Discussiones Mathematicae
General Algebra and Applications 33 (2013) 109–115
doi:10.7151/dmgaa.1196

THE INERTIA OF UNICYCLIC GRAPHS AND BICYCLIC GRAPHS

Ying Liu

School of Mathematics and Information
Shanghai Lixin University of Commerce
Shanghai, 201620, China

e-mail: lymaths@126.com

Abstract

Let G be a graph with n vertices and $\nu(G)$ be the matching number of G. The inertia of a graph G, $In(G) = (n_+, n_-, n_0)$ is an integer triple specifying the numbers of positive, negative and zero eigenvalues of the adjacency matrix $A(G)$, respectively. Let $\eta(G) = n_0$ denote the nullity of G (the multiplicity of the eigenvalue zero of G). It is well known that if G is a tree, then $\eta(G) = n - 2\nu(G)$. Guo *et al.* [Ji-Ming Guo, Weigen Yan and Yeong-Nan Yeh. On the nullity and the matching number of unicyclic graphs, Linear Algebra and its Applications, 431 (2009), 1293–1301.] proved if G is a unicyclic graph, then $\eta(G)$ equals $n - 2\nu(G) - 1, n - 2\nu(G)$ or $n - 2\nu(G) + 2$. Barrett *et al.* determined the inertia sets for trees and graphs with cut vertices. In this paper, we give the nullity of bicyclic graphs $\mathcal{B}_n^{++}$. Furthermore, we determine the inertia set in unicyclic graphs and $\mathcal{B}_n^{++}$, respectively.

Keywords: matching number, inertia, nullity, unicyclic graph, bicyclic graph.

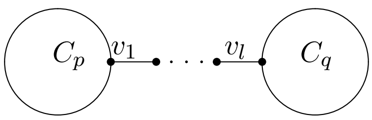
2010 Mathematics Subject Classification: 05C50.

1. Introduction

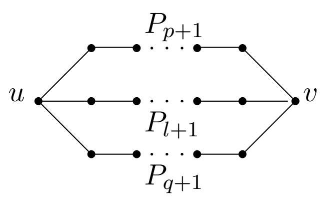
Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. The inertia of a graph G, $In(G) = (n_+, n_-, n_0)$ is an integer triple specifying the numbers of positive, negative and zero eigenvalues of the adjacency matrix $A(G)$, respectively. It is well known if G is a bipartite graph, then $n_+ = n_-$. Barrett, Hall, and Loewy [1] determined the inertia sets for trees and graphs with cut vertices. The nullity of G, denoted by $\eta = \eta(G) = n_0$, is the multiplicity of the number zero in the spectrum of G. Then $n_+ + n_- = n - r(A(G)) = \eta$. The nullity of graphs is of interest in chemistry since the occurrence of a zero eigenvalue of a bipartite graph (corresponding to an alternant hydrocarbon) indicates the chemical instability of the molecule which such a graph represents. The question is of interest also for non-alternant hydrocarbons (non-bipartite graph), but a direct connection with the chemical stability in these cases is not so straightforward. The nullity has been determined for trees, unicyclic graphs and bicyclic graphs, respectively [4, 5, 6]. Recently, Gutman and Borovićanin give a survey on the nullity of graphs.

A unicyclic graph is a simple connected graph with equal numbers of vertices and edges. For the sake of a convenient description, let $\mathcal{U}_n$ be the set of unicyclic graphs with n vertices. A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one.

Let C_p and C_q be two vertex-disjoint cycles. Suppose that $v_1 \in C_p, v_l \in C_q$. Joining v_1 and v_l by a path $v_1v_2 \cdots v_l$ of length $l-1$, where $l \geq 1$ and $l = 1$ means identifying v_1 with v_l, resultant graph, denoted by $\infty(p, l, q)$, is called an ∞-graph. Let P_{l+1}, P_{p+1} and P_{q+1} be three vertex-disjoint paths, where $\min\{p, l, q\} \geq 1$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, resultant graph, denoted by $\theta(p, l, q)$, is called a θ-graph (see Figure 1).



$\infty(p, l, q)$



$\theta(p, l, q)$

Figure 1

Let $\mathcal{B}_n$ be the set of all bicyclic graphs of order n. $\mathcal{B}_n$ consists of three types of graphs: the first type denoted by $\mathcal{B}_n^+$ is the set of those graphs each of which is an ∞-graph with trees attached when $l > 1$; the second type denoted by $\mathcal{B}_n^{++}$ is the set of those graphs each of which is an ∞-graph with trees attached when $l = 1$; the third type denoted by θ_n is the set of those graphs each of which is an θ-graph with trees attached.

