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FUZZY n-FOLD INTEGRAL FILTERS IN BL-ALGEBRAS

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Abstract

In this paper, we introduce the notion of fuzzy n-fold integral filter in BL-algebras and we state and prove several properties of fuzzy n-fold integral filters. Using a level subset of a fuzzy set in a BL-algebra, we give a characterization of fuzzy n-fold integral filters. Also, we prove that the homomorphic image and preimage of fuzzy n-fold integral filters are also fuzzy n-fold integral filters. Finally, we study the relationship among fuzzy n-fold obstinate filters, fuzzy n-fold integral filters and fuzzy n-fold fantastic filters

Keywords: BL-algebra, fuzzy n-fold obstinate filter, n-fold obstinate BL-algebra, n-fold integral BL-algebra and fuzzy n-fold integra filter.

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1. Introduction

BL-algebras are the algebraic structure for Hájek basic logic [14] in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Łukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (*BL* for short) is proposed as "the most general"many-valued logic with truth values in [0, 1] and *BL*-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0, 1]. The most familiar example of a *BL*-algebra is the unit interval [0, 1] endowed with the structure induced by a continuous t-norm. The concept of an *MV*-algebra is introduced by Chang [3]. Turunen [15] introduced the notion of an implicative filter and *Boolean* filter and proved that these notions are equivalent in *BL*-algebras. *Boolean* filters are an important class of filters, because the quotient *BL*-algebra induced by these filters are *Boolean* algebras. Heveshki and Eslami [7] introduced the notions of n-fold implicative filter and n-fold positive implicative filter and they prove some relations between these filters and construct quotient algebras via these filters in 2008. Also, Motamed and Borumand Saeid [8] introduced the notion of n-fold obstinate filter in 2011. Moreover, Lele [9, 10] studied the notion of fuzzy n-fold (positive) implicative filter and fuzzy n-fold obstinate filter in *BL*-algebras. In 2012, Borzooei and Paad [1], introduced the notions of n-fold integral filter and n-fold integral *BL* algebra and they studied n-fold obstinate *BL* algebras in [2]. Now, in this paper, we define the concepts of fuzzy n-fold integral filters and we state and prove several properties of fuzzy n-fold integral filters. Using a level subset of a fuzzy set in a *BL*-algebra, we give a characterization of fuzzy n-fold integral filters. In the following, we make a link between fuzzy n-fold integral filters and fuzzy $(n+1)$-fold integral filters and we show that extension property holds for this class of fuzzy filters. Also, we prove that a *BL*-algebra L, is an n-fold integral *BL*-algebra if and only if any fuzzy filter of L is a fuzzy n-fold integral filter. We prove that the homomorphic image and preimage of fuzzy n-fold integral filters are also fuzzy n-fold integral filters. Finally, we study the relationship among fuzzy n-fold obstinate filters, fuzzy n-fold integral filters and fuzzy n-fold fantastic filters.

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, see to the references.

Definition [14]. A *BL*-algebra is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

$(BL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

$(BL2)$ $(L, \odot, 1)$ is a commutative monoid,

$(BL3)$ $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,

$(BL4)$ $x \wedge y = x \odot (x \to y)$,

$(BL5)$ $(x \to y) \vee (y \to x) = 1$.

We denote $x^n = \overbrace{x \odot ... \odot x}^{n-times}$, if $n > 0$ and $x^0 = 1$. Also, we denote $(\underbrace{x \to (...(x \to (x}_{n-times} \to y)))...)$ by $x^n \to y$, for all $x, y \in L$.

A BL-algebra L is called a Gödel algebra, if $x^2 = x \odot x = x$, for all $x \in L$ and a BL-algebra L is called an MV-algebra, if $(x^-)^- = x$, for all $x \in L$, where $x^- = x \to 0$.

Proposition 1 [4, 5, 14]**.** *In any BL-algebra the following hold:*

$(BL6)$ $x \leq y$ *if and only if* $x \to y = 1$,

$(BL7)$ $x^{n+1} \leq x^n$, $\forall n \in \mathbb{N}$,

$(BL8)$ $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,

$(BL9)$ $0 \leq x$ *and* $x \odot x^- = 0$,

$(BL10)$ $1 \to x = x$ *and* $x \to 1 = 1$, *for all* $x, y, z \in L$.

The following theorems and definitions are from [1, 2, 4, 6, 7, 10, 11, 14, 16] and we refer the reader to them, for more details.

Definition. Let L be a BL-algebra and F be a non-empty subset of L. Then

(i) F is called a *filter* of L, if $x \odot y \in F$, for any $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$.

