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DISCRETE APPROXIMATIONS OF GENERALIZED RBSDE WITH RANDOM TERMINAL TIME

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Abstract

The convergence of discrete approximations of generalized reflected backward stochastic differential equations with random terminal time in a general convex domain is studied. Applications to investigation obstacle elliptic problem with Neumann boundary condition for partial differential equations are given.

Keywords: generalized reflected BSDE, discrete approximation methods, viscosity solution.

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1. Introduction

Pardoux and Peng in [13] have introduced nonlinear backward stochastic differential equations (BSDEs for short). Since then many papers have been devoted to the study of BSDEs, mainly due to their applications. The main aim of studying BSDEs was to give a probabilistic interpretation for solutions of partial differential equations (PDEs for short).

In [5] the authors put some constrains on the solution of BSDE. They assumed that the first component of the solution takes its values in a given convex set $D \subset \mathbb{R}^d$ and the problem was called the reflected BSDE (RBSDE for short). Later, Pardoux and Răşcanu in [14] considered RBSDE with random terminal time and pointed out the connection between RBSDEs and variational inequalities.

The notion of generalized BSDEs was introduced in [15]. By generalized it is meant that to the stochastic equation an additional component – an integral with respect to one dimensional increasing process – is added. Recently papers about the generalized RBSDEs on a finite time interval have appeared. The paper [16] treats one dimensional case and [7] treats d-dimensional case. The existence and uniqueness of the generalized RBSDE (GRBSDE for short) with random terminal time and its connection with PDEs with an obstacle problem and Neumann boundary condition was shown in [8].

In the literature we can find many papers related to discrete approximations of backward stochastic equations. We should list here [1, 3, 4, 10] that treat the case of BSDEs with deterministic terminal time and [18] that treats the case of BSDEs with random terminal time. Discrete approximations of RBSDEs firstly were proposed in one-dimensional case (e.g. [1, 11]) and later also in d-dimensional case (see [6]). In [7] an approximation scheme for the solution of generalized RBSDE with deterministic terminal time was given.

In the present paper we propose the discrete approximation of the solution of the following GRBSDE with random terminal time τ:

$$
\begin{aligned}
(1) \qquad Y_{t\wedge\tau} &= \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)ds + \int_{t\wedge\tau}^{\tau} \varphi(s, Y_s)d\Lambda_s \\
&\quad - \int_{t\wedge\tau}^{\tau} Z_s dW_s + K_\tau - K_{t\wedge\tau}, \quad t \in \mathbb{R}^+,
\end{aligned}
$$

where $W = (W_t)_{t\in\mathbb{R}^+}$ is m-dimensional Wiener process and $\Lambda = (\Lambda_t)_{t\in\mathbb{R}^+}$ is one dimensional continuous and increasing process, $\Lambda_0 = 0$. Moreover, we show the convergence of the proposed scheme to the solution of (1).

The paper is organized as follows. In Section 2 we give a definition of a solution of GRBSDE with random terminal time. We formulate here a theorem about existence and uniqueness of the solution of (1). In the next section we construct a discrete approximation scheme for solving (1) and give its properties. Moreover, this section is devoted to proving the main theorem - theorem about convergence of the proposed scheme. Finally, in the last section we show the application of the constructed scheme to numerical solving of the obstacle problem for PDE with Neumann boundary condition.

Throughout the paper we will use the following notations. By $|x|$ we mean an Euclidean norm in $\mathbb{R}^d$, $x \in \mathbb{R}^d$, $\|x\|$ stands for $(\mathrm{trace}(x^*x))^{1/2}$, where x^* is a transposition of a matrix $x \in \mathbb{R}^{d\times m}$. For a process $K = (K^1, ..., K^d)$ by $|K|_t = \sum_{i=1}^d |K^i|_t$ we denote its variation on $[0, t]$, where $|K^i|_t$ is a total variation of K^i on $[0, t]$.

2. DEFINITION

Let $(\Omega, \mathcal{G}, \mathcal{P})$ be a complete probability space carrying a standard m-dimensional Wiener process $W = (W_t)_{t\in\mathbb{R}^+}$. Let $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}^+}$ be the usual augmentation of the filtration generated by W and assume that $\Lambda = (\Lambda_t)_{t\in\mathbb{R}^+}$ is an adapted, one dimensional continuous and increasing process, $\Lambda_0 = 0$.

Let τ be an almost surely finite $\mathcal{F}$ stopping time and let ξ be an $\mathcal{F}_\tau$ measurable, square integrable random variable with values in $\bar{D}$, where D is a convex subset of $\mathbb{R}^d$. Suppose that functions $f : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \to \mathbb{R}^d$ and $\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable.

Definition. A solution of the generalized reflected backward stochastic differential equation (GRBSDE) with random terminal time associated with data $(\tau, \xi, f, \varphi, \Lambda)$ is a triple $(Y, Z, K) = (Y_t, Z_t, K_t)_{t\in\mathbb{R}^+}$ of $\mathcal{F}$ progressively measurable processes in $\bar{D} \times \mathbb{R}^{d\times m} \times \mathbb{R}^d$ satisfying

$$\begin{aligned} Y_t &= \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)ds + \int_{t\wedge\tau}^{\tau} \varphi(s, Y_s)d\Lambda_s \\ &\quad - \int_{t\wedge\tau}^{\tau} Z_s dW_s + K_\tau - K_{t\wedge\tau}, \quad t \in \mathbb{R}^+ \end{aligned} \tag{2}$$

and such that for some $\lambda \in \mathbb{R}$

$$E\biggl(\sup_{t\le\tau} e^{\lambda t}|Y_t|^2 + \int_0^\tau e^{\lambda t}\|Z_t\|^2 dt + \int_0^\tau e^{\lambda t}|Y_t|^2 d\Lambda_t\biggr) < \infty,$$

where K is a continuous process with locally finite variation, $K_0 = 0$ and

$$\int_0^\tau (Y_t - S_t)dK_t \le 0, \tag{3}$$

for every $\mathcal{F}$ progressively measurable process $S = (S_t)_{t\in\mathbb{R}^+}$ with values in $\bar{D}$.

Moreover, on the set $\{t \ge \tau\}$ we have $Y_t = \xi$, $Z_t = 0$, $K_t = K_\tau$.