In Section 3, we study the inertia in $\mathcal{U}_n$. In Section 4, we give the nullity and the inertia sets in $\mathcal{B}_n^{++}$, respectively.

2. Main Lemmas

A matching of G is a collection of independent edges of G. A maximum matching is a matching with the maximum possible number of independent edges. The size of a maximum matching of G, i.e., the maximum number of independent edges of G, is denoted by $\nu = \nu(G)$.

Denote by $\varphi(x) = \varphi_G(x)$ the characteristic polynomial of G. Let

$$\varphi(x) = |xI - A| = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n. \tag{1}$$

Then [2]

$$a_i = \sum_U (-1)^{p(U)} 2^{c(U)} \quad (i = 1, 2, \ldots, n), \tag{2}$$

where the sum is over all subgraphs U of G consisting of disjoint edges and cycles and having exactly i vertices (called "basic figures"). If U is such a subgraph, then $p(U)$ is the number of its components, of which $c(U)$ components are cycles.

Example 1. Let G is a bipartite graph, then G does not contain an odd cycle, so $a_{2i+1} = 0$ $(i \geq 1)$.

Example 2. Considering equation (1) with equation (2), it is easy to obtain $a_1 = 0$ and $a_2 = 2m$ (m is the number of edges of G). In the following, we calculate a_3 and a_4. The subgraphs U of G having exactly 3 vertices consist of only the cycle C_3. Suppose that n_Δ is the number of the cycles C_3 in G, then $a_3 = -2n_\Delta$. Let $n_\Box$ and $\nu_2(G)$be the number of the cycles C_4, and two mutually disjoint edges in G, respectively, then $a_4 = \nu_2(G) - 2n_\Box$.

Next, we introduce the well-known Cauchy's interlacing theorem in matrix theory.

Lemma 3 [2]**.** *Let A be symmetric and A' be one of its principal submatrices. Let $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda'_1 \geq \cdots \geq \lambda'_m$ be the eigenvalues of A and A', respectively. Then the inequality $\lambda_i \geq \lambda'_i \geq \lambda_{n-m+i}$ holds for all $i = 1, 2, \ldots, m$.*

Applying the Cauchy's interlacing theorem to the adjacency matrix $A(G)$ of the graph G, we have the following corollary.

Corollary 4. *Let V_0 be the k-subset of $G = (V, E)$ with n vertices $(0 \leq k \leq n-1)$, and $G - V_0$ be the subgraph induced by removing the vertices V_0 and their incident edges. Then $\lambda_i(G) \geq \lambda_i(G - V_0) \geq \lambda_{i+k}(G) \quad (1 \leq i \leq n - k)$.*

The next lemma is useful to the proof of our main results.

Lemma 5 [2]**.** *For a graph G containing a pendent vertex, if the induced subgraph H of G is obtained by deleting this vertex together with the vertex adjacent to it, then the relation $\eta(H) = \eta(G)$ holds.*

3. The inertia of unicyclic graphs

n this section, we determine the inertia in $\mathcal{U}_n$. In order to prove our result, the following lemma is necessary.

Lemma 6 [5]**.** *Suppose $G \in \mathcal{U}_n$ with the cycle C_l. Then*

(1) $\eta(G) = n - 2\nu(G) - 1$, *if* $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$;

(2) $\eta(G) = n - 2\nu(G) + 2$, *if G satisfies:* $\nu(G) = \frac{l}{2} + \nu(G - C_l)$, $l \equiv 0$ *(mod 4) and no maximum matching contains an edge incident to* C_l;

(3) $\eta(G) = n - 2\nu(G)$, *otherwise.*

If $G \in \mathcal{U}_n$ is a bipartite graph, we know $n_+ = n_-$ and $n_+ + n_- = n - \eta(G)$, then $In(G) = (\nu(G) - 1, \nu(G) - 1, n - 2\nu(G) + 2)$ or $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$, So we only consider those graphs $G \in \mathcal{U}_n$ which are non-bipartite.

Lemma 7. *If $G \in \mathcal{U}_n$ is a non-bipartite graph, then* $In(G) = (\nu(G) + 1, \nu(G), n - 2\nu(G) - 1)$, $In(G) = (\nu(G), \nu(G) + 1, n - 2\nu(G) - 1)$ *or* $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$.

