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ASYMPTOTIC BEHAVIOUR IN THE SET OF NONHOMOGENEOUS CHAINS OF STOCHASTIC OPERATORS [1]

Małgorzata Pułka

Department of Mathematics
Gdańsk University of Technology
ul. Gabriela Narutowicza 11/12, 80-233, Gdańsk, Poland

e-mail: mpulka@mif.pg.gda.pl

Abstract

We study different types of asymptotic behaviour in the set of (infinite dimensional) nonhomogeneous chains of stochastic operators acting on $L^1(\mu)$ spaces. In order to examine its structure we consider different norm and strong operator topologies. To describe the nature of the set of nonhomogeneous chains of Markov operators with a particular limit behaviour we use the category theorem of Baire. We show that the geometric structure of the set of those stochastic operators which have asymptotically stationary density differs depending on the considered topologies.

Keywords: Markov operator, asymptotic stability, residuality, dense G_δ.

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1. Introduction

The study of chains of Markov operators has become a subject of interest in regard to their applications in many different areas of science and technology. Markov operators are commonly used to describe phenomena involving a law of conservation of a certain quantity, e.g. mass, energy, the number of particles in physical or chemical processes. Typical questions appear in the context of probability

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theory and concern the evolution of a density (probability distribution) of such a quantity. The case when the chain is homogeneous in time is well-understood and has a comprehensive literature (cf. [7, 8, 10]). In particular, the asymptotic behaviour of iterates of Markov operators has been intensively studied. The ergodic structure of homogeneous chains is fully described including probabilistic, lattice and spectral conditions for convergence of iterates with respect to all standard topologies. In the case of the class of chains nonhomogeneous in time the situation is not so transparent, since no proper notion of a stationary density can be defined (in general). Thus, in order to describe the properties of a nonhomogeneous chain one may study its asymptotic behaviour, which is understood as the study of a "generalized concept" of stationarity. Namely, we may ask whether there exists a common limiting density or, at least, if the influence of the state of the process at the time m on its future states decreases to zero with the passage of time. Various gradations of this asymptotic properties may be considered depending on the mode of convergence of the iterates of the Markov operator. In this paper we focus solely on the uniform and strong modes of convergence.

Differences between the classes of homogeneous and nonhomogeneous chains attracted the attention of probabilists in the second half of the 20th century. For example, in [6] Iosifescu observed that unlike the homogenous case, uniform asymptotic stability (strong ergodicity) is not a "natural" concept for nonhomogeneous chains. Thus, given a class of possible evolutions of Markov operators, i.e., a class of nonhomogeneous chains of Markov operators with a particular asymptotic behaviour, one may ask about its topological size. Such a description is based on the category theorem of Baire. Namely, the set is recognized as a large object if it is residual (it contains a dense G_δ set). Thus, generic evolutions are those which belong to a residual subset. The aim of this paper is to define different types of asymptotic behaviour of nonhomogeneous chains of Markov operators acting on $L^1(\mu)$ spaces and to determine which one of them is prevalent.

The geometric structure of infinite dimensional nonhomogeneous Markov chains defined on the ℓ^1 space of all absolutely summable real sequences was intensively studied in [12]. Since $\ell^1 = L^1(\mathbb{N}, 2^{\mathbb{N}}$, counting measure), then the results included in this article are generalizations of the results obtained in [12]. For the convenience of the reader most of the theorems are proved in full detail. This paper may be considered as the first step to generalizations of some results included in [3]. It is worth noticing that in [11] the asymptotic properties of nonhomogeneous discrete Markov processes with general state space $L^1(\mu)$ were studied and the results obtained were applied in the investigation of the limit behaviour of the so-called quadratic stochastic processes which are concerned with genetic models.

Let $(X, \mathcal{A}, \mu)$ be a separable σ-finite measure space. Throughout the paper we consider the (separable) Banach lattice of real and $\mathcal{A}$-measurable functions

f such that $|f|$ is μ-integrable and we denote it by $L^1(\mu)$. By $\|\cdot\|_1$ we denote the relevant norm. We say that a linear operator $P\colon L^1(\mu) \to L^1(\mu)$ is *Markov* (or *stochastic*) if

$$Pf \geq 0 \quad \text{and} \quad \|Pf\|_1 = \|f\|_1$$

for all $f \geq 0$, $f \in L^1(\mu)$. By $\mathcal{D} = \mathcal{D}(X, \mathcal{A}, \mu)$ we denote the set of all *densities* on X, i.e.,

$$\mathcal{D} = \left\{ f \in L^1(\mu) : f \geq 0, \|f\|_1 = 1 \right\}.$$

In view of stochasticity of P we have that $|||P||| = 1$ (where $|||\cdot|||$ stands for the norm operator) and $P(\mathcal{D}) \subset \mathcal{D}$. The sequence of such operators denoted by $\mathbf{P} := (P^{m,m+1})_{m\geq 0}$ is called a *discrete time nonhomogeneous chain of Markov operators*. For any natural numbers $0 \leq m < n$ we set

$$P^{m,n} = P^{m,m+1} \circ P^{m+1,m+2} \circ \cdots \circ P^{n-1,n}.$$

If for each $m \geq 0$ one has $P^{m,m+1} = P$, then $\mathbf{P} = (P)_{m\geq 0}$ is called a *homogeneous chain of Markov operators*. The set of all chains of Markov operators (including homogeneous) will be denoted by $\mathscr{S}$, i.e.,

$$\mathscr{S} = \left\{ \mathbf{P} = \left(P^{m,m+1}\right)_{m\geq 0} : P^{m,m+1} \text{ are Markov operators } \right\}.$$

Let $t \in [0,1]$ be given. A *convex combination* $\mathbf{T}(t)$ of two chains of Markov operators $\mathbf{P}$ and $\mathbf{R} \in \mathscr{S}$ is defined as follows:

$$T^{m,m+1}(t) = tP^{m,m+1} + (1-t)\, R^{m,m+1}.$$

Note that $\mathbf{T}(t) \in \mathscr{S}$ for every $t \in [0,1]$ and that a mapping $[0,1] \ni t \mapsto \mathbf{T}(t) \in \mathscr{S}$ is continuous when $\mathscr{S}$ is equipped with suitable topology. Moreover, $\mathbf{T}(0) = \mathbf{R}$ and $\mathbf{T}(1) = \mathbf{P}$. Thereby, $\mathscr{S}$ has an affine structure and it is arcwise connected.

