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SOME RESULTS IN BIPOLAR-VALUED FUZZY ORDERED $\mathcal{AG}$-GROUPOIDS

Faisal

Department of Mathematics
COMSATS Institute of Information Technology, Attock, Pakistan
e-mail: yousafzaimath@yahoo.com

Naveed Yaqoob

Department of Mathematics
Quaid-i-Azam University, Islamabad, Pakistan
e-mail: nayaqoob@ymail.com

and

Arsham Borumand Saeid

Department of Mathematics
Shahid Bahonar University of Kerman, Kerman, Iran
e-mail: arsham@mail.uk.ac.ir

Abstract

In this paper, we introduce the concept of bipolar-valued fuzzification of ordered $\mathcal{AG}$-groupoids and discuss some structural properties of bipolar-valued fuzzy two-sided ideals of an intra-regular ordered $\mathcal{AG}$-groupoid.

Keywords: Ordered $\mathcal{AG}$-groupoid, intra-regular ordered $\mathcal{AG}$-groupoid, bipolar-valued fuzzy two-sided ideal, (strong) negative s-cut, (strong) positive t-cut.

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1. Introduction

A fuzzy subset f of a set S is an arbitrary mapping $f : S \to [0, 1]$, where $[0, 1]$ is the unit segment of a real line. This fundamental concept of fuzzy set was given by Zadeh [25] in 1965. Fuzzy groups have been first considered by Rosenfeld [17] and fuzzy semigroups by Kuroki [11]. Yaqoob and others [20] applied rough set theory and fuzzy set theory to ordered ternary semigroups.

There are many kinds of extensions in the fuzzy set theory, like intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, etc. Bipolar-valued fuzzy set is another extension of fuzzy set theory. Lee [12] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 indicate that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ assign that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ assign that elements somewhat satisfy the implicit counter-property. The concept of bipolar-valued fuzzification in an LA-semigroup was first introduced by Yaqoob [21]. Also Abdullah [1, 2, 3], Faisal [5, 6, 10], Yaqoob [22, 23, 24] and others added many results to the theory of fuzzy LA-semigroups ($\mathcal{AG}$-groupoid). In [4], Borumand Saeid introduced the concept of bipolar-valued fuzzy BCK/BCI-algebras.

The concept of an Abel-Grassmann's groupoid ($\mathcal{AG}$-groupoid) [8] was first studied by Kazim and Naseeruddin in 1972 and they called it left almost semigroup (LA-semigroup). Holgate called it left invertive groupoid [7]. An $\mathcal{AG}$-groupoid is a groupoid having the left invertive law

$$(ab)c = (cb)a, \tag{1}$$

for all $a, b, c \in S$. In an $\mathcal{AG}$-groupoid, the medial law [8] holds

$$(ab)(cd) = (ac)(bd), \tag{2}$$

for all $a, b, c, d \in S$. In an $\mathcal{AG}$-groupoid S with left identity, the paramedial law [14] holds

$$(ab)(cd) = (dc)(ba), \tag{3}$$

for all $a, b, c, d \in S$. If an $\mathcal{AG}$-groupoid contain a left identity, then by using medial law, the following law [14] holds

$$a(bc) = b(ac), \tag{4}$$

for all $a, b, c \in S$. An $\mathcal{AG}$-groupoid is a non-associative and non-commutative algebraic structure mid way between a groupoid and a commutative semigroup, nevertheless, it posses many interesting properties which we usually find in associative and commutative algebraic structures. The left identity in an $\mathcal{AG}$-groupoid if exists is unique [14]. The connection of a commutative inverse semigroup with an $\mathcal{AG}$-groupoid has been given in [15] as, a commutative inverse semigroup

$(S, \circ)$ becomes an $\mathcal{AG}$-groupoid $(S, \cdot)$ under $a \cdot b = b \circ a^{-1}$, for all $a, b \in S$. An $\mathcal{AG}$-groupoid S with left identity becomes a semigroup $(S, \circ)$ defined as, for all $x, y \in S$, there exists $a \in S$ such that $x \circ y = (xa)y$ [18]. An $\mathcal{AG}$-groupoid is the generalization of a semigroup theory and has vast applications in collaboration with semigroup like other branches of mathematics. An $\mathcal{AG}$-groupoid has wide range of applications in theory of flocks [16].

The concept of an ordered $\mathcal{AG}$-groupoid was first given by Khan and Faisal in [9] which is infect the generalization of an ordered semigroup.

2. Preliminaries and basic definitions

Throughout the paper S will be considered as an ordered $\mathcal{AG}$-groupoid unless otherwise specified.

Definition [9]**.** An ordered $\mathcal{AG}$-groupoid (po-$\mathcal{AG}$-groupoid) is a structure $(S, ., \leq)$ in which the following conditions hold:

(i) $(S, .)$ is an $\mathcal{AG}$-groupoid.

(ii) $(S, \leq)$ is a poset (reflexive, anti-symmetric and transitive).

(iii) For all a, b and $x \in S$, $a \leq b$ implies $ax \leq bx$ and $xa \leq xb$.

Example 1 [9]**.** Consider an open interval $\mathbb{R}_{\mathbb{O}} = (0, 1)$ of real numbers under the binary operation of multiplication. Define $a * b = ba^{-1}r^{-1}$, for all $a, b, r \in \mathbb{R}_{\mathbb{O}}$, then it is easy to see that $(\mathbb{R}_{\mathbb{O}}, *, \leq)$ is an ordered $\mathcal{AG}$-groupoid under the usual order "$\leq$" and we have called it a real ordered $\mathcal{AG}$-groupoid.