Proof. Since $G \in \mathcal{U}_n$ with the cycle C_l is a non-bipartite graph, then l is odd. Let $v_i \in V(C_l)$ and $d_i \geq 3$. Suppose that $T_1, \ldots, T_{d_i}$ are the components of $G - v_i$ where $d_i = d(v_i)$. Let $V(T_j) = n_j$ and $\nu_j = \nu(T_j)$ $(j = 1, \ldots, d_i)$, so we have $\sum_{j=1}^{d_i} n_j = n - 1$ and $\sum_{j=1}^{d_i} \nu_j = \nu(G)$ or $\nu(G) - 1$. And $In(T_j) = (\nu_j, \nu_j, n_j - 2\nu_j)$. We discuss two cases in the following.

(1) $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$, then $\eta(G) = n - 2\nu(G) - 1$ and $\sum_{j=1}^{d_i} \nu_j = \nu(G)$. We know $\eta(G - v_i) = \sum_{j=1}^{d_i} \eta(T_j) = n - 1 - 2\sum_{j=1}^{d_i} \nu_j = n - 2\nu(G) - 1$. Let $\lambda'_1, \ldots, \lambda'_{\nu(G)}, \underbrace{\lambda'_{\nu(G)+1}, \ldots, \lambda'_{n-1-\nu(G)}}_{n-2\nu(G)-1}, \lambda'_{n-\nu(G)}, \ldots, \lambda'_{n-1}$ be the eigenvalues of $G - v_i$ according to nondecreasing order. By Corollary 4, we have $\lambda_{n-\nu(G)+1}(G) \leq \lambda'_{n-\nu(G)} < 0$ and $\lambda_{\nu(G)}(G) \geq \lambda'_{\nu(G)} > 0$. So $In(G) = (\nu(G) + 1, \nu(G), n - 2\nu(G) - 1)$ or $In(G) = (\nu(G), \nu(G) + 1, n - 2\nu(G) - 1)$.

(2) $\nu(G) \neq \frac{l-1}{2} + \nu(G - C_l)$, then $\eta(G) = n - 2\nu(G)$ and $\sum_{j=1}^{d_i} \nu_j = \nu(G) - 1$. We know $\eta(G - v_i) = \sum_{j=1}^{d_i} \eta(T_j) = n - 1 - 2\sum_{j=1}^{d_i} \nu_j = n - 2\nu(G) + 1$. Let $\lambda'_1, \ldots, \lambda'_{\nu(G)}, \underbrace{\lambda'_{\nu(G)+1}, \ldots, \lambda'_{n-\nu(G)+1}}_{n-2\nu(G)+1}, \lambda'_{n-\nu(G)+2}, \ldots, \lambda'_{n-1}$ be the eigenvalues of $G - v_i$ according to nondecreasing order. By Corollary 4, we have $\lambda_{n-\nu(G)+2}(G) \leq \lambda'_{n-\nu(G)+1} < 0$ and $\lambda_{\nu(G)}(G) \geq \lambda'_{\nu(G)} > 0$. And $\eta(G) = n - 2\nu(G)$, so $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$. ∎

Basing on the above detailed account, we obtain the next theorem.

Theorem 8. *If* $G \in \mathcal{U}_n$, *then* $In(G) = (\nu(G)-1, \nu(G)-1, n-2\nu(G)+2)$, $(\nu(G), \nu(G), n-2\nu(G))$, $(\nu(G)+1, \nu(G), n-2\nu(G)-1)$ *or* $(\nu(G), \nu(G)+1, n-2\nu(G)-1)$.

4. The inertia of bicyclic graphs

In this section, we only consider $\mathcal{B}_n^{++}$. For $G \in \mathcal{B}_n^{++}$, we give the nullity of G and determine the inertia of G according to $\nu(G)$, respectively.

Lemma 9. *The graph* $\infty(p, 1, q)$ *is defined as above, then*

(1) $\eta(\infty(4s, 1, 4t+2)) = 1$ $(s, t \geq 1)$;

(2) $\eta(\infty(4s, 1, 4t)) = 3$ $(s, t \geq 1)$.