Throughout the paper we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We will endow the set $\mathscr{S}$ with metric structures. Given $\mathbf{P}, \mathbf{R} \in \mathscr{S}$ let us consider:

(1) the sup norm operator topology induced by the metric

$$d_{n.\sup}(\mathbf{P}, \mathbf{R}) = \sup_m \left|\left|\left|P^{m,m+1} - R^{m,m+1}\right|\right|\right|,$$

(2) the $\sum$ norm operator topology induced by the metric

$$d_{n.\sum}(\mathbf{P}, \mathbf{R}) = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \left|\left|\left|P^{m,m+1} - R^{m,m+1}\right|\right|\right|,$$

(3) the $\sum$sup strong operator topology induced by the metric

$$d_{so.\sup}(\mathbf{P},\mathbf{R}) = \sum_{l=0}^{\infty} \frac{1}{2^l} \sup_m \left\| P^{m,m+1} f_l - R^{m,m+1} f_l \right\|_1 ,$$

where $\{f_0, f_1, \ldots\}$ is a fixed countable and linearly dense subset of $\mathcal{D}$,

(4) the $\sum\sum$ strong operator topology induced by the metric

$$d_{so.\sum}(\mathbf{P},\mathbf{R}) = \sum_{m,l=0}^{\infty} \frac{1}{2^{m+l+1}} \left\| P^{m,m+1} f_l - R^{m,m+1} f_l \right\|_1 ,$$

where $\{f_0, f_1, \ldots\}$ is a fixed countable and linearly dense subset of $\mathcal{D}$.

Note that $d_{so.\sup}(\mathbf{P}_k,\mathbf{R}) \to 0$ as $k \to \infty$ if and only if for every $f \in L^1(\mu)$ ($f \in \mathcal{D}$) and any $m \in \mathbb{N}_0$ one has $\lim_{k\to\infty} \sup_m \|P_k^{m,m+1} f - R^{m,m+1} f\|_1 = 0$. Moreover, the topologies generated by $d_{so.\sup}$ and $d_{so.\sum}$ do not depend on the choice of a sequence $\{f_0, f_1, \ldots\}$.

Clearly, $d_{n.\sup}$ generates the strongest topology and $d_{so.\sum}$ generates the weakest. However, it should be emphasized that metrics $d_{n.\sum}$ and $d_{so.\sup}$ are not comparable. In order to observe it, consider $\mathbf{P}_j = (P_j^{m,m+1})_{m\geq 0} \in \mathscr{S}$ defined as follows:

$$P_j^{m,m+1} = \begin{cases} P, & \text{if } 0 \leq m < j, \\ I, & \text{if } m \geq j, \end{cases}$$

where I stands for the identity operator and $\mathbf{P} = (P)_{m\geq 0}$ is such that $P \neq I$. Then

$$\begin{aligned} d_{n.\sum}(\mathbf{P}_j,\mathbf{P}) &= \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \left|\left|\left| P_j^{m,m+1} - P^{m,m+1} \right|\right|\right| \\ &= \sum_{m=0}^{j-1} \frac{1}{2^{m+1}} |||P - P||| + \sum_{m=j}^{\infty} \frac{1}{2^{m+1}} |||I - P||| \\ &= \frac{1}{2^j} |||I - P||| \to 0 \text{ as } j \to \infty. \end{aligned}$$

On the other hand, for a given fixed countable and dense subset $\{f_0, f_1, \ldots\}$ of $\mathcal{D}$ we have

$$\begin{aligned} d_{so.\sup}(\mathbf{P}_j,\mathbf{P}) &= \sum_{l=0}^{\infty} \frac{1}{2^l} \sup_m \left\| P_j^{m,m+1} f_l - P^{m,m+1} f_l \right\|_1 \\ &= \sum_{l=0}^{\infty} \frac{1}{2^l} \|f_l - P f_l\|_1 > 0. \end{aligned}$$

Thus $d_{so.\sup}(\mathbf{P}_j, \mathbf{P}) \nrightarrow 0$ as $j \to \infty$. Thereby, $d_{n.\sum}$ is not stronger than $d_{so.\sup}$. Now we shall see that $d_{so.\sup}$ is not stronger than $d_{n.\sum}$. In order to prove it observe that since the measure μ is σ-finite, there exists a sequence $\{B_k\}$, $B_k \in \mathcal{A}$, such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and

$$X = \bigcup_{k=0}^{\infty} B_k, \quad 0 < \mu(B_k) < \infty \quad \text{ for all } k \in \mathbb{N}_0.$$

Let $g_k \in \mathcal{D}$ be such that the essential support $\mathit{supp}\ g_k := \{x \in X : g_k(x) \neq 0\} \subseteq B_k$ for any $k \in \mathbb{N}_0$. For any $f \in L^1(\mu)$ denote

$$a_k(f) = \int_{B_k} f d\mu.$$

Note that $\sum_{k=0}^{\infty} a_k(f) = 1$ if $f \in \mathcal{D}$. Then we can define $\mathbf{P}_j = (P_j^{m,m+1})_{j \geq 0} \in \mathscr{S}$ as follows:

$$P_j^{m,m+1} f = P_j f = f \mathbf{1}_{\bigcup_{k=0}^{j} B_k} + \sum_{k=j+1}^{\infty} a_k(f) \cdot g_0.$$