(ii) F is called an *n-fold implicative filter* of L, if $1 \in F$ and

$$x^n \to (y \to z) \in F \text{ and } x^n \to y \in F \text{ imply } x^n \to z \in F, \text{ for all } x, y, z \in L.$$

(iii) F is called an *n-fold positive implicative filter* of L, if $1 \in F$ and

$$x \to ((y^n \to z) \to y) \in F \text{ and } x \in F \text{ imply } y \in F, \text{ for all } x, y, z \in L.$$

(iv) F is called an *n-fold fantastic filter*, if $1 \in F$ and

$$z \to (y \to x) \in F \text{ and } z \in F \text{ imply } (((x^n \to y) \to y) \to x) \in F,$$
$$\text{for all } x, y, z \in L.$$

(v) A filter F is called an *n-fold obstinate filter*, if whenever $x, y \notin F$, then

$$x^n \to y \in F \text{ and } y^n \to x \in F, \text{ for all } x, y \in L.$$

(vi) A filter F is called an *n-fold integral filter*, if

$$(x^n \odot y^n)^- \in F \text{ implies } (x^n)^- \in F \text{ or } (y^n)^- \in F, \quad \text{for all } x, y \in L.$$

Note. 1-fold integral filter is an integral filter.

Definition. Let L be a BL-algebra. Then

(i) L is called an *n-fold integral BL-algebra*, if

$$(x^n \odot y^n) = 0 \text{ implies that } x^n = 0 \text{ or } y^n = 0, \quad \text{for all } x, y \in L$$

(ii) L is called an n-fold obstinate BL-algebra, if L is an MV-algebra and $x^n = 0$, for all $x \in L \backslash \{1\}$.

Theorem 2. *Let F be a filter of BL-algebra L. Then the binary relation $\equiv_F$ on L which is defined by*

$$x \equiv_F y \text{ if and only if } x \to y \in F \text{ and } y \to x \in F$$

is a congruence relation on L. Define $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ on $\frac{L}{F}$, the set of all congruence classes of L, as follows:

$$[x] \cdot [y] = [x \odot y], \; [x] \rightharpoonup [y] = [x \to y], \; [x] \sqcup [y] = [x \vee y], \; [x] \sqcap [y] = [x \wedge y].$$

Then $(\frac{L}{F}, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a BL-algebra which is called quotient BL-algebra with respect to F.

Theorem 3. *Let $F \subseteq G$, where F be an n-fold integral filter and G be a filter of L. Then G is an n-fold integral filter.*

Theorem 4. *In any BL-algebra L, the following conditions are equivalent:*

(i) *$\{1\}$ is an n-fold integral filter,*

(ii) *any filter of L is an n-fold integral filter,*

(iii) *L is an n-fold integral BL-algebra.*

Theorem 5.

(i) *Let F be a filter of L. Then F is an n-fold obstinate filter of L if and only if F is an n-fold integral and n-fold fantastic filter.*

(ii) *Let F be a filter of L. Then F is an n-fold obstinate filter of L if and only if $\frac{L}{F}$ is an n-fold obstinate BL-algebra.*

(iii) *Let F be a filter of L. Then F is an n-fold obstinate filter if and only if*

$$x \notin F \ \ \textit{implies that} \ (x^n)^- \in F, \ \textit{for all}\ x \in L.$$

Definition. Let L_1 and L_2 be two BL-algebras. Then the map $f : L_1 \to L_2$ is called a BL-algebra *homomorphism* if and only if it satisfies the following conditions, for every $x, y \in L_1$:

(i) $f(0) = 0$,

(ii) $f(x \odot y) = f(x) \odot f(y)$,

(iii) $f(x \to y) = f(x) \to f(y)$.

If f is a bijective, then the homomorphism f is called BL-algebra *isomorphism*. In this case we write $L_1 \cong L_2$.

In the following, we give some fuzzy algebraic results on BL-algebras that come from references [9, 12, 13].

Definition. Let L be a BL-algebra and $\mu : L \to [0, 1]$ be a fuzzy set on L. Then

(i) μ is called a *fuzzy filter* on L, if and only if $\mu(x) \leq \mu(1)$ and $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$, for all $x, y \in L$.

(ii) μ is called a *fuzzy n-fold implicative filter* on L, if and only if $\mu(x) \leq \mu(1)$ and

$$\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z), \text{ for all } x, y, z \in L.$$

(iii) μ is called a *fuzzy n-fold positive implicative filter* on L, if and only if $\mu(x) \leq \mu(1)$ and

$$\mu(x \to ((y^n \to z) \to y)) \wedge \mu(x) \leq \mu(y), \text{ for all } x, y, z \in L.$$

(iv) A fuzzy filter μ is called a *fuzzy n-fold obstinate filter* on L, if and only if

$$min\{\mu(x^n \to y), \mu(y^n \to x)\} \geq min\{1 - \mu(x), 1 - \mu(y)\}, \text{ for all } x, y \in L.$$

Lemma 6. *Let L be a BL-algebra and μ be a fuzzy filter on L. Then the following properties hold:*

(i) *if $x \leq y$, then $\mu(x) \leq \mu(y)$, that is μ is order-preserving,*

(ii) *if $\mu(x \to y) = \mu(1)$, then $\mu(x) \leq \mu(y)$, for all $x, y \in L$.*

Definition. Let L_1 and L_2 be two BL-algebras, μ a fuzzy subset of L_1, η a fuzzy subset of L_2 and $f : L_1 \to L_2$ a BL-algebra homomorphism. The image of μ under f denoted by $f(\mu)$ is a fuzzy set of L_2 defined by:

$$f(\mu)(y) = \sup_{x \in f^{-1}(y)} \mu(x) \text{ if } f^{-1}(y) \neq \emptyset \text{ and } f(\mu)(y) = 0 \text{ if } f^{-1}(y) = \emptyset$$

for all $y \in L_2$.