For a non-empty subset A of an ordered $\mathcal{AG}$-groupoid S, and for some $a \in A$, we define

$$(A] = \{t \in S \mid t \leq a\}.$$

For $A = \{a\}$, we usually write it as $(a]$.

Definition [9]**.** A non-empty subset A of an ordered $\mathcal{AG}$-groupoid S is called a left (right) ideal of S if

(i) $SA \subseteq A$ $(AS \subseteq A)$.

(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Equivalently, a non-empty subset A of an ordered $\mathcal{AG}$-groupoid S is called a left (right) ideal of S if $(SA] \subseteq A$ $((AS] \subseteq A)$. A non-empty subset A of an ordered $\mathcal{AG}$-groupoid S is called a two sided ideal of S if it is both a left and a right ideal of S.

Definition. A subset A of S is called semiprime if $a^2 \in A$ implies $a \in A$.

Definition [9]**.** An element a of an ordered $\mathcal{AG}$-groupoid S is called intra-regular if there exist $x, y \in S$ such that $a \leq (xa^2)y$, and S is called intra-regular if every element of S is intra-regular or equivalently, $A \subseteq ((SA^2)S]$ for all $A \subseteq S$ and $a \in ((Sa^2)S]$ for all $a \in S$.

Definition. A fuzzy subset f is a class of objects with grades of membership having the form

$$f = \{(x, f(x))/x \in S\}.$$

Definition. A bipolar-valued fuzzy set (briefly, BVF-subset) $\mathcal{B}$ in a non-empty set S is an object having the form

$$\mathcal{B} = \left\{(x, \mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))/x \in S\right\}.$$

Where $\mu^+_{\mathcal{B}} : S \longrightarrow [0,1]$ and $\mu^-_{\mathcal{B}} : S \longrightarrow [-1,0]$.

The positive membership degree $\mu^+_{\mathcal{B}}$ denote the satisfaction degree of an element x to the property corresponding to a BVF-subset $\mathcal{B}$, and the negative membership degree $\mu^-_{\mathcal{B}}$ denotes the satisfaction degree of x to some implicit counter property of BVF-subset $\mathcal{B}$. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different from each other [12, 13].

For the sake of simplicity, we will use the symbol $\mathcal{B} = (\mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}})$ for a BVF-subset $\mathcal{B} = \left\{(x, \mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))/x \in S\right\}$.

Let $\Gamma = \left\{(x, \mathcal{S}^+_\Gamma(x), \mathcal{S}^-_\Gamma(x))/\mathcal{S}^+_\Gamma(x) = 1, \mathcal{S}^-_\Gamma(x) = -1/x \in S\right\} = (\mathcal{S}^+_\Gamma, \mathcal{S}^-_\Gamma)$ be a BVF-subset, then Γ will be carried out in operations with a BVF-subset $\mathcal{B} = (\mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}})$ such that $\mathcal{S}^+_\Gamma$ and $\mathcal{S}^-_\Gamma$ will be used in collaboration with $\mu^+_{\mathcal{B}}$ and $\mu^-_{\mathcal{B}}$, respectively.

Let $x \in S$, then $A_x = \{(y, z) \in S \times S : x \leq yz\}$.

Let $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ and $\mathcal{B} = (\mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}})$ be any two BVF-subsets of an ordered $\mathcal{AG}$-groupoid S, then for some $a, b, c \in S$, the product $\mathcal{A} \circ \mathcal{B}$ is defined by,

$$\left(\mu^+_{\mathcal{A}} \circ \mu^+_{\mathcal{B}}\right)(a) = \begin{cases} \bigvee\limits_{(b,c)\in A_a} \left\{\mu^+_{\mathcal{A}}(b) \wedge \mu^+_{\mathcal{B}}(c)\right\} & \text{if } a \leq bc\,(A_a \neq \emptyset). \\ 0 & \text{otherwise.} \end{cases}$$

$$\left(\mu^-_{\mathcal{A}} \circ \mu^-_{\mathcal{B}}\right)(a) = \begin{cases} \bigwedge\limits_{(b,c)\in A_a} \left\{\mu^-_{\mathcal{A}}(b) \vee \mu^-_{\mathcal{B}}(c)\right\} & \text{if } a \leq bc\ (A_a \neq \emptyset) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be any two BVF-subsets of an ordered $\mathcal{AG}$-groupoid S, then $\mathcal{A} \subseteq \mathcal{B}$ means that

$$\mu_{\mathcal{A}}^{+}(x) \leq \mu_{\mathcal{B}}^{+}(x) \text{ and } \mu_{\mathcal{A}}^{-}(x) \geq \mu_{\mathcal{B}}^{-}(x)$$

for all x in S. Let $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ and $\mathcal{B} = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$ be BVF-subsets of an ordered $\mathcal{AG}$-groupoid S. The symbol $\mathcal{A} \cap \mathcal{B}$ will mean the following BVF-subset of S