Let $\mathbf{I} = (I, I, \ldots) \in \mathscr{S}$, where as before I stands for the identity operator. Then

$$\begin{aligned}
d_{so.\sup}(\mathbf{P}_j, \mathbf{I}) &= \sum_{l=0}^{\infty} \frac{1}{2^l} \sup_m \left\| P_j^{m,m+1} f_l - I f_l \right\|_1 \\
&= \sum_{l=0}^{\infty} \frac{1}{2^l} \left\| P_j f_l - f_l \right\|_1 \\
&= \sum_{l=0}^{\infty} \frac{1}{2^l} \left\| f_l \mathbf{1}_{\bigcup_{k=0}^{j} B_k} + \sum_{k=j+1}^{\infty} a_k(f_l)\, g_0 - f_l \mathbf{1}_{\bigcup_{k=0}^{j} B_k} - f_l \mathbf{1}_{\bigcup_{k=j+1}^{\infty} B_k} \right\|_1 \\
&\leq \sum_{l=0}^{\infty} \frac{1}{2^l} \left(\sum_{k=j+1}^{\infty} a_k(f_l) \left\| g_0 \right\|_1 + \left\| f_l \mathbf{1}_{\bigcup_{k=j+1}^{\infty} B_k} \right\|_1 \right) \\
&= \sum_{l=0}^{\infty} \frac{1}{2^l} \left(\sum_{k=j+1}^{\infty} a_k(f_l) + \int_{\bigcup_{k=j+1}^{\infty} B_k} f_l d\mu \right) \\
&= \sum_{l=0}^{\infty} \frac{1}{2^l} \cdot 2 \sum_{k=j+1}^{\infty} a_k(f_l) \\
&= \sum_{l=0}^{\infty} \frac{1}{2^{l-1}} \left(1 - \sum_{k=1}^{j} a_k(f_l) \right) \to 0 \text{ as } j \to \infty.
\end{aligned}$$

On the other hand,

$$d_{n.\sum}(\mathbf{P}_j, \mathbf{I}) = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} |||P_j^{m,m+1} - I|||$$
$$= \left(\sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \right) |||P_j - I||| = 1 \cdot 2 = 2 \nrightarrow 0 \text{ as } j \to \infty.$$

Therefore $d_{so.\sup}$ is not stronger than $d_{n.\sum}$. It follows that the metrics $d_{n.\sum}$ and $d_{so.\sup}$ are not comparable. The relationships between the considered metrics are illustrated in the Figure 1.

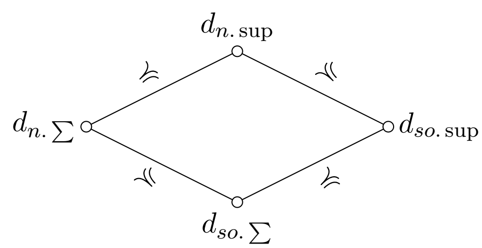

Figure 1. The relationships between the metrics $d_{n.\sup}$, $d_{n.\sum}$, $d_{so.\sup}$, $d_{so.\sum}$.

Let us note that in the class of homogeneous chains of Markov operators, metrics $d_{n.\sup}$ and $d_{n.\sum}$ are equivalent. In fact, if $\mathbf{P} = (P)_{m\geq 0}$, $\mathbf{R} = (R)_{m\geq 0} \in \mathscr{S}$, then $d_{n.\sup}(\mathbf{P}, \mathbf{R}) = d_{n.\sum}(\mathbf{P}, \mathbf{R}) = |||P - R|||$. Similarly we find that in the homogeneous case metrics $d_{so.\sup}$ and $d_{so.\sum}$ are equivalent and for any $\mathbf{P} = (P)_{m\geq 0}$, $\mathbf{R} = (R)_{m\geq 0} \in \mathscr{S}$ one has $d_{so.\sup}(\mathbf{P}, \mathbf{R}) = d_{so.\sum}(\mathbf{P}, \mathbf{R}) = \sum_{l=0}^{\infty} \frac{1}{2^l} \|Pf_l - Rf_l\|_1$, where $\{f_0, f_1, \ldots\}$ is a fixed countable and linearly dense subset of $\mathcal{D}$. This supports our remark that the nonhomogeneous case is more complex than the homogeneous one.

In what follows we study different types of asymptotic behaviour of nonhomogeneous chains of stochastic operators as well as residualities in the set $\mathscr{S}$. We shall see that the geometric structure of the set of those stochastic operators which have asymptotically stationary density differs depending on the considered topologies. We prove that the set of those Markov operators which do not possess limiting density is dense and its interior is nonempty in the topology induced by the metric $d_{n.\sup}$. On the other hand, it occurs that the set of those operators for which the limiting density exists is dense while $\mathscr{S}$ is endowed with topology induced by the metric $d_{n.\sum}$. We also examine the set of Markov operators which we call (uniformly or strongly, if studied in norm or strong operator topology respectively) almost asymptotically stable and we prove that it forms a residual subset for both norm and strong operator topologies.

Note that $d_{n.\sup}$ is the most relevant metric (topology) in studying the limit behaviour of nonhomogeneous chains of stochastic operators. It should be emphasized that, in contrast to the homogeneous case, the property of denseness of the set of nonhomogeneous chains of stochastic operators with a particular asymptotic behaviour does not suffice to understand its "size". It derives from the fact that in the case of $\sum \sup$ and $\sum\sum$ strong operator topologies the denseness of the complement of the set mentioned above can always be proved by modifying on the tail so-called sweeping operators or a fixed stochastic projection. Therefore, in order to describe the nature of the set we use the category theorem of Baire. This is because the space $\mathscr{S}$ equipped with any of the metrics (1)–(4) is complete and the classical Baire theorem is applicable.

The Baire category of asymptotic stability for homogeneous Markov chains was worked out in e.g. [2, 9, 13]. It should be clearly understood that our results are not a direct analogy of what was obtained in these works. In particular, it was proved in [13] that uniformly asymptotically stable (quasi-compact) homogeneous Markov chains form a dense G_δ subset in norm operator topology. In our nonhomogeneous case, the set of those chains of operators which are not uniformly asymptotically stable has a nonempty interior.

There are more relevant works in the literature dealing with the topic of the limit behaviour of nonhomogeneous chains of Markov operators. The reader should be warned that authors do not always use the same names for the same notions. For example, in [5] and [6] strong ergodicity is what we call uniform asymptotic stability and weak ergodicity is what we refer to as almost uniform asymptotic stability. Some authors apply the terminology derived from the ergodic theory to the theory of stochastic processes and denominate what we call asymptotic stability by mixing (cf. [2, 3, 9, 12]). See [5] for detailed classification of different types of asymptotic behaviour of nonhomogeneous Markov chains.