Assume that there exist constants $L, \kappa > 0$, $\beta < 0$, $\mu \in \mathbb{R}$ such that for any $t \in \mathbb{R}^+$, $y, y' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d\times m}$

(A1)
- $|f(t, y, z) - f(t, y', z')| \le L(|y - y'| + \|z - z'\|)$,
- $\langle y - y', f(t, y, z) - f(t, y', z)\rangle \le \mu|y - y'|^2$,

(A2)
- $|\varphi(t, y) - \varphi(t, y')| \le L|y - y'|$,
- $\langle y - y', \varphi(t, y) - \varphi(t, y')\rangle \le \beta|y - y'|^2$,
- $|\varphi(\cdot, \cdot, \cdot)| \le \kappa$,

(A3) for some real numbers λ and ν such that $\lambda > 2\mu + L^2$, $\nu > \beta$

$$E\left(\int_0^\tau e^{\lambda t+\nu\Lambda_t}|f(t,0,0)|^2 dt + \int_0^\tau e^{\lambda t+\nu\Lambda_t}|\varphi(t,0)|^2 d\Lambda_t\right) < \infty,$$

(A4) ξ is $\mathcal{F}_\tau$ measurable random variable with values in $\bar{D}$, $Ee^{\lambda\tau+\nu\Lambda_\tau}(|\xi|^2+1) < \infty$ and

$$E\left(\int_0^\tau e^{\lambda t+\nu\Lambda_t}|f(t,\xi_t,\zeta_t)|^2 dt + \int_0^\tau e^{\lambda t+\nu\Lambda_t}|\varphi(t,\xi_t)|^2 d\Lambda_t\right) < \infty,$$

where $\xi_t = E(\xi|\mathcal{F}_t)$, ζ is $\mathcal{F}$ progressively measurable $d\times m$-dimensional process such that $E\int_0^\infty \|\zeta_t\|^2 dt < \infty$ and $\xi = E\xi + \int_0^\infty \zeta_t dW_t$,

(A5) there exists $q \geq 2$ such that for every $M > 0$

$$E\int_0^M |f(t,0,0)|^{2q} dt < \infty.$$

Theorem 1 [8, Theorem 2.2 and Proposition 4.1]**.** *Let τ be an almost surely finite $\mathcal{F}$ stopping time and let assumptions* (A1)–(A5) *hold. Then there exists a unique solution of* (2).

Moreover, for any $a \in D$ there exists $C > 0$ such that for $\lambda > 2\mu + L^2$

$$E\left(\sup_{t\leq\tau} e^{\lambda t}|Y_t - a|^2 + \int_0^\tau e^{\lambda t}|Y_t - a|^2 d\Lambda_t + \int_0^\tau e^{\lambda t}\|Z_t\|^2 dt + \int_0^\tau e^{\lambda t} d|K|_t\right)$$
$$\leq CE\left(e^{\lambda\tau}|\xi - a|^2 + \int_0^\tau e^{\lambda t}|f(t,a,0)|^2 dt + \int_0^\tau e^{\lambda t}|\varphi(t,a)|^2 d\Lambda_t\right).$$

3. Convergence of the approximation scheme

We will give an approximation scheme for (2) based on an approximation of a Wiener process by a random walk. In a construction of this scheme and in the proof of its convergence we will use results from [7], where a scheme for (2) when τ is a deterministic terminal time was given.

Set $W_t^n = (\sqrt{n})^{-1}\sum_{j=1}^{[nt]}\varepsilon_j^n$, $t \in \mathbb{R}^+$, where for each $n \in \mathbb{N}$, $\{\varepsilon_j^n\}_{j\in\mathbb{N}}$ is a sequence of independent symmetric Bernoulli random variables, and by $\mathcal{F}^n = \{\mathcal{F}_t^n\}_{t\in\mathbb{R}^+}$ denote the natural filtration of W^n.

By $\Lambda^{n,L}$ we will denote a process with bounded jumps such that $\Lambda_t^{n,L} = \sum_{j=1}^{[nt]}\Delta\Lambda_{(j-1)/n}^{n,L}$, where $\Delta\Lambda_{j/n}^{n,L} = \min((2L)^{-1}, \Delta\Lambda_{j/n}^n)$ and $\Delta\Lambda_{j/n}^n = \Lambda_{j/n}^n -$

$\Lambda^n_{(j-1)/n}$. Note that since Λ has continuous trajectories, $\max_{j\le[nT_n]}|\Delta\Lambda^n_{j/n}|\to 0$ and $\sup_{t\le T_n}|\Lambda^{n,L}_t-\Lambda_t|\le\sum_{j=1}^{[nT]}|1/(2L)-\Delta\Lambda^n_{j/n}|\mathbf{1}_{\{\Delta\Lambda^n_{j/n}>1/2L\}}\to 0,\ a.s.$

Now we need to approximate the stopping time τ by a sequence of bounded stopping times $\{\tau^n\}_n$. Since for every $n\in\mathbb{N}$, τ^n is bounded, we can find $T_n\in\mathbb{N}$ such that $\tau^n\le T_n$. Denote $\tau^n_j=j/n\wedge[n\tau^n]/n$, $j\in\mathbb{N}$.

In order to define the discrete GRBSDE with random terminal time we take $j=nT_n,\dots,0$ and on the set $\{\tau^n\le j/n\}$ put $y^n_{\tau^n_j}=y^n_{\tau^n}=\xi^n=E(\xi|\mathcal{F}^n_{\tau^n})$ and $z^n_{\tau^n_j}=z^n_{\tau^n}=0$, $\Delta k^n_{\tau^n_{j+1}}=0$. Next, on the set $\{[n\tau^n]>j\}$ we solve

$$
\begin{aligned}
(4)\qquad y^n_{j/n}&=y^n_{\tau^n_{j+1}}+\frac{1}{n}f(j/n,y^n_{j/n},z^n_{j/n})\mathbf{1}_{\{[n\tau^n]>j\}}\\
&\quad+\varphi(j/n,y^n_{j/n})\mathbf{1}_{\{[n\tau^n]>j\}}\Delta\Lambda^{n,L}_{j/n}-\frac{1}{\sqrt{n}}z^n_{j/n}\varepsilon^n_{j+1}+\Delta k^n_{\tau^n_{j+1}}.
\end{aligned}
$$

By a solution of (4) we mean a triple $(Y^n,Z^n,K^n)=(Y^n_t,Z^n_t,K^n_t)_{t\in[0,T_n]}$ of $\mathcal{F}^n$ adapted processes in $\bar{D}\times\mathbb{R}^{d\times m}\times\mathbb{R}^d$ such that $K^n_0=0$ and $\int_0^{\tau^n}(Y^n_{t-}-S^n_{t-})dK^n_t\le 0$ for every $\mathcal{F}^n$ adapted process S^n with values in $\bar{D}$, where $Y^n_t=y^n_{\tau^n_{[nt]}}$, $Z^n_t=z^n_{\tau^n_{[nt]}}$, $K^n_t=\sum_{j=1}^{[nt]}\Delta k^n_{\tau^n_j}$. Moreover, on the set $\{t\ge\tau^n\}$, $Y^n_t=Y^n_{\tau^n}$, $Z^n_t=0$ and $K^n_t=K^n_{\tau^n}$.