The preimage of η under f denoted by $f^{-1}(\eta)$ is a fuzzy set of L_1 defined by: $f^{-1}(\eta)(x) = \eta(f(x))$, for all $x \in L_1$.

Also a fuzzy subset μ of X has a sup property, if for any nonempty subset Y of X, there exists $y_0 \in Y$, such that $\mu(y_0) = \sup_{y \in Y} \mu(y)$.

Theorem 7. *Let L be a BL-algebra, μ be a fuzzy set on L and $\mu_t = \{x \in L \mid \mu(x) \geq t\}$, $\forall\ t \in [0,1]$. Then*

(i) *μ is a fuzzy filter on L if and only if $\forall\ t \in [0,1]$, $\emptyset \neq \mu_t$ is a filter of L.*

(ii) *μ is a fuzzy n-fold fantastic filter on L if and only if $\forall\ t \in [0,1]$, $\emptyset \neq \mu_t$ is a n-fold fantastic filter of L.*

(iii) *μ is a fuzzy n-fold positive implicative filter on L if and only if μ is a fuzzy filter and $\mu((x^n \to 0) \to x) \leq \mu(x)$, for all $x \in L$.*

(iv) *μ is a fuzzy n-fold obstinate filter on L if and only if μ is a fuzzy filter and $\mu((x^n)^-) \geq 1 - \mu(x)$, for all $x \in L$.*

Note. In the rest of the paper we assume that L is a BL-algebra. Unless otherwise is stated.

3. Fuzzy n-fold integral filters in BL-algebras

Definition. Let μ be a fuzzy filter on L. Then μ is called a *fuzzy n-fold integral filter*, if for all $x, y \in L$, it satisfies:

$$\mu((x^n \odot y^n)^-) = \mu((x^n)^-) \vee \mu((y^n)^-)$$

Example 8. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\to$ are defined as the following tables:

Table 1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

Table 2

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra. Now, let the fuzzy set μ on L is defined by

$$\mu(1) = t_2,\ \mu(0) = \mu(a) = \mu(b) = t_1\ (0 \leq t_1 < t_2 \leq 1).$$

It is easy to check that μ is a fuzzy filter and it is a fuzzy 3-fold integral filter. But, it is not a fuzzy 2-fold integral filter. Because, $\mu((b^2 \odot b^2)^-) = \mu((a \odot a)^-) = \mu(0^-) = \mu(1) = t_2$ and $\mu((b^2)^-) = \mu(a^-) = \mu(b) = t_1$. Hence, $\mu((b^2 \odot b^2)^-) \neq \mu((b^2)^-) \vee \mu((b^2)^-)$.

Theorem 9. *A non empty subset F of L is an n-fold integral filter if and only if the characteristic function χ_F is a fuzzy n-fold integral filter on L.*

Proof. Let F be an n-fold integral filter. Then we show that $\chi_F((x^n \odot y^n)^-) = \chi_F((x^n)^-) \vee \chi_F((y^n)^-)$. If $(x^n \odot y^n)^- \in F$, then $\chi_F((x^n \odot y^n)^-) = 1$ and since F is an n-fold integral filter, then $(x^n)^- \in F$ or $(y^n)^- \in F$ and so $\chi_F((x^n)^-) = 1$ or $\chi_F((y^n)^-) = 1$. Hence, $\chi_F((x^n \odot y^n)^-) = \chi_F((x^n)^-) \vee \chi_F((y^n)^-) = 1$. Now, let $(x^n \odot y^n)^- \notin F$. Then $(x^n)^- \notin F$ and $(y^n)^- \notin F$. Indeed by $(BL7)$ and $(BL8)$, $(x^n)^-$and $(y^n)^- \leq (x^n \odot y^n)^-$ and if $(x^n)^- \in F$ or $(y^n)^- \in F$, then $(x^n \odot y^n)^- \in F$ and it is impossible. Hence, $(x^n)^- \notin F$ and $(y^n)^- \notin F$ and so $\chi_F((x^n)^-) = 0$ and $\chi_F((y^n)^-) = 0$. Therefore, $\chi_F((x^n \odot y^n)^-) = \chi_F((x^n)^-) \vee \chi_F((y^n)^-) = 0$.

