

Replacing x by $x\gamma d^2(w)$ in relation (5), where $w \in U$ and $\gamma \in \Gamma$, and using $d^2(U) \subset Z$ and M is 2-torsion free, then the relation (5) reduced to

$$[x, u]_\alpha \gamma d^3(w)\beta d^3(v) \in Z, \text{ for all } v, u, w \in U,\ x \in M \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

Since $d^2(U) \subset Z$ and $d(U) \subset U$, then $d^3(U) \subset Z$ and thus by Lemma 2(ii) we have $d^3(U) = 0$ or $[x, u]_\alpha \subset Z$. Therefore if $d^3(U) = 0$, hence by Lemma 10 yields $U \subset Z$. If $[M, U]_\alpha \subset Z$, then by Lemma 4 we get $U \subset Z$. ■

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