Theorem 2. *Assume* (A1)–(A5). *Let $\{\tau^n\}$ be a sequence of $\mathcal{F}^n$ stopping times such that $\sup_n Ee^{\lambda\tau^n}(|\xi^n|^2+1)<\infty$ and $\tau^n\xrightarrow{\mathcal{P}}\tau$. Then for every $T\in\mathbb{R}^+$*

$$
\begin{aligned}
\lim_{n\to\infty}E\Big(\sup_{t\le T}|Y^n_{t\wedge\tau^n}-Y_{t\wedge\tau}|^2&+\int_0^T\|Z^{n,\tau^n}_{t-}-Z^\tau_t\|^2dt+\int_0^T|Y^{n,\tau^n}_{t-}-Y^\tau_t|d\Lambda_t\\
&+\sup_{t\le T}|K^n_{t\wedge\tau^n}-K_{t\wedge\tau}|^2\Big)=0.
\end{aligned}
$$

Before proving the above theorem we will give some properties of the discrete scheme. First note, that the triple (Y^n,Z^n,K^n) is a unique solution of (4), that is equivalent to

$$
\begin{aligned}
(5)\qquad Y^n_{t\wedge\tau^n}&=\xi^n+\int_{t\wedge\tau^n}^{\tau^n}f(\varrho^n_{s-},Y^n_{s-},Z^n_{s-})d\varrho^n_s+\int_{t\wedge\tau^n}^{\tau^n}\varphi(\varrho^n_{s-},Y^n_{s-})d\Lambda^{n,L}_s\\
&\quad-\int_{t\wedge\tau^n}^{\tau^n}Z^n_{s-}dW^n_s+K^n_{\tau^n}-K^n_{t\wedge\tau^n},\quad t\in\mathbb{R}^+,
\end{aligned}
$$

where $\varrho^n_t=[nt]/n$. Now, note that solving (4) relies on finding on the set $\{[n\tau^n]>j\}$

$$\begin{aligned}
z^n_{j/n} &= z^n_{j/n}\mathbf{1}_{\{[n\tau^n]>j\}} = \sqrt{n}E(y^n_{\tau^n_{j+1}}\varepsilon^n_{j+1}|\mathcal{F}^n_{j/n})\mathbf{1}_{\{[n\tau^n]>j\}},\\
h^n_{j/n} &= E(y^n_{\tau^n_{j+1}}|\mathcal{F}^n_{j/n}) + \frac{1}{n}f(j/n,\pi(h^n_{j/n}),z^n_{j/n})\mathbf{1}_{\{[n\tau^n]>j\}}\\
&\quad + \varphi(j/n,\pi(h^n_{j/n}))\mathbf{1}_{\{[n\tau^n]>j\}}\Delta\Lambda^{n,L}_{j/n},\\
y^n_{j/n} &= \pi(h^n_{j/n}),\\
\Delta k^n_{\tau^n_{j+1}} &= y^n_{j/n} - h^n_{j/n}.
\end{aligned}$$

Since f and φ are Lipschitz, $h^n_{j/n}$ is well defined for $n > 2L$. Moreover processes defined above satisfy

$$\int_0^T (Y^n_{t-} - S^n_{t-})dK^n_t \le 0. \tag{6}$$

Indeed, since D is a convex set $\langle \pi(h)-x', \pi(h)-h\rangle \le 0, \quad h\in\mathbb{R}^d,\ x'\in\bar{D}$ (see ex. [12]). In particular, $\langle y^n_{(j-1)/n} - x', \Delta k^n_{j/n}\rangle \le 0$ for any $x'\in\bar{D}$ and $j=1,\dots,[nT_n]$ and as a consequence, for any $\mathcal{F}^n$ adapted process S^n with values in $\bar{D}$

$$\int_0^T (Y^n_{t-} - S^n_{t-})dK^n_t = \sum_{j=1}^{[nT]}\langle y^n_{(j-1)/n} - S^n_{(j-1)/n}, \Delta k^n_{j/n}\rangle \le 0.$$

It can be shown that $E\sup_{t\le\tau^n} e^{\lambda t}|Y^n_t|^2 < \infty$ and for $t\in\mathbb{R}^+$, $E(\|Z^n_t\|^2 + |K^n|^2_t)$ $< \infty$.

Proposition 3. *Assume that* (A1)–(A4) *are satisfied and* τ^n *is a sequence of* $\mathcal{F}^n$ *stopping times such that* $\sup_n Ee^{\lambda\tau^n}(1+|\xi^n|^2) < \infty$.

(a) *There exists a constant* $C>0$ *such that for any* $n\in\mathbb{N}$ *and* $a\in D$

$$\begin{aligned}
&E\biggl(\sup_{t\le\tau^n} e^{\lambda t}|Y^n_t - a|^2 + \int_0^{\tau^n} e^{\lambda\varrho^n_t}\|Z^n_{t-}\|^2 d\varrho^n_t + \int_0^{\tau^n} e^{\lambda\varrho^n_t}|Y^n_{t-} - a|^2 d\Lambda^{n,L}_t\biggr)\\
&\le CE\biggl(e^{\lambda\tau^n}|\xi^n - a|^2 + \int_0^{\tau^n} e^{\lambda\varrho^n_t}|f(\varrho^n_{t-},a,0)|^2 d\varrho^n_t + \int_0^{\tau^n} e^{\lambda\varrho^n_t}|\varphi(\varrho^n_{t-},a)|^2 d\Lambda^{n,L}_t\biggr)
\end{aligned}$$

and

$$E\left(\sup_{t\le\tau^n} e^{\lambda t}|K_t^n|^2 + e^{\lambda\tau^n}|K^n|^2_{\tau^n}\right) \le CE\Bigg(1+e^{\lambda\tau^n}|\xi^n-a|^2$$

$$\int_0^{\tau^n} e^{\lambda\varrho_t^n}|f(\varrho_{t-}^n,a,0)|^2 d\varrho_t^n + \int_0^{\tau^n} e^{\lambda\varrho_t^n}|\varphi(\varrho_{t-}^n,a)|^2 d\Lambda_t^{n,L}\Bigg).$$

(b) *The solution of* (5) *is unique.*

Proof. (a) By the Itô formula

$$\begin{aligned} e^{\lambda(t\wedge\tau^n)} \quad & |Y^n_{t\wedge\tau^n}-a|^2 + \lambda\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda s}|Y^n_{s-}-a|^2 d\varrho^n_s + \int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s} d[Y^n]_s \\ = \quad & e^{\lambda\tau^n}|\xi^n-a|^2 + 2\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}(Y^n_{s-}-a)f(\varrho^n_{s-},Y^n_{s-},Z^n_{s-})d\varrho^n_s \\ & +2\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}(Y^n_{s-}-a)\varphi(\varrho^n_{s-},Y^n_{s-})d\Lambda^{n,L}_s + 2\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}(Y^n_{s-}-a)dK^n_s \\ & -2\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}(Y^n_{s-}-a)Z^n_{s-}dW^n_s. \end{aligned}$$