Proof. Let $\varphi_1(x) = |xI - A| = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{4s+4t}x + a_{4s+4t+1}$ and $\varphi_2(x) = |xI - B| = x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{4s+4t-2}x + b_{4s+4t-1}$ be the polynomials of $\infty(4s, 1, 4t+2)$ and $\infty(4s, 1, 4t)$, respectively. Since $\infty(4s, 1, 4t+2)$ and $\infty(4s, 1, 4t)$ are bipartite graph, so by the equation (2), we have $a_{2i+1} = 0$ and $b_{2i+1} = 0$ for $i \geq 1$. First of all, we consider a_{4s+4t} using the equation (2), then $a_{4s+4t} = 2m_1(-1)^{2t+1} + 2m_2(-1)^{2s} + (2m_1 + 2m_2) \neq 0$, where m_1 is the number of methods picking up $2t$ disjoint edges from P_{4t+1} and m_2 is the number of methods picking up $2s-1$ disjoint edges from P_{4s-1}. So $\eta(\infty(4s, 1, 4t+2)) = 1$.

Next, we prove $b_{4s+4t-2} = 0$ and $b_{4s+4t-4} \neq 0$. Using the similar method as above, we have $b_{4s+4t-2} = 2m_1(-1)^{2t} + 2m_2(-1)^{2s} - (2m_1 + 2m_2) = 0$, where m_1 is the number of methods picking up $2t-1$ disjoint edges from P_{4t-1} and m_2 is the number of methods picking up $2s-1$ disjoint edges from P_{4s-1}. And $b_{4s+4t-4} \geq m_3 > 0$ where m_3 is the number of methods picking up $2t-1$ disjoint edges from P_{4t} and picking up $2s-1$ disjoint edges from P_{4s-1}. So we complete the proof. ■

Using the similar method of proof in Lemma 9 and the equation (2), we obtain the following lemma.

Lemma 10. *The graph* $\infty(p, 1, q)$ *is defined as above, then*

(1) $\eta(\infty(2s+1, 1, 4t)) = \eta(\infty(4s+1, 1, 4t+3)) = 1$;

(2) $\eta(\infty(2s+1, 1, 4t+2)) = \eta(\infty(4s+1, 1, 4t+1)) = 0$.

Lemma 11 [3]**.** *If a bipartite graph* G *with* $n \geq 1$ *vertices does not contain any cycle of length* $4s$ $(s \geq 1)$, *then* $\eta(G) = n - 2\nu(G)$.

In accordance with Lemma 11, it is easy to know for $G \in \mathcal{B}_n^{++}$ is a bipartite graph with not containing cycle C_{4s} $(s \geq 1)$, then $\eta(G) = n - 2\nu(G)$, so $In(G) = (\nu(G), \nu(G), n - 2\nu(G))$. Hence in the following, we discuss the case $G \in \mathcal{B}_n^{++}$ is a bipartite graph with containing cycles C_{4s} $(s \geq 1)$.

Lemma 12. *If $G \in \mathcal{B}_n^{++}$ is a bipartite graph with containing cycle C_{4s} $(s \geq 1)$, then $\eta(G) = n - 2\nu(G)$ or $\eta(G) = n - 2\nu(G) + 2$.*

Proof. Putting to use the Lemma 5 a times, we can obtain the following cases:

(1) T_i $(1 \leq i \leq s)$ are the components where T_i $(1 \leq i \leq s)$ are trees with n_i vertices. Then $\eta(G) = \sum_{i=1}^{s} \eta(T_i) = \sum_{i=1}^{s}(n_i - 2\nu(T_i)) = n - a - 2(\nu(G) - a) = n - 2\nu(G)$.

(2) U_0, T_i $(1 \leq i \leq s)$ are the components where T_i $(1 \leq i \leq s)$ are trees with n_i vertices and U_0 is a unicyclic graph with n_0 vertices. By Lemma 6, we know $\eta(U_0) = n_0 - 2\nu(U_0)$ or $n_0 - 2\nu(U_0) + 2$, so $\eta(G) = \eta(U_0) + \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G)$ or $n - 2\nu(G) + 2$.

(3) $\infty(p, 1, q), T_i$ $(1 \leq i \leq s)$ are the components where T_i $(1 \leq i \leq s)$ are trees with n_i vertices and $\infty(p, 1, q)$ is a bicyclic graph with n_0 vertices. By Lemma 9, we have $\eta(\infty(4s, 1, 4t + 2)) = 1$ or $\eta(\infty(4s, 1, 4t)) = 3$. Then $\eta(G) = \eta(\infty(p, 1, q)) + \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G)$ or $n - 2\nu(G) + 2$. ■

Combining Lemmas 10 and 12, we obtain the following theorem.