$$\begin{aligned}(\mu_{\mathcal{A}}^{+} \cap \mu_{\mathcal{B}}^{+})(x) &= \min\{\mu_{\mathcal{A}}^{+}(x), \mu_{\mathcal{B}}^{+}(x)\} = \mu_{\mathcal{A}}^{+}(x) \wedge \mu_{\mathcal{B}}^{+}(x)\\ (\mu_{\mathcal{A}}^{-} \cup \mu_{\mathcal{B}}^{-})(x) &= \max\{\mu_{\mathcal{A}}^{-}(x), \mu_{\mathcal{B}}^{-}(x)\} = \mu_{\mathcal{A}}^{-}(x) \vee \mu_{\mathcal{B}}^{-}(x),\end{aligned}$$

for all x in S. The symbol $\mathcal{A} \cup \mathcal{B}$ will mean the following BVF-subset of S

$$\begin{aligned}(\mu_{\mathcal{A}}^{+} \cup \mu_{\mathcal{B}}^{+})(x) &= \max\{\mu_{\mathcal{A}}^{+}(x), \mu_{\mathcal{B}}^{+}(x)\} = \mu_{\mathcal{A}}^{+}(x) \vee \mu_{\mathcal{B}}^{+}(x)\\ (\mu_{\mathcal{A}}^{-} \cap \mu_{\mathcal{B}}^{-})(x) &= \min\{\mu_{\mathcal{A}}^{-}(x), \mu_{\mathcal{B}}^{-}(x)\} = \mu_{\mathcal{A}}^{-}(x) \wedge \mu_{\mathcal{B}}^{-}(x),\end{aligned}$$

for all x in S. Let S be an ordered $\mathcal{AG}$-groupoid and let $\emptyset \neq W \subseteq S$, then the bipolar-valued characteristic function $\Omega_W = (\mu_{\Omega_w}^{+}, \mu_{\Omega_w}^{-})$ of W is defined as

$$\mu_{\Omega_w}^{+}(x) = \begin{cases} 1 & \text{if } x \in W \\ 0 & \text{if } x \notin W \end{cases} \quad \text{and} \quad \mu_{\Omega_w}^{-}(x) = \begin{cases} -1 & \text{if } x \in W \\ 0 & \text{if } x \notin W\,. \end{cases}$$

3. Bipolar-valued fuzzy ideals in ordered $\mathcal{AG}$-groupoids

Definition. A BVF-subset $\mathcal{B} = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$ of an ordered $\mathcal{AG}$-groupoid S is called a bipolar-valued fuzzy left ideal of S if

(i) $x \leq y \Rightarrow \mu_{\mathcal{B}}^{+}(x) \geq \mu_{\mathcal{B}}^{+}(y)$ and $\mu_{\mathcal{B}}^{-}(x) \leq \mu_{\mathcal{B}}^{-}(y)$ for all $x, y \in S$.

(ii) $\mu_{\mathcal{B}}^{+}(xy) \geq \mu_{\mathcal{B}}^{+}(y)$ and $\mu_{\mathcal{B}}^{-}(xy) \leq \mu_{\mathcal{B}}^{-}(y)$ for all $x, y \in S$.

Definition. A BVF-subset $\mathcal{B} = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$ of an ordered $\mathcal{AG}$-groupoid S is called a bipolar-valued fuzzy right ideal of S if

(i) $x \leq y \Rightarrow \mu_{\mathcal{B}}^{+}(x) \geq \mu_{\mathcal{B}}^{+}(y)$ and $\mu_{\mathcal{B}}^{-}(x) \leq \mu_{\mathcal{B}}^{-}(y)$ for all $x, y \in S$.

(ii) $\mu_{\mathcal{B}}^{+}(xy) \geq \mu_{\mathcal{B}}^{+}(x)$ and $\mu_{\mathcal{B}}^{-}(xy) \leq \mu_{\mathcal{B}}^{-}(x)$ for all $x, y \in S$.

A BVF-subset $\mathcal{B} = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$ of an ordered $\mathcal{AG}$-groupoid S is called a bipolar-valued fuzzy two-sided ideal of S if it is both a bipolar-valued fuzzy left and a bipolar-valued fuzzy right ideal of S.

Example 2. Let $S = \{1, 2, 3, 4, 5\}$ be an ordered $\mathcal{AG}$-groupoid with left identity 4 with the following multiplication table and order below.

.	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	4	5	3
4	1	2	3	4	5
5	1	2	5	3	4

$$\leq := \{(1,1), (1,2), (2,2), (3,3), (4,4), (5,5)\}$$

It is easy to see that S is intra-regular. Define a BVF-subset $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ of S as follows:

$$\mu_{\mathcal{A}}^{+}(1) = 1,\ \mu_{\mathcal{A}}^{+}(2) = \mu_{\mathcal{A}}^{+}(3) = \mu_{\mathcal{A}}^{+}(4) = \mu_{\mathcal{A}}^{+}(5) = 0,$$

and

$$\mu_{\mathcal{A}}^{-}(1) = -0.6,\ \mu_{\mathcal{A}}^{-}(2) = -0.4 \text{ and } \mu_{\mathcal{A}}^{-}(3) = \mu_{\mathcal{A}}^{-}(4) = \mu_{\mathcal{A}}^{-}(5) = -0.2,$$

then by routine calculation one can easily verify that $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is a bipolar-valued fuzzy two-sided ideal of S.

Let $BVF(S)$ denote the set of all BVF-subsets of an ordered $\mathcal{AG}$-groupoid S.