Since exponent function is nonnegative and by (6), $\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}(Y^n_{s-}-a)dK^n_s \le 0$. Using assumptions on f and φ and by the equality $[Y^n]_t = \int_0^t \|Z^n_{s-}\|^2 d\varrho^n_s$

$$\begin{aligned} e^{\lambda(t\wedge\tau^n)} \quad & |Y^n_{t\wedge\tau^n}-a|^2 + \lambda\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-}-a|^2 d\varrho^n_s + \int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}\|Z^n_{s-}\|^2 d\varrho^n_s \\ \le \quad & e^{\lambda\tau^n}|\xi^n-a|^2 + (2\mu+L^2/\varepsilon+\eta)\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-}-a|^2 d\varrho^n_s \\ & +\varepsilon\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}\|Z^n_{s-}\|^2 d\varrho^n_s + 1/\eta\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|f(\varrho^n_{s-},a,0)|^2 d\varrho^n_s \\ & +\beta\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-}-a|^2 d\Lambda^{n,L}_s + 1/|\beta|\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|\varphi(\varrho^n_{s-},a)|^2 d\Lambda^{n,L}_s \\ & -2\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}(Y^n_{s-}-a)Z^n_{s-}dW^n_s. \end{aligned}$$

Let $\varepsilon, \eta > 0$ be such that $\tilde{\varepsilon} = 1 - \varepsilon > 0$ and $\tilde{\lambda} = \lambda - (2\mu + L^2/\varepsilon + \eta) > 0$. Since $\beta < 0$,

$$
\begin{aligned}
(7)\quad e^{\lambda(t\wedge\tau^n)} \quad & |Y^n_{t\wedge\tau^n} - a|^2 + \tilde{\lambda}\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-} - a|^2 d\varrho^n_s \\
& + |\beta|\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-} - a|^2 d\Lambda^{n,L}_s + \tilde{\varepsilon}\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}\|Z^n_{s-}\|^2 d\varrho^n_s \\
\leq \quad & e^{\lambda\tau^n}|\xi^n - a|^2 + C\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|f(\varrho^n_{s-}, a, 0)|^2 d\varrho^n_s \\
& + C\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|\varphi(\varrho^n_{s-}, a)|^2 d\Lambda^{n,L}_s - 2\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}(Y^n_{s-} - a)Z^n_{s-}dW^n_s,
\end{aligned}
$$

where by C we denoted a constant which values may vary from line to line. Since $\{\int_0^{\cdot} e^{\lambda s}(Y_s - a)Z_s dW_s\}$ is a martingale, after integrating the above inequality we get

$$
\begin{aligned}
(8)\, E\Big(e^{\lambda(t\wedge\tau^n)}|Y^n_{t\wedge\tau^n} - a|^2 \; + & \int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-} - a|^2 d\Lambda^{n,L}_s + \int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}\|Z^n_{s-}\|^2 d\varrho^n_s\Big) \\
\leq \; & CE\Big(e^{\lambda\tau^n}|\xi^n - a|^2 + \int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|f(\varrho^n_{s-}, a, 0)|^2 d\varrho^n_s \\
& + \int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|\varphi(\varrho^n_{s-}, a)|^2 d\Lambda^{n,L}_s\Big).
\end{aligned}
$$

Note that

$$
\begin{aligned}
E\sup_{t\leq\tau^n}\left|\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}(Y^n_{s-} - a)Z^n_{s-}dW^n_s\right| & \leq CE\left(\int_0^T e^{2\lambda\varrho^n_s}|Y^n_{s-} - a|^2\|Z^n_{s-}\|^2 d\varrho^n_s\right)^{1/2} \\
& \leq \frac{1}{2}E\sup_{t\leq\tau^n} e^{\lambda t}|Y^n_t - a|^2 + CE\int_0^T e^{\lambda\varrho^n_t}\|Z^n_{t-}\|^2 d\varrho^n_t.
\end{aligned}
$$

Therefore taking supremum in (7) and using (8) we get estimate on $E\sup_{t\leq\tau^n} e^{\lambda t}|Y^n_{t\wedge\tau^n} - a|^2$.

Now, note that from

$$
\begin{aligned}
K^n_{t\wedge\tau^n} \; = \; & Y^n_0 - Y^n_{t\wedge\tau^n} - \int_0^{t\wedge\tau^n} f(\varrho^n_{s-}, Y^n_{s-}, Z^n_{s-})d\varrho^n_s \\
& - \int_0^{t\wedge\tau^n} \varphi(\varrho^n_{s-}, Y^n_{s-})d\Lambda^{n,L}_s + \int_0^{t\wedge\tau^n} Z^n_{s-}dW^n_s
\end{aligned}
$$

and the previous estimates it follows that

$$E\sup_{t\le\tau^n} e^{\lambda t}|K^n_t|^2 + Ee^{\lambda\tau^n}|K^n|_{\tau^n} \le CE\Big(e^{\lambda\tau^n}|\xi^n-a|^2+(\Lambda^{n,L}_{\tau^n})^2$$
$$+\int_0^{\tau^n} e^{\lambda\varrho^n_s}|f(\varrho^n_{s-},a,0)|^2 d\varrho^n_s+\int_0^{\tau^n} e^{\lambda\varrho^n_s}|\varphi(\varrho^n_{s-},a)|^2 d\Lambda^{n,L}_s\Big),$$

which shows the part (a).

(b) Let (Y^n,Z^n,K^n), $(\tilde Y^n,\tilde Z^n,\tilde K^n)$ be two solutions of (5). By the Itô formula,

$$e^{\lambda(t\wedge\tau^n)}|Y^n_{t\wedge\tau^n}-\tilde Y^n_{t\wedge\tau^n}|^2+\lambda\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-}-\tilde Y^n_{s-}|^2 d\varrho^n_s+\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}\|Z^n_{s-}-\tilde Z^n_{s-}\|^2 d\varrho^n_s$$
$$\le (2\mu+L^2/\varepsilon)\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-}-\tilde Y^n_{s-}|^2 d\varrho^n_s+\varepsilon\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}\|Z^n_{s-}-\tilde Z^n_{s-}\|^2 d\varrho^n_s$$
$$+2\beta\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-}-\tilde Y^n_{s-}|^2 d\Lambda^{n,L}_s-2\int_{t\wedge\tau^n}^{\tau^n} e^{\lambda\varrho^n_s}(Y^n_{s-}-\tilde Y^n_{s-})(Z^n_{s-}-\tilde Z^n_{s-})dW^n_s.$$

Integrating and choosing $\varepsilon<1$ such that $2\mu+L^2/\varepsilon<\lambda$ we complete the proof. ∎

***Proof of Theorem* 2.** The proof of the theorem we divide into three steps.

Step 1. For every natural M we construct a solution (Y^M,Z^M,K^M) of GRBSDE

$$Y^M_t = \xi+\int_{t\wedge\tau}^{M\wedge\tau} f(s,Y^M_s,Z^M_s)ds+\int_{t\wedge\tau}^{M\wedge\tau}\varphi(s,Y^M_s)d\Lambda_s$$
$$-\int_{t\wedge\tau}^{\tau} Z^M_s dW_s+K^M_{M\wedge\tau}-K^M_{t\wedge\tau},\quad t\in\mathbb{R}^+ \tag{9}$$

in the following way. Let $\xi_M=E(\xi|\mathcal{F}_M)$ and Λ^τ be a process stopped in the stopping time τ, i.e., $\Lambda^\tau_t=\Lambda_{t\wedge\tau}$.