Conversely, let χ_F is a fuzzy n-fold integral filter on L and $(x^n \odot y^n)^- \in F$. Then $\chi_F((x^n \odot y^n)^-) = \chi_F((x^n)^-) \vee \chi_F((y^n)^-) = 1$ and so $\chi_F((x^n)^-) = 1$ or $\chi_F((y^n)^-) = 1$. Hence, $(x^n)^- \in F$ or $(y^n)^- \in F$. Therefore, F is an n-fold integral filter. ∎

Theorem 10. *Let μ be a fuzzy filter on L. Then μ is a fuzzy n-fold integral filter if and only if for each $t \in [0, 1]$, $\emptyset \neq \mu_t$ is an n-fold integral filter.*

Proof. Let μ be a fuzzy n-fold integral filter and $(x^n \odot y^n)^- \in \mu_t$, for $t \in [0, 1]$ and $x, y \in L$. Then $\mu((x^n \odot y^n)^-) \geq t$. Since $\mu((x^n \odot y^n)^-) = \mu((x^n)^-) \vee \mu((y^n)^-)$, then $\mu((x^n)^-) \vee \mu((y^n)^-) \geq t$. Now, by contradiction if $(x^n)^- \notin \mu_t$ and $(y^n)^- \notin \mu_t$, then $\mu((x^n)^-) < t$ and $\mu((y^n)^-) < t$. Hence, $\mu((x^n)^-) \vee \mu((y^n)^-) < t$ and it is a contradiction. Therefore, $(x^n)^- \in \mu_t$ or $(y^n)^- \in \mu_t$ and so μ_t is an n-fold integral filter.

Conversely, Since μ is a fuzzy filter on L, then assume that for each $t \in [0, 1]$, $\emptyset \neq \mu_t$ is an n-fold integral filter. Now, we prove that μ is a fuzzy n-fold integral filter. Since by $(BL7)$, $x^n \odot y^n \leq x^n$, then by $(BL8)$, $(x^n)^- \leq (x^n \odot y^n)^-$ and so by Lemma 6, $\mu((x^n)^-) \leq \mu((x^n \odot y^n)^-)$. By similar way we have $\mu((y^n)^-) \leq \mu((x^n \odot y^n)^-)$ and so $\mu((x^n)^-) \vee \mu((y^n)^-) \leq \mu((x^n \odot y^n)^-)$. Now, we show, $\mu((x^n \odot y^n)^-) \leq \mu((x^n)^-) \vee \mu((y^n)^-)$. In the other wise, there exist $a, b \in L$ such that

$$\mu((a^n \odot b^n)^-) > \mu((a^n)^-) \vee \mu((b^n)^-).$$

Let

$$t_0 = \mu((a^n)^-) \vee \mu((b^n)^-) + 1/2\{\mu((a^n \odot b^n)^-) - \mu((a^n)^-) \vee \mu((b^n)^-)\}.$$

Then we have

$$\mu((a^n)^-) \vee \mu((b^n)^-) < t_0 < \mu((a^n \odot b^n)^-)$$

and so $(a^n \odot b^n)^- \in \mu_{t_0}$. Now, since μ_{t_0} is an n-fold integral filter, then $(a^n)^- \in \mu_{t_0}$ or $(b^n)^- \in \mu_{t_0}$. Hence, $\mu((a^n)^-) \geq t_0$ or $\mu((b^n)^-) \geq t_0$ and so $\mu((a^n)^-) \vee \mu((b^n)^-) \geq t_0$, it is a contradiction. Therefore, $\mu((x^n \odot y^n)^-) = \mu((x^n)^-) \vee \mu((y^n)^-)$ and so μ is a fuzzy n-fold integral filter. ■

In the following theorem, we make a link between fuzzy n-fold integral filters and fuzzy $(n+1)$-fold integral filters.

Theorem 11. *Let μ be a fuzzy n-fold integral filter. Then μ is a fuzzy $(n+1)$-fold integral filter.*

Proof. Let μ be a fuzzy n-fold integral filter. Then it is easy to check that $\mu((x^{n+1} \odot y^{n+1})^-) \geq \mu((x^{n+1})^-) \vee \mu((y^{n+1})^-)$, for all $x, y \in L$. Now, since by $(BL7)$, $x^{n+n} \odot y^{n+n} \leq x^{n+1} \odot y^{n+1}$, then by $(BL8)$, $(x^{n+1} \odot y^{n+1})^- \leq (x^{n+n} \odot y^{n+n})^-$ and by Lemma 6, $\mu((x^{n+1} \odot y^{n+1})^-) \leq \mu((x^{n+n} \odot y^{n+n})^-)$. Since μ is a fuzzy n-fold integral filter, then

$$\begin{aligned}
\mu((x^{n+n} \odot y^{n+n})^-) &= \mu((x^{2n} \odot y^{2n})^-) \\
&= \mu((x^2)^n \odot (y^2)^n)^-) \\
&= \mu(((x^2)^n)^-) \vee \mu(((y^2)^n)^-) \\
&= \mu((x^{n+n})^-) \vee \mu((y^{n+n})^-) \\
&= \mu((x^n \odot x^n)^-) \vee \mu((y^n \odot y^n)^-) \\
&= \mu((x^n)^-) \vee \mu((x^n)^-) \vee \mu((y^n)^-) \vee \mu((y^n)^-) \\
&= \mu((x^n)^-) \vee \mu(y^n)^-) \\
&\leq \mu((x^{n+1})^-) \vee \mu((y^{n+1})^-), \text{ by } (BL8) \text{ and Lemma 6.}
\end{aligned}$$

Hence, $\mu((x^{n+1} \odot y^{n+1})^-) = \mu((x^{n+1})^-) \vee \mu((y^{n+1})^-)$ and so μ is a fuzzy $(n+1)$-fold integral filter. ■

By mathematical induction, we can prove that every fuzzy n-fold integral filter is a fuzzy $(n+k)$-fold integral filter, for any integer $k \geq 0$.