2. Uniform asymptotic stability

In this section we examine the strongest case of asymptotic stability of chains of Markov operators, i.e., uniform asymptotic stability. We start with

Definition. A nonhomogeneous chain of Markov operators $\mathbf{P}$ is called uniformly asymptotically stable if there exists a unique $f_* \in \mathcal{D}$ such that for every $m \in \mathbb{N}_0$

$$\lim_{n\to\infty} \sup_{f\in\mathcal{D}} \|P^{m,n}f - f_*\|_1 = 0.$$

The set of all uniformly asymptotically stable chains of Markov operators is denoted by $\mathscr{S}_{uas}$.

Note that uniformly asymptotically stable chains of operators possess common limiting density and the mode of convergence is uniform. The following theorem is concerned with the prevalence problem in the set $\mathscr{S}$.

Theorem 1. *The set $\mathscr{S}_{uas}^{c}$ of all Markov operators which are not uniformly asymptotically stable is a* sup *norm topology dense subset of $\mathscr{S}$ (i.e., in $d_{n.\sup}$). Moreover, in this case its interior Int$\mathscr{S}_{uas}^{c} \neq \emptyset$.*

Proof. Let $\mathbf{P} \in \mathscr{S}$ and $0 < \varepsilon < 1$ be taken arbitrarily. As before, since the measure μ is σ-finite, there exists a sequence $\{B_k\}$, $B_k \in \mathcal{A}$, such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and

$$X = \bigcup_{k=0}^{\infty} B_k, \quad 0 < \mu(B_k) < \infty \quad \text{for all } k \in \mathbb{N}_0.$$

Let $g_k \in \mathcal{D}$ be such that *supp* $g_k = \{x \in X : g_k(x) \neq 0\} \subseteq B_k$ for any $k \in \mathbb{N}_0$. Then we can define $\mathbf{R} \in \mathscr{S}$ as follows: for any $f \in L^1(\mu)$,

$$R^{m,m+1} f = \sum_{j=m+1}^{\infty} a_{j-m-1}(f) \cdot g_j,$$

where $a_j(f) = \int_{B_j} f d\mu$. Note that $\sum_{j=0}^{\infty} a_j(f) = 1$ if $f \in \mathcal{D}$. Consider a convex combination

$$P_{\varepsilon}^{m,m+1} = (1-\varepsilon) P^{m,m+1} + \varepsilon R^{m,m+1}.$$

Clearly, $\mathbf{P}_{\varepsilon} = (P_{\varepsilon}^{m,m+1})_{m \geq 0} \in \mathscr{S}$. We have

$$\begin{aligned} d_{n.sup}(\mathbf{P}_{\varepsilon}, \mathbf{P}) &= \sup_m \left|\left|\left|(1-\varepsilon) P^{m,m+1} + \varepsilon R^{m,m+1} - P^{m,m+1}\right|\right|\right| \\ &= \varepsilon \sup_m \left|\left|\left|P^{m,m+1} - R^{m,m+1}\right|\right|\right| \leq 2\varepsilon. \end{aligned}$$

It remains to show that $\mathbf{P}_{\varepsilon}$ is not uniformly asymptotically stable. Suppose that, on the contrary, there exists $f_* \in \mathcal{D}$ such that for every $f \in \mathcal{D}$ we have $\lim_{n\to\infty} P_{\varepsilon}^{m,n} f = f_*$. Since $f \in \mathcal{D}$, there exists $M \in \mathbb{N}_0$ such that

$$\int_{\bigcup_{k=0}^{M} B_k} f_* d\mu > 1 - \varepsilon.$$

Hence

$$\int_{\bigcup_{k=0}^{M} B_k} P_{\varepsilon}^{m,n} f d\mu \xrightarrow{n\to\infty} \int_{\bigcup_{k=0}^{M} B_k} f_* d\mu > 1 - \varepsilon.$$

On the other hand, if $n > m > M$, then

$$\begin{aligned}
\int_{\bigcup\limits_{k=0}^{M} B_k} P_\varepsilon^{m,n+1} f d\mu &= 1 - \int_{\bigcup\limits_{k=M+1}^{\infty} B_k} P_\varepsilon^{m,n+1} f d\mu \\
&= 1 - \int_{\bigcup\limits_{k=M+1}^{\infty} B_k} \left((1-\varepsilon) P^{n,n+1} + \varepsilon R^{n,n+1} \right) \left(P_\varepsilon^{m,n} f \right) d\mu \\
&\leq 1 - \varepsilon \int_{\bigcup\limits_{k=M+1}^{\infty} B_k} R^{n,n+1} \left(P_\varepsilon^{m,n} f \right) d\mu = 1 - \varepsilon.
\end{aligned}$$

It follows that $\mathscr{S}_{uas}^c$ is $d_{n.\sup}$ dense in $\mathscr{S}$.

To show that $Int \mathscr{S}_{uas}^c \neq \emptyset$ for the sup norm topology (i.e., in $d_{n.\sup}$) consider the open ball

$$K(\mathbf{R}, 1) := \{ \mathbf{T} \in \mathscr{S} : d_{n.\sup}(\mathbf{T}, \mathbf{R}) < 1 \},$$

where $\mathbf{R}$ is defined as before. If $\mathbf{T} \in K(\mathbf{R}, 1)$, then for some $0 < \varepsilon < 1$

$$\sup_{f \in \mathcal{D}} \left\| T^{m,m+1} f - R^{m,m+1} f \right\|_1 \leq d_{n.\sup}(\mathbf{T}, \mathbf{R}) = 1 - \varepsilon.$$

Hence for every $m + 1 > M$ and every $f \in \mathcal{D}$,

$$\int_{\bigcup\limits_{k=0}^{M} B_k} T^{0,m+1} f d\mu \leq d_{n.\sup}(\mathbf{T}, \mathbf{R}) = 1 - \varepsilon.$$

Thus,

$$\sup_{M \in \mathbb{N}} \limsup_{m \to \infty} \int_{\bigcup\limits_{k=0}^{M} B_k} T^{0,m+1} f d\mu \leq d_{n.\sup}(\mathbf{T}, \mathbf{R}) = 1 - \varepsilon < 1,$$

and therefore $\mathbf{T}$ has no "invariant" densities (common limiting density). It follows that $\mathbf{T} \in \mathscr{S}_{uas}^c$. ■