For $t\in[0,M]$ consider

$$Y^M_t = \xi_M+\int_t^M \mathbf{1}_{[0,\tau]}(s)f(s,Y^M_s,Z^M_s)ds+\int_t^M\varphi(s,Y^M_s)d\Lambda^\tau_s$$
$$-\int_t^M Z^M_s dW_s+K^M_M-K^M_t. \tag{10}$$

Since ξ_M is $\mathcal{F}_M$ measurable, from [7] it follows that there exists a unique solution of (10) on a deterministic interval $[0, M]$. Note that, on the set $\{t \geq \tau\}$, $\xi_M = \xi$ and

$$Y_t^M = \xi + 0 - \int_t^M Z_s^M dW_s + K_M^M - K_t^M.$$

By the uniqueness of the solution of BSDE, increments of the process K^M are constant, i.e., $K_M^M = K_t^M = K_\tau^M$. Therefore $Y_t^M = \xi - \int_t^M Z_s^M dW_s$ and in particular $Y_\tau^M = E(\xi|\mathcal{F}_\tau) = \xi$. On the other hand, by the Itô formula

$$E\Big(|Y_\tau^M|^2 + \int_\tau^M \|Z_s^M\|^2 ds\Big) = E|\xi|^2.$$

As a consequence, on the set $\{t \geq \tau\}$ $Y_t^M = \xi$, and $Z_t^M = 0$.

For $t > M$ put $Y_t^M = \xi_t$, $Z_t^M = \zeta_t$ and $K_t^M = K_M^M$. These processes satisfy

$$Y_t^M = \xi - \int_t^\tau Z_s^M dW_s,$$

and on the set $\{t \geq \tau\}$, $Y_t^M = \xi_t = \xi$ and $Z_t^M = 0$.

It can be shown that (compare [8])

$$\begin{aligned} &E\Big(\sup_{t\leq\tau} e^{\lambda t}|Y_t^M - a|^2 + \int_0^\tau e^{\lambda t}(|Y_t^M - a|^2 d\Gamma_t + \|Z_t^M\|^2 dt) + \int_0^\tau e^{\lambda t} d|K^M|_t\Big) \\ &\qquad \leq CE\Big(e^{\lambda\tau}|\xi - a|^2 + \int_0^\tau e^{\lambda t}|f(t,a,0)|^2 dt + \int_0^\tau e^{\lambda t}|\varphi(t,a)|^2 d\Lambda_t\Big), \end{aligned} \tag{11}$$

where $\Gamma_t = \Lambda_t + t$. It is clear that the solution of (9) converges to the solution of (2) when $M \to \infty$.

Step 2. For every $M \in \mathbb{N}$ we construct a sequence $(Y^{n,M}, Z^{n,M}, K^{n,M})$ of solutions of GRBSDE on $[0, M]$ which approximate the solution of (10) (compare [7]). We will show that this sequence converges to the solution of (5), when $M \to \infty$. Let $\xi_M^n = E(\xi^n|\mathcal{F}_M^n)$. For $j = nM$ we put $y_{\tau_j^n}^{n,M} = \xi_M^n$, $z_{\tau_j^n}^{n,M} = 0$, $\Delta k_{\tau_{j+1}^n}^{n,M} = 0$. Next, for $j = nM - 1, \ldots, 0$ we solve

$$\begin{aligned} y_{\tau_j^n}^{n,M} &= y_{\tau_{j+1}^n}^{n,M} + \frac{1}{n} f\Big(j/n, y_{\tau_j^n}^{n,M}, z_{\tau_j^n}^{n,M}\Big)\mathbf{1}_{\{[n\tau^n]>j\}} \\ &\quad + \varphi\Big(j/n, y_{\tau_j^n}^{n,M}\Big)\Delta\Lambda_{\tau_j^n}^{n,L}\mathbf{1}_{\{[n\tau^n]>j\}} - z_{\tau_j^n}^{n,M}\Delta W_{\tau_{j+1}^n}^n + \Delta k_{\tau_{j+1}^n}^{n,M}. \end{aligned}$$

If we put $Y_t^{n,M} = y^{n,M}_{\tau^n_{[nt]}}$, $Z_t^{n,M} = z^{n,M}_{\tau^n_{[nt]}}$ and $K_t^{n,M} = \sum_{j\le[nt]}\Delta k^{n,M}_{\tau^n_j}$, then the triple $(Y^{n,M},Z^{n,M},K^{n,M})$ is a unique (compare Proposition (3)) solution of

$$
\begin{aligned}
Y_t^{n,M} &= \xi^n_M + \int_{t\wedge\tau^n}^{M\wedge\tau^n} f(\varrho^n_{s-},Y^{n,M}_{s-},Z^{n,M}_{s-})d\varrho^n_s + \int_{t\wedge\tau^n}^{M\wedge\tau^n}\varphi(\varrho^n_{s-},Y^{n,M}_{s-})d\Lambda^{n,L}_s \\
&\quad - \int_{t\wedge\tau^n}^{M} Z^{n,M}_{s-}dW^n_s + K^{n,M}_{M\wedge\tau^n} - K^{n,M}_{t\wedge\tau^n}, \quad t\in[0,M].
\end{aligned}
\tag{12}
$$

Let (Y^n,Z^n,K^n) be a solution of (5). For $t\in[0,M]$ we have

$$
\begin{aligned}
Y^n_{t\wedge\tau^n} &= Y^n_{M\wedge\tau^n} + \int_{t\wedge\tau^n}^{M\wedge\tau^n} f(\varrho^n_{s-},Y^n_{s-},Z^n_{s-})d\varrho^n_s + \int_{t\wedge\tau^n}^{M\wedge\tau^n}\varphi(\varrho^n_{s-},Y^n_{s-})d\Lambda^{n,L}_s \\
&\quad - \int_{t\wedge\tau^n}^{M\wedge\tau^n} Z^n_{s-}dW^n_s + K^n_{M\wedge\tau^n} - K^n_{t\wedge\tau^n}.
\end{aligned}
$$

By the Itô formula and assumptions,

$$
\begin{aligned}
&e^{\lambda(t\wedge\tau^n)}|Y^n_t - Y^{n,M}_t|^2 + \int_{t\wedge\tau^n}^{M\wedge\tau^n} e^{\lambda\varrho^n_s}(\lambda|Y^n_{s-}-Y^{n,M}_{s-}|^2 + \|Z^n_{s-}-Z^{n,M}_{s-}\|^2)d\varrho^n_s \\
&\le e^{\lambda(M\wedge\tau^n)}|Y^n_{M\wedge\tau^n}-\xi^n_M|^2 + (2\mu+L^2/\varepsilon)\int_{t\wedge\tau^n}^{M\wedge\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-}-Y^{n,M}_{s-}|^2 d\varrho^n_s \\
&\quad + \varepsilon\int_{t\wedge\tau^n}^{M\wedge\tau^n} e^{\lambda\varrho^n_s}\|Z^n_{s-}-Z^{n,M}_{s-}\|^2 d\varrho^n_s + 2\beta\int_{t\wedge\tau^n}^{M\wedge\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-}-Y^{n,M}_{s-}|^2 d\Lambda^{n,L}_s \\
&\quad - 2\int_{t\wedge\tau^n}^{M\wedge\tau^n} e^{\lambda\varrho^n_s}(Y^n_{s-}-Y^{n,M}_{s-})(Z^n_{s-}-Z^{n,M}_{s-})dW^n_s.
\end{aligned}
\tag{13}
$$