Note. Example 8 shows that the converse of Theorem 11 is not correct in general.

Theorem 12. (*Extension property for fuzzy n-fold integral filters*) *Let μ and η be two fuzzy filters such that $\mu \subseteq \eta$ and $\mu(1) = \eta(1)$. If μ is a fuzzy n-fold integral filter, then η is a fuzzy n-fold integral filter too.*

Proof. Let μ be a fuzzy n-fold integral filter. Then by Theorem 10, $\emptyset \neq \mu_t$ is an n-fold integral filter, for each $t \in [0, 1]$ and since $\mu \subseteq \eta$, then $\mu(x) \leq \eta(x)$, for all $x \in L$. Now, if $x \in \mu_t$, then $\mu(x) \geq t$ and so $\eta(x) \geq t$. Hence, $x \in \eta_t$ and $\mu_t \subseteq \eta_t$. If $\emptyset \neq \eta_t$, since $\mu(1) = \eta(1)$ then $\emptyset \neq \mu_t$. Now, by Theorem 3, since μ_t is an n-fold integral filter, then η_t is an n-fold integral filter, for each $t \in [0, 1]$. Hence, by Theorem 10, η is a fuzzy n-fold integral filter. ■

Theorem 13. *Let μ be a fuzzy set on L defined by*

$$\mu(x) = \begin{cases} 0, & x \neq 1, \\ \alpha, & x = 1. \end{cases}$$

For fixed $\alpha \in (0, 1]$. Then the following are equivalent:

(i) *L is an n-fold integral BL-algebra,*

(ii) *Any fuzzy filter is a fuzzy n-fold integral filter,*

(iii) *μ is a fuzzy n-fold integral filter.*

Proof. (i)⇒(ii): Let L be an n-fold integral BL-algebra and η be a fuzzy filter on L. Then by Theorem 4, every filter of L is an n-fold integral filter. Now, since η is a fuzzy filter by Theorem 7(i), for each $t \in [0, 1]$, $\emptyset \neq \eta_t$ is a filter and so η_t is an n-fold integral filter of L. Theref ore, by Theorem 10, η is a fuzzy n-fold integral filter on L.

(ii)⇒(iii): First, we will prove that μ is a fuzzy filter. By definition of μ, for any $x \in L$, $\mu(x) \leq \mu(1)$. Now, let $x, y \in L$. We consider two following cases for y. If $y = 1$, then

$$\mu(x \to y) \wedge \mu(x) \leq \alpha = \mu(1) = \mu(y).$$

If $y \neq 1$, then we consider two following cases for x. If $x = 1$, then by $(BL10)$

$$\mu(x \to y) \wedge \mu(x) = \mu(1 \to y) \wedge \mu(1) = \mu(y) \wedge \mu(1) = \mu(y) \leq \mu(y).$$

If $x \neq 1$, then

$$\mu(x \to y) \wedge \mu(x) = \mu(x \to y) \wedge 0 = 0 \leq \mu(y).$$

Hence, μ is a fuzzy filter and so by (ii), it is a fuzzy n-fold integral filter.

(iii)⇒(i): Since μ is a fuzzy n-fold integral filter, then by Theorem 10, $\mu_\alpha = \{x \in L \mid \mu(x) \geq \alpha\} = \{1\}$ is an n-fold integral filter and so by Theorem 4, L is an n-fold integral BL-algebra. ■

Corollary 14. *Let μ be a fuzzy set on L defined by*

$$\mu(x) = \begin{cases} 0, & x \neq 1, \\ 1, & x = 1. \end{cases}$$

Then the following are equivalent:

(i) *L is an integral BL-algebra,*

(ii) *Any fuzzy filter is a fuzzy integral filter,*

(iii) *μ is a fuzzy integral filter.*

Proof. Let $n = 1$ in Theorem 13. Then the proof is clear. ■

Theorem 15. *Let $f : L_1 \to L_2$ be a BL-algebra homomorphism and μ be a fuzzy n-fold integral filter on L_2. Then $f^{-1}(\mu)$ is a fuzzy n-fold integral filter on L_1.*

Proof. First, we show that $f^{-1}(\mu)$ is a fuzzy filter on L_1. Since $f(x) \leq f(1)$, for all $x \in L_1$ and by Lemma 6,

$$f^{-1}(\mu)(x) = \mu(f(x)) \leq \mu(f(1)) = f^{-1}(\mu)(1).$$

Also, since μ is a fuzzy filter on L_2, then for all $x, y \in L_1$,

$$\begin{aligned} f^{-1}(\mu)(x \to y) \wedge f^{-1}(\mu)(x) &= \mu(f(x) \to f(y)) \wedge \mu(f(x)) \\ &\leq \mu(f(y)) \\ &= f^{-1}(\mu)(y). \end{aligned}$$