Theorem 13. *If $G \in \mathcal{B}_n^{++}$ is a bipartite graph, then $\eta(G) = n - 2\nu(G)$ or $\eta(G) = n - 2\nu(G) + 2$.*

Lemma 14. *If $G \in \mathcal{B}_n^{++}$ is a non-bipartite graph, then $\eta(G) = n - 2\nu(G) - 1,\ n - 2\nu(G),\ n - 2\nu(G) + 1$ or $\eta(G) = n - 2\nu(G) + 2$.*

Proof. Putting to use the Lemma 5 b times, we can obtain the following cases:

(1) T_i $(1 \leq i \leq s)$ are the components where T_i $(1 \leq i \leq s)$ are trees with n_i vertices. Then $\eta(G) = \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G)$.

(2) U_0, T_i $(1 \leq i \leq s)$ are the components where T_i $(1 \leq i \leq s)$ are trees with n_i vertices and U_0 is a unicyclic graph with n_0 vertices. By Lemma 6, we know $\eta(U_0) = n_0 - 2\nu(U_0), n_0 - 2\nu(U_0) + 2$ or $n_0 - 2\nu(U_0) - 1$, so $\eta(G) = \eta(U_0) + \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G),\ n - 2\nu(G) + 2$ or $n - 2\nu(G) - 1$.

(3) $\infty(p,1,q), T_i$ $(1 \le i \le s)$ are the components where T_i $(1 \le i \le s)$ are trees with n_i vertices and $\infty(p,1,q)$ is a bicyclic graph with n_0 vertices. By Lemma 10, we have $\eta(\infty(2t+1,1,4s)) = 1$, $\eta(\infty(2t+1,1,4s+2)) = 0$, $\eta(\infty(4s+1,1,4t+1)) = 0$ or $\eta(\infty(4s+1,1,4t+3)) = 1$. Then $\eta(G) = \eta(\infty(p,1,q)) + \sum_{i=1}^{s} \eta(T_i) = n - 2\nu(G) + 1$, $n - 2\nu(G)$ or $n - 2\nu(G) - 1$. ■

Using the similar method of Lemma 7 and Lemma 14, we have the next lemma.

Lemma 15. *If $G \in \mathcal{B}_n^{++}$ is a non-bipartite graph, then* $In(G) = (\nu(G), \nu(G)+1, n-2\nu(G)-1)$, $(\nu(G)+1, \nu(G), n-2\nu(G)-1)$, $(\nu(G), \nu(G), n-2\nu(G))$, $(\nu(G), \nu(G)-1, n-2\nu(G)+1)$, $(\nu(G)+1, \nu(G)-2, n-2\nu(G)+1)$, $(\nu(G), \nu(G)-2, n-2\nu(G)+2)$.

So we obtain our main result.

Theorem 16. *If $G \in \mathcal{B}_n^{++}$, then* $In(G) = (\nu(G), \nu(G)+1, n-2\nu(G)-1)$, $(\nu(G)+1, \nu(G), n-2\nu(G)-1)$, $(\nu(G), \nu(G), n-2\nu(G))$, $(\nu(G), \nu(G)-1, n-2\nu(G)+1)$, $(\nu(G)+1,\ \nu(G)-2, n-2\nu(G)+1)$, $(\nu(G), \nu(G)-2, n-2\nu(G)+2)$.

Remark 17. The paper is supported by the National Natural Science Foundation 205 for Young Scholar of China (11101284), China Scholarship Council (201208310422) and Shanghai Municipal Natural Science Foundation (11ZR1425100).

REFERENCES

[1] W. Barrett, H. Tracy Hall and R. Loewy, *The inverse inertia problem for graphs: Cut vertices, trees, and a counterexample*, Linear Algebra and its Applications **431** (2009) 1147–1191. doi:10.1016/j.laa.2009.04.007

[2] D. Cvetkocić, M. Doob and H. Sachs, Spectra of Graphs - Theory and Application (Academic Press, New York, 1980).

[3] D. Cvetkocić, I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals II*, Croat.Chem. Acta **44** (1972) 365–374.

[4] S. Fiorini, I. Gutman and I. Sciriha, *Trees with maximum nullity*, Linear Algebra and its Applications **397** (2005) 245–252. doi:10.1016/j.laa.2004.10.024

[5] Ji-Ming Guo, Weigen Yan and Yeong-Nan Yeh, *On the nullity and the matching number of unicyclic graphs*, Linear Algebra and its Applications **431** (2009) 1293–1301. doi:10.1016/j.laa.2009.04.026

[6] Shengbiao Hu, Tan Xuezhong and Bolian Liu, *On the nullity of bicyclic graphs*, Linear Algebra and its Applications **429** (2008) 1387–1391. doi:10.1016/j.laa.2007.12.007

Received 8 March 2013
Revised 26 March 2013