Choosing $\varepsilon<1$ such that $2\mu+L^2/\varepsilon<\lambda$ and after integrating we get

$$
\begin{aligned}
&E\Big(e^{\lambda(t\wedge\tau^n)}|Y^n_t-Y^{n,M}_t|^2 + \int_{t\wedge\tau^n}^{M\wedge\tau^n} e^{\lambda\varrho^n_s}|Y^n_{s-}-Y^{n,M}_{s-}|^2 d\Lambda^{n,L}_s \\
&\qquad + \int_{t\wedge\tau^n}^{M\wedge\tau^n} e^{\lambda\varrho^n_s}(|Y^n_{s-}-Y^{n,M}_{s-}|^2 + \|Z^n_{s-}-Z^{n,M}_{s-}\|^2)d\varrho^n_s\Big) \\
&\le CEe^{\lambda(M\wedge\tau^n)}|Y^n_{M\wedge\tau^n}-\xi^n_M|^2.
\end{aligned}
$$

Moreover, taking supremum in (13) we get

$$E \sup_{t \le M} e^{\lambda(t\wedge\tau^n)}|Y_t^n - Y_t^{n,M}|^2 \le C E e^{\lambda(M\wedge\tau^n)}|Y^n_{M\wedge\tau^n} - \xi^n_M|^2.$$

Similarly we can compute for λ' such $2\mu + L^2/\varepsilon = \lambda' < \lambda$. Since

$$\sup_n E e^{\lambda(M\wedge\tau^n)}|Y^n_{M\wedge\tau^n} - \xi^n_M|^2 \le 2\sup_n E \sup_{t\le\tau^n} e^{\lambda t}|Y_t^n|^2 + 2\sup_n E e^{\lambda\tau^n}|\xi^n|^2 < \infty$$

and $E e^{\lambda'(M\wedge\tau^n)}|Y^n_{M\wedge\tau^n} - \xi^n_M|^2 \le C e^{(\lambda'-\lambda)M} E e^{\lambda(M\wedge\tau^n)}|Y^n_{M\wedge\tau^n} - \xi^n_M|^2$, we have

$$\lim_{M\to\infty} \sup_n E\Big(\sup_{t\le\tau^n} e^{\lambda' t}|Y_t^n - Y_t^{n,M}|^2 + \int_0^{M\wedge\tau^n} e^{\lambda'\varrho_s^n}\|Z^n_{s-} - Z^{n,M}_{s-}\|^2 d\varrho_s^n$$
$$+ \int_0^{M\wedge\tau^n} e^{\lambda'\varrho^n_s}|Y^n_{s-} - Y^{n,M}_{s-}|^2 d\Lambda_s^{n,L} + \sup_{t\le\tau^n}|K_t^n - K_t^{n,M}|^2\Big) = 0.$$

Step 3. Using the proof of Theorem 4.1 in [7] we get

$$\lim_{n\to\infty} E\Big(\sup_{t\le M}|Y^{n,M}_{t\wedge\tau^n} - Y^M_{t\wedge\tau}|^2 + \int_0^M \|Z^{n,M,\tau^n}_{t-} - Z^{M,\tau}_t\|^2 dt$$
$$+ \int_0^M |Y^{n,M,\tau^n}_{t-} - Y^{M,\tau}_t|^2 d\Lambda_t + \sup_{t\le M}|K^{n,M}_{t\wedge\tau^n} - K^M_{t\wedge\tau}|^2\Big) = 0.$$

Combining arguments from steps 1–3 we complete the proof. ∎

4. Partial differential equations

In [7] and [16] it was shown that GRBSDE with deterministic terminal time gives a probabilistic formula for the viscosity solution to an obstacle problem for parabolic PDE with Neumann boundary condition. Moreover, in [7] the application of the discrete approximation in solving appropriate PDE was given. In [8] the connection between GRBSDE with random terminal time and an obstacle problem for elliptic PDE with Neumann boundary condition was shown. Here we will give an application of the numerical scheme in solving PDE.

Let $\mathcal{O}$, G be open connected bounded and smooth subsets of $\mathbb{R}^m$ such that $G \cap \mathcal{O} \ne \emptyset$ and $\partial\mathcal{O} \cap G \ne \emptyset$, $\partial G \cap \mathcal{O} \ne \emptyset$.

Assume that $b : \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ are Lipschitz functions, i.e. for some $L' > 0$, all $x, x' \in \mathbb{R}^m$

(B1) $$|b(x) - b(x')| + \|\sigma(x) - \sigma(x')\| \le L'|x - x'|$$

and let (X^x, A^x) be a solution of SDE with reflection, i.e.,

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dW_s + A_t^x, \quad t \in \mathbb{R}^+, \tag{14}$$

where $P(X^x \in \bar{\mathcal{O}}) = 1$, A^x is a process with locally finite variation $|A^x|$, that increases only if $X_t^x \in \partial\mathcal{O}$; $A_0^x = 0$, $X_0 = x \in \mathcal{O} \cap G$ (for a unique existence see [9]).

In particular, if $\mathcal{O} = \{x;\ \phi(x) > 0\}$, $\partial\mathcal{O} = \{x;\ \phi(x) = 0\}$ for some $\phi \in C_b^2(\mathbb{R}^m)$ such that $|\nabla\phi(x)| = 1$ for $x \in \partial\mathcal{O}$, then

$$A_t^x = \int_0^t \nabla\phi(X_s^x)d|A^x|_s = \int_0^t \nabla\phi(X_s^x)\mathbf{1}_{\{X_s^x \in \partial\mathcal{O}\}}d|A^x|_s.$$

Define $\tau^x = \inf\{t \geq 0;\ X_t^x \notin G\}$ and assume that

$$E\tau^x < \infty. \tag{B2}$$

Suppose that $D = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_d, b_d)$ and $g:\ \partial G \cap \bar{\mathcal{O}} \to \bar{D}$. Let (Y^x, Z^x, K^x) be a solution of GRBSDE with data $(\tau^x, g(X_{\tau^x}^x), F, \Phi, |A^x|)$, where $F(t, \omega, y, z) = f(X_t^x(\omega), y, z)$, $\Phi(t, \omega, y) = \varphi(X_t^x(\omega), y)$, $t \in \mathbb{R}^+$, $\omega \in \Omega$, $y \in \mathbb{R}^d$, $z \in \mathbb{R}^{d\times m}$, i.e.