Theorem 3. *The family of bipolar-valued fuzzy right (left, two-sided) ideals of an ordered $\mathcal{AG}$-groupoid S forms a complete distributive lattice under the ordering of bipolar-valued fuzzy set inclusion $\subset$.*

Proof. Let $\{B_i \mid i \in I\}$ be a family of bipolar-valued fuzzy right ideals of an ordered $\mathcal{AG}$-groupoid S. Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering in $[0, 1]$, it is sufficient to show that $\bigcap B_i = (\bigvee \mu_{\mathcal{B}i}^{+}, \bigwedge \mu_{\mathcal{B}i}^{-})$ is a bipolar-valued fuzzy subalgebra of X. It is clear that if $x \leq y$, then $\mu_{\mathcal{B}}^{+}(x) \geq \mu_{\mathcal{B}}^{+}(y)$ and $\mu_{\mathcal{B}}^{-}(x) \leq \mu_{\mathcal{B}}^{-}(y)$. Also

$$\left(\bigvee \mu_{\mathcal{B}i}^{+}\right)(xy) = \sup\{\mu_{\mathcal{B}i}^{+}(xy) \mid i \in I\} \geq \sup\{\mu_{\mathcal{B}i}^{+}(y) \mid i \in I\} = \bigvee \mu_{\mathcal{B}i}^{+}(y),$$

also we have

$$\left(\bigwedge \mu_{\mathcal{B}i}^{-}\right)(xy) = \inf\{\mu_{\mathcal{B}i}^{-}(xy) \mid i \in I\} \leq \inf\{\mu_{\mathcal{B}i}^{-}(x) \mid i \in I\} = \bigwedge \mu_{\mathcal{B}i}^{-}(y)).$$

Hence $\bigcap B_i = (\bigvee \mu_{\mathcal{B}i}^{+}, \bigwedge \mu_{\mathcal{B}i}^{-})$ is a bipolar-valued fuzzy subalgebra of X. ∎

Definition. Let $B = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$ be a bipolar-valued fuzzy set and $(s, t) \in [-1, 0] \times [0, 1]$. Define:

(1) the sets $B_t^{+} = \{x \in X \mid \mu^{+}(x) \geq t\}$ and $B_s^{-} = \{x \in G \mid \nu^{-}(x) \leq s\}$, which are called positive t-*cut* of $B = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$ and the negative s-*cut* of $B = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$, respectively,

(2) the sets ${}^{>}B_t^{+} = \{x \in X \mid \mu_{\mathcal{B}}^{+}(x) > t\}$ and ${}^{<}B_s^{-} = \{x \in G \mid \mu_{\mathcal{B}}^{-}(x) < s\}$, which are called strong positive t-*cut* of $B = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$ and the strong negative s-*cut* of $B = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$, respectively,

(3) the set $X_B^{(t,s)} = \{x \in X \mid \mu_{\mathcal{B}}^{+}(x) \geq t,\ \mu_{\mathcal{B}}^{-}(x) \leq s\}$ is called an (s, t)-level subset of B,

(4) the set ${}^{S}X_B^{(t,s)} = \{x \in X \mid \mu_{\mathcal{B}}^{+}(x) > t,\ \mu_{\mathcal{B}}^{-}(x) < s\}$ is called a strong (s, t)-level subset of B,

(5) the set of all $(s, t) \in Im(\mu_{\mathcal{B}}^{+}) \times Im(\mu_{\mathcal{B}}^{-})$ is called the image of $B = (\mu^{+}, \nu^{-})$.

Theorem 4. *Let B be a bipolar-valued fuzzy subset of S such that the least upper bound t_0 of $Im(\mu_{\mathcal{B}}^{+})$ and the greatest lower bound s_0 of $Im(\mu_{\mathcal{B}}^{-})$ exist. Then the following condition are equivalent:*

(i) *B is a bipolar-valued fuzzy subalgebra of S,*

(ii) *For all $(s, t) \in Im(\mu_{\mathcal{B}}^{-}) \times Im(\mu_{\mathcal{B}}^{+})$, the non-empty level subset $X_B^{(t,s)}$ of B is a (crisp) subalgebra of S.*

(iii) *For all $(s, t) \in Im(\mu_{\mathcal{B}}^{-}) \times Im(\mu_{\mathcal{B}}^{+}) \setminus (s_0, t_0)$, the non-empty strong level subset ${}^{S}X_B^{(t,s)}$ of B is a (crisp) subalgebra of S.*

(iv) *For all $(s, t) \in [-1, 0] \times [0, 1]$, the non-empty strong level subset ${}^{S}X_B^{(t,s)}$ of B is a (crisp) subalgebra of S.*

(v) *For all $(s, t) \in [-1, 0] \times [0, 1]$, the non-empty level subset $X_B^{(t,s)}$ of B is a (crisp) subalgebra of S.*

Proof. (i)→(iv) Let B be a bipolar-valued fuzzy subalgebra of S, $(s, t) \in [0, 1] \times [0, 1]$ and $x, y \in^{S} X_B^{(t,s)}$. Then we have

$$\mu_{\mathcal{B}}^{+}(xy) \geq \mu_{\mathcal{B}}^{+}(y) \geq t \quad \text{and} \quad \mu_{\mathcal{B}}^{-}(xy) \leq \mu_{\mathcal{B}}^{-}(y) < s,$$

thus $xy \in^{S} X_B^{(t,s)}$. Hence ${}^{S}X_B^{(t,s)}$ is a (crisp) subalgebra of S.

(iv)→(iii) It is clear.