Then $f^{-1}(\mu)$ is a fuzzy filter on L_1. Now, let μ be a fuzzy n-fold integral filter on L_2 and $x, y \in L_1$. Then

$$\begin{aligned} f^{-1}(\mu)((x^n \odot y^n)^-) &= \mu(f((x^n \odot y^n)^-)) \\ &= \mu((f(x)^n \odot f(y)^n)^-) \\ &= \mu((f(x)^n)^-) \vee \mu((f(y)^n)^-) \\ &= \mu(f((x^n)^-)) \vee \mu(f((y^n)^-)) \\ &= f^{-1}(\mu)((x^n)^-) \vee f^{-1}(\mu)((y^n)^-). \end{aligned}$$

Therefore, $f^{-1}(\mu)$ is a fuzzy n-fold integral filter on L_1. ■

Lemma 16. *Let $f : L_1 \to L_2$ be a BL-algebra isomorphism and μ be a fuzzy filter on L_1. Then $f(\mu)$ is a fuzzy filter on L_2.*

Proof. Since μ is a fuzzy filter on L_1, then $\mu(x) \leq \mu(1)$, for all $x \in L_1$. Now, for all $y \in L_2$,

$$f(\mu)(y) = \sup\{\mu(x) \mid x \in f^{-1}(y)\} \leq \sup\{\mu(1) \mid 1 \in f^{-1}(1)\} = f(\mu)(1).$$

Thus, $f(\mu)(y) \leq f(\mu)(1)$, for all $y \in L_2$. Now, suppose that $y_1, y_2 \in L_2$. Since f is a BL-algebra isomorphism, then there exist $x_1, x_2 \in L_1$, such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Now,

$$f(\mu)(y_1 \to y_2) = \sup\{\mu(z) \mid z \in f^{-1}(y_1 \to y_2)\}.$$

Also, since f is a BL-algebra isomorphism and $z \in f^{-1}(y_1 \to y_2)$, then

$$f(z) = y_1 \to y_2 = f(x_1) \to f(x_2) = f(x_1 \to x_2).$$

And so $z = x_1 \to x_2$. Therefore,

$$\begin{aligned} f(\mu)(y_1 \to y_2) &= \sup\{\mu(x_1 \to x_2) \mid x_1 \to x_2 \in f^{-1}(y_1 \to y_2)\} \\ &= \mu(x_1 \to x_2). \end{aligned}$$

By similar way, we have

$$f(\mu)(y_1) = \sup\{\mu(x_1) \mid x_1 \in f^{-1}(y_1)\} = \mu(x_1)$$
$$f(\mu)(y_2) = \sup\{\mu(x_2) \mid x_1 \in f^{-1}(y_2)\} = \mu(x_2).$$

Moreover, since μ is a fuzzy filter on L_1, then

$$\begin{aligned} f(\mu)(y_1 \to y_2) \wedge f(\mu)(y_1) &= \mu(x_1 \to x_2) \wedge \mu(x_1) \\ &\leq \mu(x_2) \\ &= f(\mu)(y_2). \end{aligned}$$

Therefore, $f(\mu)$ is a fuzzy filter on L_2. ■

Theorem 17. *Let $f : L_1 \to L_2$ be a BL-algebra isomorphism and μ be a fuzzy n-fold integral filter on L_1 with sup property. Then $f(\mu)$ is a fuzzy n-fold integral filter on L_2.*

Proof. By Lemma 16, $f(\mu)$ is a fuzzy filter on L_2. Now, we show that $f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) \leq f(\mu)((y_1^n \odot y_2^n)^-)$. Also,

$$f(\mu)((y_1^n \odot y_2^n)^-) = \sup_{t \in f^{-1}((y_1^n \odot y_2^n)^-)} \mu(t) \text{ and } f(\mu)((y_1^n)^-) = \sup_{t \in f^{-1}((y_1^n)^-)} \mu(t).$$

Since f is a BL-algebra isomorphism and μ has sup property, then there exist $x_1 \in f^{-1}((y_1^n)^-)$ and $x_3 \in f^{-1}((y_1^n \odot y_2^n)^-)$ such that $\sup\limits_{t\in f^{-1}((y_1^n)^-)} \mu(t) = \mu(x_1)$ and $\sup\limits_{t\in f^{-1}((y_1^n\odot y_2^n)^-)} \mu(t) = \mu(x_3)$. Since $x_1 \in f^{-1}((y_1^n)^-)$, then $f(x_1) = (y_1^n)^- \leq (y_1^n\odot y_2^n)^- = f(x_3)$. Now, since f^{-1} is a BL-algebra homomorphism, then $x_1 \leq x_3$ and so by Lemma 6, $\mu(x_1) \leq \mu(x_3)$. Hence, $f(\mu)((y_1^n)^-) \leq f(\mu)((y_1^n \odot y_2^n)^-)$. By similar way, we have $f(\mu)((y_2^n)^-) \leq f(\mu)((y_1^n \odot y_2^n)^-)$ and so $f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) \leq f(\mu)((y_1^n \odot y_2^n)^-)$.