In the next result we shall see that topologies on $\mathscr{S}$ generated by $d_{n.\sup}$ and $d_{n.\sum}$ differ. Namely $\mathscr{S}_{uas}$ is large for $d_{n.\sum}$. In fact, we have

Proposition 2. *The set $\mathscr{S}_{uas}$ is $\sum$ norm topology dense in $\mathscr{S}$ (i.e., in $d_{n.\sum}$).*

Proof. Let $\mathbf{P} \in \mathscr{S}$ and $0 < \varepsilon < 1$ be taken arbitrarily. There exists $M \in \mathbb{N}_0$ such that $\frac{1}{2^M} < \varepsilon$. Define $\mathbf{P}_\varepsilon \in \mathscr{S}$ as follows:

$$P_\varepsilon^{m,m+1} = \begin{cases} P^{m,m+1}, & \text{if } m \leq M, \\ E, & \text{if } m > M, \end{cases}$$

where $Ef = (\int_X f d\mu)g$ for some fixed $g \in \mathcal{D}$ and any $f \in L^1(\mu)$. Obviously, $\mathbf{E} = (E^{m,m+1})_{m\geq 0} \in \mathscr{S}$, where for every $m \in \mathbb{N}_0$, $E^{m,m+1} = E$. Then for every $m \in \mathbb{N}_0$

$$\lim_{n\to\infty} |||P_\varepsilon^{m,n} - E||| = 0.$$

Therefore $\mathbf{P}_\varepsilon$ is uniformly asymptotically stable. Clearly,

$$d_{n.\sum}(\mathbf{P}, \mathbf{P}_\varepsilon) = \sum_{m=M+1}^{\infty} \frac{1}{2^{m+1}} \left|\left|\left|P^{m,m+1} - E\right|\right|\right| \leq \frac{1}{2^M} < \varepsilon,$$

which completes the proof. ■

We will now discuss a weaker case of asymptotic stability of chains of Markov operators, i.e., almost uniform asymptotic stability. We begin with

Definition. A nonhomogeneous chain of Markov operators $\mathbf{P}$ is said to be almost uniformly asymptotically stable if for every $m \in \mathbb{N}_0$

$$\lim_{n\to\infty} \sup_{f,g\in\mathcal{D}} \|P^{m,n}f - P^{m,n}g\|_1 = 0.$$

The set of all almost uniformly asymptotically stable Markov operators is denoted by $\mathscr{S}_{a.uas}$.

Repeating arguments from [1] or following the proof of Theorem 4.6 in [12] (cf. [8]), we obtain a useful characterization of almost uniformly asymptotically stable nonhomogeneous chains of Markov operators.

Theorem 3. *Let* $\mathbf{P} \in \mathscr{S}$. *If there exists a sequence* $(\lambda_n)_{n\in\mathbb{N}_0}$, $0 \leq \lambda_n < 1$, *satisfying*

$$\sum_{n=0}^{\infty} \lambda_n = \infty$$

and such that for every f, $g \in \mathcal{D}$ *we have*

$$\left\|P^{n,n+1}f \wedge P^{n,n+1}g\right\|_1 \geq \lambda_n \qquad \textit{for all } n \in \mathbb{N}_0,$$

then $\mathbf{P}$ *is almost uniformly asymptotically stable* (*here* $\wedge$ *stands for the ordinary minimum in* $L^1(\mu)$).

Almost uniform asymptotic stability means that the influence of the state of the process at the time m on its future states decreases (uniformly) to zero with the passage of time. Thus, in the case of nonhomogeneous chains of Markov operators this property is essentially weaker than the uniform asymptotic stability which additionally claims the existence of a "stationary" (common limiting) density. In the class of homogeneous chains of Markov operators notions of uniform asymptotic stability and almost uniform asymptotic stability coincide. Indeed, if there exists $\varepsilon > 0$ such that for some n_0 and every $f, g \in \mathcal{D}$ we have $\|P^{n_0} f \wedge P^{n_0} g\|_1 \geq \varepsilon$, then repeating arguments from [1] we obtain that $\|P^{n_0} f - P^{n_0} g\|_1 \leq (1-\varepsilon)\|f-g\|_1$ and we conclude that the mapping P is a strict contraction. Applying the Banach fixed point theorem there exists a unique P - invariant density $f_* \in \mathcal{D}$ such that $\lim_{n\to\infty} \|P^n f - f_*\|_1 = 0$, where $f \in \mathcal{D}$ is arbitrary. It follows that

$$\sup_{f\in\mathcal{D}} \|P^n f - f_*\|_1 \leq (1-\varepsilon)^{\left\lfloor \frac{n}{n_0} \right\rfloor} \cdot \|f - f_*\| \leq 2\,(1-\varepsilon)^{\left\lfloor \frac{n}{n_0} \right\rfloor} \to 0$$

uniformly for $f \in \mathcal{D}$. Hence in the class of homogeneous chains of Markov operators

$$\mathscr{S}_{a.uas} = \mathscr{S}_{uas} = \{\mathbf{P} \in \mathscr{S} : \exists_n \ \exists_{\varepsilon>0} \ \forall_{f,g\in\mathcal{D}} \ \|P^n f \wedge P^n g\|_1 \geq \varepsilon\},$$

which implies that in the homogeneous case the set $\mathscr{S}_{a.uas}$ is norm open.

The following theorem states that almost uniformly asymptotically stable nonhomogeneous chains of Markov operators are generic. Its proof may be partially derived from Theorem 3, but for the convenience of the reader we give it in full detail.