$$\begin{aligned} Y_{t\wedge\tau^x}^x \ &= g(X_{\tau^x}^x) + \int_{t\wedge\tau^x}^{\tau^x} f(X_\theta^x, Y_\theta^x, Z_\theta^x)d\theta + \int_{t\wedge\tau^x}^{\tau^x} \varphi(X_\theta^x, Y_\theta^x)d|A^x|_\theta \\ &\quad - \int_{t\wedge\tau^x}^{\tau^x} Z_\theta^x dW_\theta + K_{\tau^x}^x - K_{t\wedge\tau^x}^x, \quad t \in \mathbb{R}^+. \end{aligned} \tag{15}$$

Assume that functions g, f and φ are continuous and there exist constants $\kappa, p \geq 0$, $L > 0$, $\mu \in \mathbb{R}$ and $\beta < 0$ such that $\mu + L^2 < 0$ and for any $x \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d\times m}$,

$$\begin{aligned} |g(x)| \ &\leq\ \kappa(1 + |x|^p), \\ \langle y - y', f(x, y, z) - f(x, y', z)\rangle \ &\leq\ \mu|y - y'|^2, \\ |f(x, y, z) - f(x, y', z')| \ &\leq\ L\left(|y - y'| + \|z - z'\|\right), \\ |f(x, y, 0)| \leq \kappa(1 + |y|),\ |\varphi(x, y)| \ &\leq\ \kappa, \\ |\varphi(x, y) - \varphi(x, y')|\rangle \ &\leq\ L|y - y'|, \\ \langle y - y', \varphi(x, y) - \varphi(x, y')\rangle \ &\leq\ \beta|y - y'|^2, \\ E\int_0^{\tau^x} |f(X_t^x, \xi_t, \zeta_t)|^2 dt \ &<\ \infty, \end{aligned} \tag{$B3$}$$

where $\xi = g(X^x_{\tau^x})$, $\xi_t = E(g(X^x_{\tau^x})|\mathcal{F}_t)$, $\xi = E\xi + \int_0^\infty \zeta_t dW_t$.

Let $u : \bar{\mathcal{O}} \cap \bar{G} \to \bar{D}$ be a solution of the following obstacle problem for elliptic PDE with Neumann boundary condition

$$
(16)\quad \begin{cases} \min\Big(u_i(x) - a_i, \max\Big(u_i(x) - b_i, \\ \qquad -Lu_i(x) - f_i(x, u(x), (\nabla u_i \sigma)(x))\Big)\Big) = 0, & x \in \mathcal{O} \cap G \\ \min\Big(u_i(x) - a_i, \max\Big(u_i(x) - b_i, \\ \qquad -\frac{\partial u_i}{\partial n}(x) - \varphi_i(x, u(x))\Big)\Big) = 0, & x \in \partial\mathcal{O} \cap G \\ \quad u(x) = g(x), & x \in \partial G \cap \bar{\mathcal{O}}, \end{cases}
$$

for $i = 1, \dots, d$, where

$$
\begin{aligned} Lu_i(x) &= \frac{1}{2} \sum_{1 \le j,k \le m} \frac{\partial^2 u_i}{\partial x_j \partial x_k}(x)(\sigma\sigma^*)_{jk}(x) + \sum_{1 \le j \le m} \frac{\partial u_i}{\partial x_j}(x) b_j(x), \\ \frac{\partial}{\partial n} &= \sum_{j=1}^m \frac{\partial \phi}{\partial x_j}(x) \frac{\partial}{\partial x_j}, \quad x \in \partial\mathcal{O}. \end{aligned}
$$

Theorem 4 [8, Theorem 3.2]. *Assume* (B1)–(B3). *Function u defined as $u(x) = Y_0^x$, $x \in \bar{\mathcal{O}} \cap \bar{G}$ is a continuous function being a viscosity solution of* (16).

Now, using the approximation scheme from Section 3 we will give a numerical scheme for solving (16). Let $\{\tau^{x,n}\}$ be a sequence of stopping times approximating τ^x such that $(W^n, \tau^{x,n}) \xrightarrow{\mathcal{D}} (W, \tau^x)$. Let $\tau^{x,n} \le T_{x,n}$, for $T_{x,n} \in \mathbb{N}$ and let $\tau^{x,n}_j = j/n \wedge [n\tau^{x,n}]/n$, $j \in \mathbb{N}$.

Put $X^{x,n}_t = x^n_{\tau^{x,n}_{[nt]}}$, $A^{x,n}_t = \sum_{j=1}^{[nt]} \Delta a^n_{\tau^{x,n}_j}$, where $x^n_0 = x$ and for $j = 0, 1, \dots, nT_{x,n} - 1$, $(x^n_{\tau^{x,n}_{j+1}}, \Delta a^n_{\tau^{x,n}_{j+1}})$ is given by

$$
(17)\quad \begin{cases} x^n_{\tau^{x,n}_{j+1}} = \pi_{\bar{\mathcal{O}}}\Big(x^n_{\tau^{x,n}_j} + \frac{1}{n} b(x^n_{\tau^{x,n}_j}) + \frac{1}{\sqrt{n}} \sigma(x^n_{\tau^{x,n}_j}) \varepsilon^n_{j+1}\Big), \\ \Delta a^n_{(j+1)/n} = x^n_{\tau^{x,n}_j} + \frac{1}{n} b(x^n_{\tau^{x,n}_j}) + \frac{1}{\sqrt{n}} \sigma(x^n_{\tau^{x,n}_j}) \varepsilon^n_{j+1} \\ \qquad -\pi_{\bar{\mathcal{O}}}\Big(x^n_{\tau^{x,n}_j} + \frac{1}{n} b(x^n_{\tau^{x,n}_j}) + \frac{1}{\sqrt{n}} \sigma(x^n_{\tau^{x,n}_j}) \varepsilon^n_{j+1}\Big), \end{cases}
$$

where $\pi_{\bar{\mathcal{O}}}(x)$ denotes the projection of x on the set $\mathcal{O}$. Then $(X^{x,n}, A^{x,n})$ satisfies

$$X_t^{x,n} = x + \int_0^t b(X_{s-}^{x,n})d\varrho_s^n + \int_0^t \sigma(X_{s-}^{x,n})dW_s^n + A_t^{x,n}, \quad t \in \mathbb{R}^+$$

and $(X^{x,n}, A^{x,n}, W^n) \to^{\mathcal{D}} (X^x, A^x, W)$ (see e.g. [17]). Moreover, since

$$|A^x|_t = \int_0^t \nabla\phi(X_s^x)dA_s^x \quad \text{and} \quad |A^{x,n}|_t = \int_0^t \nabla\phi(X_{s-}^{x,n})dA_s^{x,n},$$

we have also $|A^{x,n}| \to^{\mathcal{D}} |A^x|$.