(iii)→(ii) Let $(s,t) \in Im(\mu_{\mathcal{B}}^{+}) \times Im(\mu_{\mathcal{B}}^{-})$. Then $X_B^{(t,s)}$ is nonempty. Since $X_B^{(t,s)} = \bigcap_{t>\beta,s<\alpha}^{S} X_B^{(\beta,\alpha)}$, where $\beta \in Im(\mu_{\mathcal{B}}^{+}) \setminus s_0$ and $\alpha \in Im(\mu_{\mathcal{B}}^{-}) \setminus t_0$. Then by (iii) we get that $X_B^{(t,s)}$ is a (crisp) subalgebra of S.

(ii)→(v) Let $(s,t) \in [0,1] \times [0,1]$ and $X_B^{(t,s)}$ be non-empty. Suppose that $x, y \in X_B^{(t,s)}$. Let $\alpha = \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y)\}$ and $\beta = \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y)\}$. It is clear that $\alpha \geq s$ and $\beta \leq t$. Thus $x, y \in X_B^{(t,s)}$ and $\alpha \in Im(\mu_{\mathcal{B}}^{+})$ and $\beta \in Im(\mu_{\mathcal{B}}^{-})$, by (ii) $X_B^{(\alpha,\beta)}$ is a subalgebra of X, hence $xy \in X_B^{(\alpha,\beta)}$. Then we have

$$\mu_{\mathcal{B}}^{+}(xy) \geq \mu_{\mathcal{B}}^{+}(y) \geq \alpha \geq s \quad \text{and} \quad \mu_{\mathcal{B}}^{-}(xy) \leq \mu_{\mathcal{B}}^{-}(y) \leq \beta \leq t.$$

Therefore $xy \in X_B^{(t,s)}$. Then $X_B^{(t,s)}$ is a (crisp) subalgebra of S.

(v)→(i) Assume that the non-empty set $X_B^{(t,s)}$ is a (crisp) subalgebra of S, for any $(s,t) \in [0,1] \times [0,1]$. In contrary, let $x_0, y_0 \in X$ be such that

$$\mu_{\mathcal{B}}^{+}(x_0y_0) < \mu_{\mathcal{B}}^{+}(y_0) \quad \text{and} \quad \mu_{\mathcal{B}}^{-}(x_0y_0) > \mu_{\mathcal{B}}^{-}(y_0)\}.$$

Let $\mu_{\mathcal{B}}^{+}(y_0) = \beta$, $\mu_{\mathcal{B}}^{+}(x_0y_0) = \lambda$, $\mu_{\mathcal{B}}^{-}(y_0) = \gamma$ and $\mu_{\mathcal{B}}^{-}(x_0y_0) = \nu$. Then $\lambda < \beta$ and $\nu > \gamma$. Put

$$\lambda_1 = \frac{1}{2}(\mu_{\mathcal{B}}^{+}(x_0y_0) + \mu_{\mathcal{B}}^{+}(y_0)) \quad \text{and} \quad \nu_1 = \frac{1}{2}(\mu_{\mathcal{B}}^{-}(x_0y_0) + \mu_{\mathcal{B}}^{-}(y_0)),$$

therefore $\lambda_1 = \frac{1}{2}(\lambda + \beta)$ and $\nu_1 = \frac{1}{2}(\nu + \gamma)$. Hence $\nu > \nu_1 = \frac{1}{2}(\nu + \gamma) > \theta$. Thus

$$\beta > \lambda_1 > \lambda = \mu_{\mathcal{B}}^{+}(x_0y_0) \quad \text{and} \quad \theta < \nu_1 < \nu = \mu_{\mathcal{B}}^{-}(x_0y_0),$$

so that $x_0y_0 \notin X_B^{(\lambda_1,\nu_1)}$. Which is a contradiction, since

$$\mu_{\mathcal{B}}^{+}(y_0) = \beta > \lambda_1 \quad \text{and} \quad \mu_{\mathcal{B}}^{-}(y_0) = \gamma < \nu_1,$$

imply that $x_0, y_0 \in X_B^{(\lambda_1,\nu_1)}$. Thus $\mu_{\mathcal{B}}^{+}(xy) \geq \mu_{\mathcal{B}}^{+}(y)$ and $\mu_{\mathcal{B}}^{-}(xy) \leq \mu_{\mathcal{B}}^{-}(y)$, for all $x, y \in S$. The proof is completed. ■

Theorem 5. *Each subalgebra of X is a level subalgebra of a bipolar-valued fuzzy subalgebra of X.*

Proof. Let Y be a subalgebra of S and B be a bipolar-valued fuzzy subset of S which is defined by:

$$\mu_{\mathcal{B}}^{+}(x) = \begin{cases} \alpha & \text{if } x \in Y \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \mu_{\mathcal{B}}^{-}(x) = \begin{cases} \beta & \text{if } x \in Y \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in [0,1]$ and $\beta \in [-1,0]$. It is clear that $X_B^{(t,s)} = Y$. Let $x, y \in X$. We consider the following cases:

Case 1. If $x, y \in Y$, then $xy \in Y$, therefore $\mu_{\mathcal{B}}^{+}(xy) = \alpha = \mu_{\mathcal{B}}^{+}(y)$ and $\mu_{\mathcal{B}}^{-}(xy) = \beta\mu_{\mathcal{B}}^{-}(y)$.