Theorem 4. *$\mathscr{S}_{a.uas}$ is a dense G_δ subset of $\mathscr{S}$ in both* sup *norm and $\sum$ norm topologies (i.e., in $d_{n.\sup}$ and $d_{n.\sum}$ respectively).*

Proof. First we will show that $\mathscr{S}_{a.uas}$ is a $d_{n.\sup}$ dense subset of $\mathscr{S}$ (the denseness in the metric $d_{n.\sum}$ follows from the fact that $d_{n.\sup}$ is stronger than $d_{n.\sum}$). To this end, given an arbitrary $\mathbf{P} \in \mathscr{S}$ and $0 < \varepsilon < 1$, consider a convex combination

$$P_\varepsilon^{m,m+1} = (1-\varepsilon)\,P^{m,m+1} + \varepsilon E,$$

where as before $\mathbf{E} = (E^{m,m+1})_{m\geq 0} \in \mathscr{S}$ is such that $E^{m,m+1} = E$ for every $m \in \mathbb{N}_0$ and $Ef = (\int_X f d\mu) g$ for some fixed $g \in \mathcal{D}$ and any $f \in L^1(\mu)$. Clearly, $\mathbf{P}_\varepsilon \in \mathscr{S}$ and $d_{n.\sup}(\mathbf{P}, \mathbf{P}_\varepsilon) < 2\varepsilon$. To prove that $\mathbf{P}_\varepsilon$ is almost uniformly asymptotically stable notice that for any densities f and $g \in \mathcal{D}$ we have

$$\begin{aligned}\left\|P_\varepsilon^{n-1,n} f - P_\varepsilon^{n-1,n} g\right\|_1 &= (1-\varepsilon)\left\|P^{n-1,n} f - P^{n-1,n} g\right\|_1 \\ &= (1-\varepsilon)\left\|P^{n-1,n}(f-g)\right\|_1 \\ &\leq (1-\varepsilon)\,\|f-g\|_1\end{aligned}$$

and therefore

$$\begin{aligned}\left\|P_\varepsilon^{m,n}f - P_\varepsilon^{m,n}g\right\|_1 &= \left\|P^{n-1,n}\left(P^{m,n-1}f - P^{m,n-1}g\right)\right\|_1 \\ &\le (1-\varepsilon)\left\|P^{m,n-1}f - P^{m,n-1}g\right\|_1 .\end{aligned}$$

Iterating the last inequality for any $f, g \in \mathcal{D}$ we have

$$\left\|P_\varepsilon^{m,n}f - P_\varepsilon^{m,n}g\right\|_1 \le (1-\varepsilon)^{n-m}\left\|f-g\right\|_1 .$$

Hence

$$\left\|P_\varepsilon^{m,n}f - P_\varepsilon^{m,n}g\right\|_1 \le 2(1-\varepsilon)^{n-m}$$

for any $f, g \in \mathcal{D}$. Thus,

$$\sup_{f,g\in\mathcal{D}}\left\|P_\varepsilon^{m,n}f - P_\varepsilon^{m,n}g\right\|_1 \le 2\,(1-\varepsilon)^{n-m} .$$

Therefore,

$$\lim_{n\to\infty}\sup_{f,g\in\mathcal{D}}\left\|P_\varepsilon^{m,n}f - P_\varepsilon^{m,n}g\right\|_1 = 0$$

and the denseness of the set $\mathscr{S}_{a.uas}$ in $\mathscr{S}$ is proved for both sup norm and $\sum$ norm topologies.

To show G_δ-ness of $\mathscr{S}_{a.uas}$ observe that

$$\begin{aligned}\left\|P^{m,n+1}f - P^{m,n+1}g\right\|_1 &= \left\|P^{n,n+1}\left(P^{m,n}f\right) - P^{n,n+1}\left(P^{m,n}g\right)\right\|_1 \\ &\le \left\|P^{m,n}f - P^{m,n}g\right\|_1 ,\end{aligned}$$

which means that the sequence $\left\|P^{m,n}f - P^{m,n}g\right\|_1$ is nonincreasing. It follows that for the fixed m the sequence $\sup_{f,g\in\mathcal{D}}\left\|P^{m,n}f - P^{m,n}g\right\|_1$ is nonincreasing as well. We obtain that

$$\begin{aligned}\mathscr{S}_{a.uas} &= \left\{\mathbf{P}\in\mathscr{S} : \forall_{m\in\mathbb{N}_0}\lim_{n\to\infty}\sup_{f,g\in\mathcal{D}}\left\|P^{m,n}f - P^{m,n}g\right\|_1 = 0\right\} \\ &= \bigcap_{m=0}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=m+1}^{\infty}\left\{\mathbf{P}\in\mathscr{S} : \sup_{f,g\in\mathcal{D}}\left\|P^{m,n}f - P^{m,n}g\right\|_1 < \frac{1}{k}\right\} .\end{aligned}$$

Note that for fixed $m < n$ the function

$$\mathscr{S} \ni \mathbf{P} \mapsto \sup_{f,g\in\mathcal{D}} \|P^{m,n}f - P^{m,n}g\|_1$$

is $d_{n.\sum}$ continuous. Hence $\mathscr{S}_{a.uas}$ is a G_δ set for the metric $d_{n.\sum}$ (and it is G_δ set for the stronger metric $d_{n.\sup}$). ■

The following example shows that (in contrast to the homogeneous case) in the class of nonhomogeneous chains of Markov operators the set $\mathscr{S}_{a.uas}$ is not open (even though it is a dense G_δ subset of $\mathscr{S}$).

Example 5. As before, since the measure μ is σ-finite, there exists a sequence $\{B_k\}$, $B_k \in \mathcal{A}$, such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $X = \bigcup_{k=0}^{\infty} B_k$, $0 < \mu(B_k) < \infty$ for all $k \in \mathbb{N}_0$. Define

$$S^{m,m+1}f = \tfrac{1}{m+1}\left(\int_X fd\mu\right) g_{m+1} + \left(1 - \tfrac{1}{m+1}\right) \sum_{j=m+1}^{\infty} a_{j-m-1}(f) \cdot g_{j+1}$$

for any $f \in L^1(\mu)$ and any $g_j \in \mathcal{D}$ such that *supp* $g_j \subseteq B_j$ and where $a_j(f) = \int_{B_j} fd\mu$. Applying Theorem 3 we obtain that $\mathbf{S} \in \mathscr{S}_{a.uas}$ as for every f, $h \in \mathcal{D}$ we have $\|S^{m,m+1}f \wedge S^{m,m+1}h\|_1 \geq \frac{1}{m+1}$ and $\sum_{m=0}^{\infty} \frac{1}{m+1} = \infty$. Fix $\varepsilon > 0$ and choose M such that $\frac{2}{M+1} < \varepsilon$. Consider $\mathbf{T} \in \mathscr{S}$ defined for $f \in L^1(\mu)$ as follows:

$$T^{m,m+1}f = \begin{cases} S^{m,m+1}f, & \text{if } m < M, \\ \sum_{j=m+1}^{\infty} a_{j-m-1}(f) \cdot g_{j+1}, & \text{if } m \geq M. \end{cases}$$