Now, we show that $f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) \geq f(\mu)((y_1^n \odot y_2^n-)$. Since f is a BL-algebra isomorphism, then there exist $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. But, since $(x_1^n)^- \in f^{-1}((y_1^n)^-)$ and $(x_2^n)^- \in f^{-1}((y_1^n)^-)$, then

$$\begin{aligned} f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) &= \sup_{t\in f^{-1}((y_1^n)^-)} \mu(t) \vee \sup_{t\in f^{-1}((y_2^n)^-)} \mu(t) \\ &\geq \mu((x_1^n)^-) \vee \mu((x_2^n)^-) \\ &= \mu((x_1^n \odot x_2^n)^-). \end{aligned}$$

By sup property for μ, there exist

$$x_3 \in f^{-1}((y_1^n \odot y_2^n)^-) \text{ such that } \sup_{t\in f^{-1}((y_1^n\odot y_2^n)^-)} \mu(t) = \mu(x_3)$$

and so $f(\mu)((y_1^n \odot y_2^n)^-) = \mu(x_3)$. Now, since f is a BL-algebra monomorphism and $f(x_3) = (y_1^n \odot y_2^n)^- = f((x_1^n \odot x_2^n)^-)$, then $x_3 = (x_1^n \odot x_2^n)^-$ and so $\mu(x_3) = \mu((x_1^n \odot x_2^n)^-)$. Hence, $\mu((x_1^n \odot x_2^n)^-) = f(\mu)((y_1^n \odot y_2^n)^-)$ and so $f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) \geq f(\mu)((y_1^n \odot y_2^n)^-)$. Therefore, $f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) = f(\mu)((y_1^n \odot y_2^n)^-)$ and so $f(\mu)$ is a fuzzy n-fold integral filter on L_2. ■

4. Fuzzy n-fold obstinate filters and fuzzy n-fold integral filters

In this section, we study relationship among fuzzy n-fold obstinate filters, fuzzy n-fold integral filters and fuzzy n-fold fantastic filters.

Theorem 18. *Let μ be a fuzzy filter on L. Then μ is a fuzzy n-fold integral filter and fuzzy n-fold fantastic filter if and only if $\emptyset \neq \mu_t$ is an n-fold obstinate filter, for any $t \in [0, 1]$.*

Proof. Let μ be a fuzzy n-fold integral filter and fuzzy n-fold fantastic filter. Then by Theorems 7(ii) and 10, μ_t is a n-fold integral filter and n-fold fantastic filter of L, for any $t \in [0,1]$. Hence, by Theorem 5(i), μ_t is an n-fold obstinate filter, for any $t \in [0,1]$. Conversely, let $\emptyset \neq \mu_t$ is an n-fold obstinate filter, for any $t \in [0,1]$. Then by Theorem 5(i), μ_t is a n-fold integral filter and n-fold fantastic filter of L, for any $t \in [0,1]$. Hence, by Theorem 7(ii) and 10, μ is a fuzzy n-fold integral filter and fuzzy n-fold fantastic filter on L. ■

Theorem 19. *Let μ be a fuzzy n-fold obstinate filter on L. Then*

(i) *for any $t \in [0,0.5]$, $\emptyset \neq \mu_t$ is an n-fold obstinate filter of L.*

(ii) *for any $t \in (0.5,1]$, if $\mu_t \neq \emptyset$, then μ_t either n-fold obstinate filter or $\mu_{1-t} = L$.*

Proof. (i) It is hold by Theorem 3.4 of [10].

(ii) Assume that $t \in (0.5,1]$ and $\mu_t \neq \emptyset$. If μ_t is an n-fold obstinate filter, then the proof is complete. Otherwise, suppose that μ_t is not an n-fold obstinate filter. Then by Theorem 5(iii), there exist $a \in L$ such that $a \notin \mu_t$ and $(a^n)^- \notin \mu_t$. Hence, $\mu(a) < t$ and $\mu((a^n)^-) < t$. Now, since μ is a fuzzy n-fold obstinate filter, then $\mu((a^n)^-) \geq 1-\mu(a)$ and so $\mu((a^n)^-) > 1-t$. Hence, $(a^n)^- \in \mu_{1-t}$ and since $t > \mu((a^n)^-) \geq 1-\mu(a)$, then $t > 1-\mu(a)$ and so $\mu(a) > 1-t$. Thus, $a \in \mu_{1-t}$ and so $a^n \in \mu_{1-t}$. Therefore, by $(BL9)$, $0 = (a^n)^- \odot (a^n) \in \mu_{1-t}$ and so $\mu_{1-t} = L$. ■

Theorem 20. *Let L be an n-fold obstinate BL-algebra. Then every fuzzy filter is a fuzzy n-fold (positive) implicative filter and fuzzy n-fold integral filter.*

Proof. Let μ be a fuzzy filter on an n-fold obstinate BL-algebra L. Then for all $1 \neq x \in L$, $x^n = 0$ and so

$$\mu((x^n \to 0) \to x) = \mu((0 \to 0) \to x) = \mu(1 \to x) = \mu(x) \leq \mu(x)$$

and for $x = 1$,

$$\mu((1^n \to 0) \to 1) = \mu((1 \to 0) \to 1) = \mu(0 \to 1) = \mu(1) \leq \mu(1).$$