Case 2. If $x, y \notin Y$, then $0 = \mu_{\mathcal{B}}^{+}(y)$ and $0 = \mu_{\mathcal{B}}^{-}(y)$ and so $\mu_{\mathcal{B}}^{+}(xy) \geq 0 = \mu_{\mathcal{B}}^{+}(y)$ and $\mu_{\mathcal{B}}^{-}(xy) \leq 0 = \mu_{\mathcal{B}}^{-}(y)$.

Case 3. If $x \in Y$ and $y \notin Y$, then $\mu_{\mathcal{B}}^{+}(y) = 0 = \mu_{\mathcal{B}}^{-}(y)$. Thus $\mu_{\mathcal{B}}^{+}(xy) \geq 0 = \mu_{\mathcal{B}}^{+}(y)$ and $\mu_{\mathcal{B}}^{-}(xy) \leq 0 = \mu_{\mathcal{B}}^{-}(y)$.

Case 4. If $x \notin Y$ and $y \in Y$, then by the same argument as in Case 3, we can conclude the results.

Therefore B is a bipolar-valued fuzzy subalgebra of S. ■

Lemma 6. *Let S be an ordered $\mathcal{AG}$-groupoid, then the set $(BVF(S), \circ, \subseteq)$ is an ordered $\mathcal{AG}$-groupoid.*

Proof. Clearly $BVF(S)$ is closed. Let $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$, $\mathcal{B} = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$ and $\mathcal{C} = (\mu_{\mathcal{C}}^{+}, \mu_{\mathcal{C}}^{-})$ be in $BVF(S)$. If $A_x = \emptyset$ for any $x \in S$, then

$$((\mu_{\mathcal{A}}^{+} \circ \mu_{\mathcal{B}}^{+}) \circ \mu_{\mathcal{C}}^{+})(x) = 0 = ((\mu_{\mathcal{C}}^{+} \circ \mu_{\mathcal{B}}^{+}) \circ \mu_{\mathcal{A}}^{+})(x),$$

and

$$((\mu_{\mathcal{A}}^{-} \circ \mu_{\mathcal{B}}^{-}) \circ \mu_{\mathcal{C}}^{-})(x) = 0 = ((\mu_{\mathcal{C}}^{-} \circ \mu_{\mathcal{B}}^{-}) \circ \mu_{\mathcal{A}}^{-})(x).$$

Let $A_x \neq \emptyset$, then there exist y and z in S such that $(y, z) \in A_x$. Therefore by using (1), we have

$$\begin{aligned}
((\mu_{\mathcal{A}}^{+}\circ\mu_{\mathcal{B}}^{+})\circ\mu_{\mathcal{C}}^{+})(x) &= \bigvee_{(y,z)\in A_x}\left\{(\mu_{\mathcal{A}}^{+}\circ\mu_{\mathcal{B}}^{+})(y)\wedge\mu_{\mathcal{C}}^{+}(z)\right\} \\
&= \bigvee_{(y,z)\in A_x}\left\{\bigvee_{(p,q)\in A_y}\left\{\mu_{\mathcal{A}}^{+}(p)\wedge\mu_{\mathcal{B}}^{+}(q)\right\}\wedge\mu_{\mathcal{C}}^{+}(z)\right\} \\
&= \bigvee_{x\leq(pq)z}\left\{\mu_{\mathcal{A}}^{+}(p)\wedge\mu_{\mathcal{B}}^{+}(q)\wedge\mu_{\mathcal{C}}^{+}(z)\right\} \\
&= \bigvee_{x\leq(zq)p}\left\{\mu_{\mathcal{C}}^{+}(z)\wedge\mu_{\mathcal{B}}^{+}(q)\wedge\mu_{\mathcal{A}}^{+}(p)\right\} \\
&= \bigvee_{(w,p)\in A_x}\left\{\bigvee_{(z,q)\in A_w}\left(\mu_{\mathcal{C}}^{+}(z)\wedge\mu_{\mathcal{B}}^{+}(q)\wedge\mu_{\mathcal{A}}^{+}(p)\right)\right\} \\
&= \bigvee_{(w,p)\in A_x}\left\{(\mu_{\mathcal{C}}^{+}\circ\mu_{\mathcal{B}}^{+})(w)\wedge f(p)\right\} \\
&= ((\mu_{\mathcal{C}}^{+}\circ\mu_{\mathcal{B}}^{+})\circ\mu_{\mathcal{A}}^{+})(x)
\end{aligned}$$