Notice that for $m \geq M$ we have $\sup_{f,h\in\mathcal{D}} \|T^{m,n}f - T^{m,n}h\|_1 = 2$ (e.g. take f, $h \in \mathcal{D}$ such that *supp* $f \subseteq B_{M+1}$ and *supp* $h \subseteq B_{M+2}$). Hence $\mathbf{T} \notin \mathscr{S}_{a.uas}$. We easily find that $d_{n.sup}(\mathbf{S}, \mathbf{T}) \leq \frac{2}{m+1} < \varepsilon$. It follows that $\mathscr{S}_{a.uas}$ is not norm open.

3. STRONG OPERATOR TOPOLOGY ASYMPTOTIC STABILITY

This section is dedicated to the study of the asymptotic stability of nonhomogeneous chains of Markov operators in strong operator topology on $\mathscr{S}$. Similar to the previous section, we introduce two types of limit behaviour with the only difference that the mode of convergence is strong. We begin with

Definition. A nonhomogeneous chain of Markov operators $\mathbf{P}$ is called strong asymptotically stable if there exists (a unique) $f_* \in \mathcal{D}$ such that for every $m \in \mathbb{N}_0$

and every $f \in \mathcal{D}$

$$\lim_{n\to\infty} \|P^{m,n} f - f_*\|_1 = 0.$$

The set of all strong asymptotically stable Markov operators is denoted by $\mathscr{S}_{sas}$.

Obviously $\mathscr{S}_{uas} \subseteq \mathscr{S}_{sas}$.

Theorem 6. *The set $\mathscr{S}_{sas}^c$ of all Markov operators which are not strong asymptotically stable is $\sum \sup$ topology dense subset of $\mathscr{S}$ (i.e., in $d_{so.\sup}$). Moreover, $\mathscr{S}_{sas}^c$ contains $\sum\sum$ dense G_δ set (i.e., in $d_{so.\sum}$).*

Proof. Similar arguments to those which were used towards the proof of Theorem 1 imply the first part of the above theorem. Therefore we only prove the second statement.

Let $\{f_0, f_1, \ldots\}$ be a fixed countable and linearly dense subset of $\mathcal{D}$. As before, we find a sequence $\{B_k\}$ such that $B_k \in \mathcal{A}$, $B_i \cap B_j = \emptyset$ for $i \neq j$ and $X = \bigcup_{k=0}^{\infty} B_k$, $0 < \mu(B_k) < \infty$ for all $k \in \mathbb{N}_0$. To see that $\mathscr{S}_{sas}^c$ contains the $\sum\sum$ dense G_δ set observe that

$$\begin{aligned}\mathscr{S}_{sas}^c &\supseteq \left\{\mathbf{P} \in \mathscr{S} : \forall_{t\in\mathbb{N}_0} \forall_{m\in\mathbb{N}_0} \forall_{j\in\mathbb{N}_0} \forall_{\imath\in\mathbb{N}} \forall_{N\in\mathbb{N}_0} \exists_{n>\max\{N,m\}} \sum_{k=0}^{t} \int_{B_k} P^{m,n} f_j d\mu < \frac{1}{\imath}\right\} \\ &= \bigcap_{t=0}^{\infty} \bigcap_{m=0}^{\infty} \bigcap_{j=0}^{\infty} \bigcap_{\imath=1}^{\infty} \bigcap_{N=0}^{\infty} \bigcup_{n>\max\{N,m\}} \left\{\mathbf{P} \in \mathscr{S} : \sum_{k=0}^{t} \int_{B_k} P^{m,n} f_j d\mu < \frac{1}{\imath}\right\}.\end{aligned}$$

Clearly, the mapping

$$\mathscr{S} \ni \mathbf{P} \mapsto \sum_{k=0}^{t} \int_{B_k} P^{m,n} f_j d\mu$$

is $d_{so.\sum}$ continuous, hence the sets $\{\mathbf{P} \in \mathscr{S} : \sum_{k=0}^{t} \int_{B_k} P^{m,n} f_j d\mu < \frac{1}{\imath}\}$ are open in the topology induced by $d_{so.\sum}$. It remains to show that the set

$$\left\{\mathbf{P} \in \mathscr{S} : \forall_{t\in\mathbb{N}_0} \forall_{m\in\mathbb{N}_0} \forall_{j\in\mathbb{N}_0} \forall_{\imath\in\mathbb{N}} \forall_{N\in\mathbb{N}_0} \exists_{n>\max\{N,m\}} \sum_{k=0}^{t} \int_{B_k} P^{m,n} f_j d\mu < \frac{1}{\imath}\right\} \quad (*)$$

is $\sum\sum$ dense. Now then, let $\mathbf{P} \in \mathscr{S}$ and $0 < \varepsilon < 1$ be taken arbitrarily. There exists $m_0 \in \mathbb{N}_0$ such that

$$\sum_{l=0}^{\infty} \sum_{m=m_0+1}^{\infty} \frac{1}{2^{m+l+1}} = \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} \sum_{m=m_0+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^{m_0}} < \frac{\varepsilon}{2}.$$