Therefore, by Theorem 7(iii), μ is a fuzzy n-fold positive implicative filter. Also, since

$$\mu((x^n \odot y^n)^-) = \mu((x^n)^-) = \mu((y^n)^-) = \mu(0^-) = \mu(1).$$

Then $\mu((x^n \odot y^n)^-) = \mu((x^n)^-) \vee \mu((y^n)^-)$. Therefore, μ is a fuzzy n-fold integral filter. ■

Theorem 21. *Let μ be a fuzzy n-fold obstinate filter on an n-fold obstinate BL-algebra L. Then for all $x \in L$*

(i) $\mu(x) \geq 1 - \mu(1)$,

(ii) $\mu(0) + \mu(1) \geq 1$.

Proof. (i) Let μ be a fuzzy n-fold obstinate filter on L. Then by Theorem 7(iv), $\mu((x^n)^-) \geq 1 - \mu(x)$ and since L is an n-fold obstinate BL-algebra, then $x^n = 0$, for all $x \in L$ and so $\mu(0^-) \geq 1 - \mu(x)$. Therefore, $\mu(x) \geq 1 - \mu(1)$.

(ii) Since μ be a fuzzy n-fold obstinate filter, then by Theorem 7(iv), $\mu((x^n)^-) \geq 1 - \mu(x)$. Now, let $x = 1$. Then $\mu((1^n)^-) \geq 1 - \mu(1)$ and so $\mu(0) \geq 1 - \mu(1)$. Therefore, $\mu(0) + \mu(1) \geq 1$. ∎

Note. The following example show that there is a fuzzy n-fold integral filter and fuzzy n-fold fantastic filter such that it is not a fuzzy n-fold obstinate filter.

Example 22. Let L be BL-algebra in Example 8. Now, let the fuzzy set μ on L is defined by

$$\mu(1) = t_2,\ \mu(b) = \mu(a) = \mu(0) = t_1\ (0 \leq t_1 \leq t_2 < 0.5 \leq 1).$$

It is easy to check that μ is a fuzzy 3-fold fantastic filter and fuzzy 3-fold integral filter and since L is an 3-fold obstinate BL-algebra and $\mu(1) < 0.5$, then by Theorem 21, μ is not a fuzzy 3-fold Obstinate filter.

References

[1] R.A. Borzooei and A. Paad, *Integral filters and Integral BL-algebras*, Italian J. Pure and Appl. Math., to appear.

[2] R.A. Borzooei and A. Paad, *n-fold integral and n-fold obstinate BL-algebras*, submitted.

[3] C.C. Chang, *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc. **88** (1958) 467–490. doi:10.1090/S0002-9947-1958-0094302-9

[4] A. Di Nola, G. Georgescu and A. Iorgulescu, *Pseudo BL-algebra*. Part I, Int. J. Mult. Val. Logic **8** (5–6) (2002) 673–714.

[5] A. Di Nola and L. Leustean, *Compact representations of BL-algebras*, Department of Computer Science, University Aarhus, BRICS Report series (2002).

[6] M. Haveshki, A. Borumand Saeid and E. Eslami, *Some types of filters in BL-algebras*, Soft. Comput. **10** (2006) 657–664. doi:10.1007/s00500-005-0534-4

[7] M. Haveshki and E. Eslami, *n-Fold filters in BL-algebras*, Math. Log. Quart **54** (2) (2008) 176–186. doi:10.1002/malq.200710029

[8] S. Motamed and A. Borumand Saeid, *n-fold obstinate filters in BL-algebras*, Neural Comput. and Applic. **20** (2011) 461–472. doi:10.1007/s00521-011-0548-z

[9] C. Lele, *Folding theory of positive implicative/fuzzy positive implicative in BL-algebras*, Journal of Fuzzy Mathematics **17** (3) (2009), Los Angeles.

[10] C. Lele, *Fuzzy n-fold obstinate filters in BL-algebras*, Afrika Mathematika (2011) (On line).

[11] C. Lele and M. Hyland, *Folding theory for fantastic filters in BL-algebra*, International Journal of Artificial Life Research **2** (4) (2011) 32–42. doi:10.4018/IJALR.2011100104

[12] L. Liu and K. Li, *Fuzzy filters of BL-algebras*, Information Sciences **173** (2005) 141–154. doi:10.1016/j.ins.2004.07.009

[13] L. Lianzhen and L. Kaitai, *Fuzzy Boolean and positive implicative filters of BL-algebras*, Fuzzy Sets and Systems **152** (2005) 333–348. doi:10.1016/j.fss.2004.10.005

[14] P. Hájek, *Metamathematics of fuzzy logic*, Trends in Logic, vol. 4, Kluwer Academic Publishers, (1998), ISBN:9781402003707. doi:10.1007/978-94-011-5300-3

[15] E. Turunen, *BL-algebras of basic fuzzy logic*, Mathware Soft. Comput. **6** (1999) 49–61.

[16] E. Turunen, Mathematics Behind Fuzzy Logic (Physica Verlag, 1999).

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