and

$$\begin{aligned}
((\mu_{\mathcal{A}}^{-}\circ\mu_{\mathcal{B}}^{-})\circ\mu_{\mathcal{C}}^{-})(x) &= \bigwedge_{(y,z)\in A_x}\left\{(\mu_{\mathcal{A}}^{-}\circ\mu_{\mathcal{B}}^{-})(y)\vee\mu_{\mathcal{C}}^{-}(z)\right\} \\
&= \bigwedge_{(y,z)\in A_x}\left\{\bigwedge_{(p,q)\in A_y}\left\{\mu_{\mathcal{A}}^{-}(p)\vee\mu_{\mathcal{B}}^{-}(q)\right\}\vee\mu_{\mathcal{C}}^{-}(z)\right\} \\
&= \bigwedge_{x\leq(pq)z}\left\{\mu_{\mathcal{A}}^{-}(p)\vee\mu_{\mathcal{B}}^{-}(q)\vee\mu_{\mathcal{C}}^{-}(z)\right\} \\
&= \bigwedge_{x\leq(zq)p}\left\{\mu_{\mathcal{C}}^{-}(z)\vee\mu_{\mathcal{B}}^{-}(q)\vee\mu_{\mathcal{A}}^{-}(p)\right\} \\
&= \bigwedge_{(w,p)\in A_x}\left\{\bigwedge_{(z,q)\in A_w}\left(\mu_{\mathcal{C}}^{-}(z)\vee\mu_{\mathcal{B}}^{-}(q)\vee\mu_{\mathcal{A}}^{-}(p)\right)\right\} \\
&= \bigwedge_{(w,p)\in A_x}\left\{(\mu_{\mathcal{C}}^{-}\circ\mu_{\mathcal{B}}^{-})(w)\vee\mu_{\mathcal{A}}^{-}(p)\right\} \\
&= ((\mu_{\mathcal{C}}^{-}\circ\mu_{\mathcal{B}}^{-})\circ\mu_{\mathcal{A}}^{-})(x).
\end{aligned}$$

Hence $(BVF(S),\circ)$ is an $\mathcal{AG}$-groupoid.

Assume that $\mathcal{A} \subseteq \mathcal{B}$, then $\mu^+_{\mathcal{A}} \subseteq \mu^+_{\mathcal{B}}$ and $\mu^-_{\mathcal{A}} \supseteq \mu^-_{\mathcal{B}}$. Let $A_x = \emptyset$ for any $x \in S$, then

$$(\mu^+_{\mathcal{A}} \circ \mu^+_{\mathcal{C}})(x) = 0 = (\mu^+_{\mathcal{B}} \circ \mu^+_{\mathcal{C}})(x) \Longrightarrow \mu^+_{\mathcal{A}} \circ \mu^+_{\mathcal{C}} \subseteq \mu^+_{\mathcal{B}} \circ \mu^+_{\mathcal{C}},$$

and

$$(\mu^-_{\mathcal{A}} \circ \mu^-_{\mathcal{C}})(x) = 0 = (\mu^-_{\mathcal{B}} \circ \mu^-_{\mathcal{C}})(x) \Longrightarrow \mu^-_{\mathcal{A}} \circ \mu^-_{\mathcal{C}} \supseteq \mu^-_{\mathcal{B}} \circ \mu^-_{\mathcal{C}},$$

thus we get $\mathcal{A} \circ \mathcal{C} \subseteq \mathcal{B} \circ \mathcal{C}$. Similarly we can show that $\mathcal{C} \circ \mathcal{A} \subseteq \mathcal{C} \circ \mathcal{B}$. Let $A_x \neq \emptyset$, then there exist y and z in S such that $(y, z) \in A_x$, therefore

$$(\mu^+_{\mathcal{A}} \circ \mu^+_{\mathcal{C}})(x) = \bigvee_{(y,z)\in A_x} \{\mu^+_{\mathcal{A}}(y) \vee \mu^+_{\mathcal{C}}(z)\} \leq \bigvee_{(y,z)\in A_x} \{\mu^+_{\mathcal{B}}(y) \vee \mu^+_{\mathcal{C}}(z)\} = (\mu^+_{\mathcal{B}} \circ \mu^+_{\mathcal{C}})(x),$$

and

$$(\mu^-_{\mathcal{A}} \circ \mu^-_{\mathcal{C}})(x) = \bigwedge_{(y,z)\in A_x} \{\mu^-_{\mathcal{A}}(y) \vee \mu^-_{\mathcal{C}}(z)\} \geq \bigwedge_{(y,z)\in A_x} \{\mu^-_{\mathcal{B}}(y) \vee \mu^-_{\mathcal{C}}(z)\} = (\mu^-_{\mathcal{B}} \circ \mu^-_{\mathcal{C}})(x),$$

thus we get $\mathcal{A} \circ \mathcal{C} \subseteq \mathcal{B} \circ \mathcal{C}$. Similarly we can show that $\mathcal{C} \circ \mathcal{A} \subseteq \mathcal{C} \circ \mathcal{B}$. It is easy to see that $BVF(S)$ is a poset. Thus $(BVF(S), \circ, \subseteq)$ is an ordered $\mathcal{AG}$-groupoid. ■

Lemma 7. *For any subset A of an ordered $\mathcal{AG}$-groupoid S, the following properties holds.*

(i) *A is an ordered AG-subgroupoid of S if and only if $\Omega_A = (\mu^+_{\Omega_A}, \mu^-_{\Omega_A})$ is a bipolar-valued fuzzy ordered AG-subgroupoid of S.*

(ii) *A is left (right, two-sided) ideal of S if and only if $\Omega_A = (\mu^+_{\Omega_A}, \mu^-_{\Omega_A})$ is a bipolar-valued fuzzy left (right, two-sided) ideal of S.*

(iii) *For any non-empty subsets A and B of an ordered $\mathcal{AG}$-groupoid S, $\Omega_A \circ \Omega_B = \Omega_{(AB]}$ holds.*

Proof. The proof is straightforward. ■

Definition. A BVF-subset $\mathcal{B} = (\mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}})$ of an ordered $\mathcal{AG}$-groupoid S is said to be idempotent if $\mu^+_{\mathcal{B}} \circ \mu^+_{\mathcal{B}} = \mu^+_{\mathcal{B}}$ and $\mu^-_{\mathcal{B}} \circ \mu^-_{\mathcal{B}} = \mu^-_{\mathcal{B}}$, that is, $\mathcal{B} \circ \mathcal{B} = \mathcal{B}$ or $\mathcal{B}^2 = \mathcal{B}$.