Let $g_k \in \mathcal{D}$ be such that $supp\ g_k = \{x \in X : g_k(x) \neq 0\} \subseteq B_k$ for any $k \in \mathbb{N}_0$. Consider $\mathbf{P}_\varepsilon \in \mathscr{S}$ defined as follows:

$$P_\varepsilon^{m,m+1} f = \begin{cases} (1-\varepsilon)\, P^{m,m+1} f + \varepsilon \left(\int_X f d\mu\right) g_{m_0+1}, & \text{if } \ 0 \le m \le m_0, \\ \left(\int_X f d\mu\right) g_{m+1}, & \text{if } \ m > m_0 \end{cases}$$

for any $f \in L^1(\mu)$. We notice that

$$\int_{\bigcup\limits_{k=0}^{t} B_k} P_\varepsilon^{m,n} f d\mu = 0$$

if $n > \max\{m_0 + 1, t\}$, so clearly $\mathbf{P}_\varepsilon = (P_\varepsilon^{m,m+1})_{m \ge 0}$ is an element of the considered subset $(*)$. Moreover, we have

$$\begin{aligned}
&d_{so.\sum}(\mathbf{P}_\varepsilon, \mathbf{P}) \\
&= \sum_{m,l=0}^{\infty} \frac{1}{2^{m+l+1}} \left\| P_\varepsilon^{m,m+1} f_l - P^{m,m+1} f_l \right\|_1 \\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{m_0} \frac{1}{2^{m+l+1}} \left\| P_\varepsilon^{m,m+1} f_l - P^{m,m+1} f_l \right\|_1 \\
&\qquad + \sum_{l=0}^{\infty} \sum_{m=m_0+1}^{\infty} \frac{1}{2^{m+l+1}} \left\| P_\varepsilon^{m,m+1} f_l - P^{m,m+1} f_l \right\|_1 \\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{m_0} \frac{1}{2^{m+l+1}} \left\| (1-\varepsilon) P^{m,m+1} f_l + \varepsilon \left(\int_X f_l d\mu \right) g_{m_0+1} - P^{m,m+1} f_l \right\|_1 \\
&\qquad + \sum_{l=0}^{\infty} \sum_{m=m_0+1}^{\infty} \frac{1}{2^{m+l+1}} \left\| \left(\int_X f_l d\mu \right) g_{m+1} - P^{m,m+1} f_l \right\|_1 \\
&= \varepsilon \sum_{l=0}^{\infty} \sum_{m=0}^{m_0} \frac{1}{2^{m+l+1}} \left\| g_{m_0+1} - P^{m,m+1} f_l \right\|_1 \\
&\qquad + \sum_{l=0}^{\infty} \sum_{m=m_0+1}^{\infty} \frac{1}{2^{m+l+1}} \left\| g_{m+1} - P^{m,m+1} f_l \right\|_1 \\
&\le \varepsilon \sum_{l=0}^{\infty} \sum_{m=0}^{m_0} \frac{1}{2^{m+l+1}} \cdot 2 + \sum_{l=0}^{\infty} \sum_{m=m_0+1}^{\infty} \frac{1}{2^{m+l+1}} \cdot 2 \\
&< \varepsilon \sum_{l=0}^{\infty} \sum_{m=0}^{m_0} \frac{1}{2^{m+l}} + \frac{\varepsilon}{2} \cdot 2 = \varepsilon \left(1 - \frac{1}{2^{m_0}} \right) \sum_{l=0}^{\infty} \frac{1}{2^l} + \varepsilon < 2\varepsilon + \varepsilon = 3\varepsilon,
\end{aligned}$$

which completes the proof. ∎

On the other hand, the fact that $\mathscr{S}_{uas} \subseteq \mathscr{S}_{sas}$ and the Proposition 2 lead to the following

Proposition 7. *The set $\mathscr{S}_{sas}$ is $\sum\sum$ topology dense in $\mathscr{S}$ (i.e., in $d_{so.\sum}$).*

Let us proceed with

Definition. A nonhomogeneous chain of Markov operators $\mathbf{P}$ is called strong almost asymptotically stable if for every $m \in \mathbb{N}_0$ and f, $g \in \mathcal{D}$

$$\lim_{n\to\infty} \|P^{m,n}f - P^{m,n}g\|_1 = 0.$$

The set of all strong almost asymptotically stable Markov operators is denoted by $\mathscr{S}_{a.sas}$.

Clearly, $\mathscr{S}_{a.uas} \subset \mathscr{S}_{a.sas}$. It should be emphasized that unlike the uniform case, in the class of homogeneous chains of Markov operators notions of strong asymptotic stability and strong almost asymptotic stability are essentially different (cf. [4]).

We easily obtain that strong almost asymptotically stable nonhomogeneous chains of Markov operators are generic.

Theorem 8. *The set $\mathscr{S}_{a.sas}$ is a dense G_δ subset of $\mathscr{S}$ in both $\sum\sup$ and $\sum\sum$ strong operator topologies (i.e., in $d_{so.\sup}$ and $d_{so.\sum}$ respectively).*

Proof. It remains to show the G_δ-ness of $\mathscr{S}_{a.sas}$. Let $\{f_0, f_1, \ldots\}$ be a fixed countable and linearly dense subset of $\mathcal{D}$. Notice that

$$\begin{aligned}\mathscr{S}_{s.aas} &= \left\{\mathbf{P} \in \mathscr{S} : \forall_{m\in\mathbb{N}_0}\, \forall_{i\in\mathbb{N}_0}\, \forall_{j\in\mathbb{N}_0}\, \lim_{n\to\infty} \|P^{m,n}f_i - P^{m,n}f_j\|_1 = 0\right\}\\ &= \bigcap_{m=0}^{\infty}\bigcap_{i=0}^{\infty}\bigcap_{j=0}^{\infty}\bigcap_{\imath=1}^{\infty}\bigcap_{N=0}^{\infty}\bigcup_{n>\max\{N,m\}} \left\{\mathbf{P} \in \mathscr{S} : \|P^{m,n}f_i - P^{m,n}f_j\|_1 < \frac{1}{\imath}\right\}.\end{aligned}$$

Observe that the sequence $n \mapsto \|P^{m,n}f_i - P^{m,n}f_j\|_1$ is nonincreasing and that for fixed $m < n$ the function $\mathscr{S} \ni \mathbf{P} \mapsto \|P^{m,n}f_i - P^{m,n}f_j\|_1$ is continuous for the metric $d_{so.\sum}$. Hence $\mathscr{S}_{a.sas}$ is a G_δ set for the metric $d_{so.\sum}$ (and it is G_δ set for the stronger metric $d_{so.\sup}$ as well). ■

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