In order to solve the discrete GRBSDE on the set $\{\tau^{x,n} \le j/n\}$ put $y^n_{\tau_j^{x,n}} = g(x^n_{\tau_j^{x,n}})$ and on the set $\{[n\tau^{x,n}] > j\}$ solve as in Section 3:

$$\begin{aligned} y^n_{j/n} &= y^n_{\tau^{x,n}_{j+1}} + \frac{1}{n} f(x^n_{j/n}, y^n_{j/n}, z^n_{j/n})\mathbf{1}_{\{[n\tau^{x,n}]>j\}} \\ &\quad + \varphi(x^n_{j/n}, y^n_{j/n})\mathbf{1}_{\{[n\tau^{x,n}]>j\}}\Delta\Lambda^{n,L}_{j/n} - \frac{1}{\sqrt{n}} z^n_{j/n}\varepsilon^n_{j+1} + \Delta k^n_{\tau^n_{j+1}}, \end{aligned}$$

where $\Delta\Lambda^n_{j/n} = E(|\Delta a^n_{(j+1)/n}||\mathcal{F}^n_{j/n})$. For $t \in [0, T_{x,n}]$ put $Y_t^{x,n} = y^n_{\tau^{x,n}_{[nt]}}$, $Z_t^{x,n} = z^n_{\tau^{x,n}_{[nt]}}$ and $K_t^{x,n} = \sum_{j=1}^{[nt]} \Delta k^n_{\tau_j^{x,n}}$.

Let us now denote

$$\begin{aligned} D^n_{\pm}u(x) &= \frac{1}{2}\, u\bigg(\pi_{\bar{\mathcal{O}}}\Big(x + \frac{1}{n}b\,(x) + \frac{1}{\sqrt{n}}\,\sigma\,(x)\Big)\bigg) \\ &\quad \pm \frac{1}{2}\, u\bigg(\pi_{\bar{\mathcal{O}}}\Big(x + \frac{1}{n}b\,(x) - \frac{1}{\sqrt{n}}\,\sigma\,(x)\Big)\bigg), \\ a(x) &= \frac{1}{2}\bigg|x + \frac{1}{n}\,b(x) + \frac{1}{\sqrt{n}}\,\sigma(x) - \pi_{\bar{\mathcal{O}}}\Big(x + \frac{1}{n}\,b(x) + \frac{1}{\sqrt{n}}\,\sigma(x)\Big)\bigg| \\ &\quad + \frac{1}{2}\bigg|x + \frac{1}{n}\,b(x) - \frac{1}{\sqrt{n}}\,\sigma(x) - \pi_{\bar{\mathcal{O}}}\Big(x + \frac{1}{n}\,b(x) - \frac{1}{\sqrt{n}}\,\sigma(x)\Big)\bigg|. \end{aligned}$$

Lemma 5. *Assume that $u^n : \bar{\mathcal{O}} \cap \bar{G} \to \bar{D}$, $n > 2L$ satisfies $u^n(x) = g(x)$ for $x \in \partial G \cap \bar{\mathcal{O}}$ and for $x \in G \cap \bar{\mathcal{O}}$ is a unique solution of*

$$\begin{aligned} u^n(x) &= \pi\bigg(D^n_+u^n(x) + \tfrac{1}{n}\, f(x, u^n(x), \sqrt{n}\, D^n_-u^n(x)) \\ &\qquad + \varphi\,(x, u^n(x)) \min\big((2L)^{-1}, a(x)\big)\bigg). \end{aligned} \tag{18}$$

Then $y^n_{\tau^{x,n}_j} = u^n(x^n_{\tau^{x,n}_j})$, $z^n_{\tau^{x,n}_j} = \sqrt{n}D^n_-\ u^n(x^n_{\tau^{x,n}_j})\mathbf{1}_{\{[n\tau^n_x]>j\}}$, $j = 0, \ldots, nT_{x,n}$.

Proof. The proof will be done by induction. First note, that if $x \in \partial G$, then $y^n_0 = g(x^n_0) = u^n(x^n_0)$. Suppose that $x \in G$ and consider the case $\{[n\tau^{x,n}] > j\}$. Let $\tau^{x,n}_{j+1}$ be such that $x^n_{\tau^{x,n}_{j+1}} \in \partial G$. Then $y^n_{\tau^{x,n}_{j+1}} = u^n(x^n_{\tau^{x,n}_{j+1}})$ and

$$z^n_{\tau^{x,n}_j} = \sqrt{n}E(u^n(x^n_{\tau^{x,n}_{j+1}})\varepsilon^n_{j+1}|\mathcal{F}^n_{j/n})\mathbf{1}_{\{[n\tau^{x,n}]>j\}} = \sqrt{n}D^n_- u^n(x^n_{\tau^{x,n}_j})\mathbf{1}_{\{[n\tau^{x,n}]>j\}}.$$

Since $E(u^n(x^n_{\tau^{x,n}_{j+1}})|\mathcal{F}^n_{j/n}) = D^n_+\ u^n(x^n_{\tau^{x,n}_j})$ and $E(|a^n_{\tau^{x,n}_{j+1}}||\mathcal{F}^n_{j/n}) = a(x^n_{\tau^{x,n}_j})$,

$$\begin{aligned}
h^n_{\tau^{x,n}_j} &= E(u^n(x^n_{\tau^{x,n}_{j+1}})|\mathcal{F}^n_{j/n}) + \frac{1}{n}f(x^n_{\tau^{x,n}_j}, \pi(h^n_{\tau^{x,n}_j}), z^n_{\tau^{x,n}_j}) + \varphi(x^n_{\tau^{x,n}_j}, \pi(h^n_{\tau^{x,n}_j}))\Delta\Lambda^{n,L}_{\tau^{x,n}_j} \\
&= D^n_+\ u^n(x^n_{\tau^{x,n}_j}) + \frac{1}{n}f(x^n_{\tau^{x,n}_j}, \pi(h^n_{\tau^{x,n}_j}), \sqrt{n}D^n_-\ u^n(x^n_{\tau^{x,n}_j})) \\
&\quad + \varphi(x^n_{\tau^{x,n}_j}, \pi(h^n_{\tau^{x,n}_j}))\min\left((2L)^{-1}, a(x^n_{\tau^{x,n}_j})\right).
\end{aligned}$$

By the fact that $y^n_{\tau^{x,n}_j} = \pi(h^n_{\tau^{x,n}_j})$ and since functions f, φ and π are Lipschitz, the solution of (18) is unique for n big enough and the lemma follows. ■

Proposition 6. $u^n(x)$ *converges to* $u(x)$ *being a solution of* (16).

Proof. Note that if $x \in \partial G \cap \bar{\mathcal{O}}$, then $u^n(x) = g(x) = u(x)$. Assume that $x \in G \cap \bar{\mathcal{O}}$. Since $u^n(x) = u^n(x^n_0) = y^n_0 = Y^{x,n}_0$, $u(x) = Y^x_0$ and $Y^{x,n}_0$, Y^x_0 are deterministic, applying Theorem 2 completes the proof. ■

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