Definition. A BVF-subset $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ of an ordered $\mathcal{AG}$-groupoid S is called bipolar-valued fuzzy semiprime if $\mu^+_{\mathcal{A}}(a) \geq \mu^+_{\mathcal{A}}(a^2)$ and $\mu^-_{\mathcal{A}}(a) \leq \mu^-_{\mathcal{A}}(a^2)$ for all a in S.

Example 8. Let us consider an ordered $\mathcal{AG}$-groupoid $S = \{a, b, c, d, e\}$ with left identity d in the following Cayley's table and order below.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

$$\leq := \{(a, a), (a, b), (a, c), (a, e), (b, b), (c, c), (d, d), (e, e)\}.$$

Let us define a BVF-subset $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ of S as follows:

$$\mu^+_{\mathcal{A}}(a) = 0.2,\ \mu^+_{\mathcal{A}}(b) = 0.5,\ \mu^+_{\mathcal{A}}(c) = 0.6,\ \mu^+_{\mathcal{A}}(d) = 0.1 \text{ and } \mu^+_{\mathcal{A}}(e) = 0.4,$$

and

$$\mu^-_{\mathcal{A}}(a) = -0.5,\ \mu^-_{\mathcal{A}}(b) = -0.8, \mu^-_{\mathcal{A}}(c) = -0.6,\ \mu^-_{\mathcal{A}}(d) = -0.4 \text{ and } \mu^-_{\mathcal{A}}(e) = -0.2,$$

by routine calculations, it is easy to see that $\mathcal{A}$ is bipolar-valued fuzzy semiprime.

Lemma 9. *Every right (left, two-sided) ideal of an ordered $\mathcal{AG}$-groupoid S is semiprime if and only if their characteristic functions are bipolar-valued fuzzy semiprime.*

Proof. Let R be any right ideal of an ordered $\mathcal{AG}$-groupoid S, then by Lemma 7, the bipolar-valued characteristic function of R, that is, $\Omega_R = (\mu^+_{\Omega_R}, \mu^-_{\Omega_R})$ is a bipolar-valued fuzzy right ideal of S. Let $a^2 \in R$, then $\mu^+_{\Omega_R}(a^2) = 1$ and assume that R is semiprime, then $a \in R$, which implies that $\mu^+_{\Omega_R}(a) = 1$. Thus we get $\mu^+_{\Omega_R}(a^2) = \mu^+_{\Omega_R}(a)$ and similarly we can show that $\mu^-_{\Omega_R}(a^2) = \mu^-_{\Omega_R}(a)$, therefore $\Omega_R = (\mu^+_{\Omega_R}, \mu^-_{\Omega_R})$ is a bipolar-valued fuzzy semiprime. The converse is simple. The same holds for left and two-sided ideal of S. ■

Corollary 10. *Let S be an ordered $\mathcal{AG}$-groupoid, then every right (left, two-sided) ideal of S is semiprime if every bipolar-valued fuzzy right (left, two-sided) ideal of S is a bipolar-valued fuzzy semiprime.*

Lemma 11. *Every bipolar-valued fuzzy right ideal of an ordered $\mathcal{AG}$-groupoid S with left identity is a bipolar-valued fuzzy left ideal of S.*

Proof. Assume that S is an ordered $\mathcal{AG}$-groupoid with left identity and let $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ be a bipolar-valued fuzzy right ideal of S, then by using (1), we have

$$\mu_{\mathcal{A}}^{+}(ab) = \mu_{\mathcal{A}}^{+}((ea)b) = \mu_{\mathcal{A}}^{+}((ba)e) \geq \mu_{\mathcal{A}}^{+}(b).$$

Similarly we can show that $\mu_{\mathcal{A}}^{-}(ab) \leq \mu_{\mathcal{A}}^{-}(b)$, which show that $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is a bipolar-valued fuzzy left ideal of S. ∎

The converse of above is not true in general.

Example 12. Consider the Cayley's table and order of Example 8 and define a BVF-subset $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ of S as follows:

$$\mu_{\mathcal{A}}^{+}(a) = 0.8,\ \mu_{\mathcal{A}}^{+}(b) = 0.5,\ \mu_{\mathcal{A}}^{+}(c) = 0.4,\ \mu_{\mathcal{A}}^{+}(d) = 0.3 \text{ and } \mu_{\mathcal{A}}^{+}(e) = 0.6,$$

and

$$\mu_{\mathcal{A}}^{-}(a) = -0.9,\ \mu_{\mathcal{A}}^{-}(b) = -0.5, \mu_{\mathcal{A}}^{-}(c) = -0.4,\ \mu_{\mathcal{A}}^{-}(d) = -0.1 \text{ and } \mu_{\mathcal{A}}^{-}(e) = -0.7,$$

then it is easy to observe that $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is a bipolar-valued fuzzy left ideal of S but it is not a bipolar-valued fuzzy right ideal of S, because $\mu_{\mathcal{A}}^{+}(bd) \ngeq \mu_{\mathcal{A}}^{+}(b)$ and $\mu_{\mathcal{A}}^{-}(bd) \nleq \mu_{\mathcal{A}}^{-}(b)$.

The proof of following Lemma is same as in [19].

Lemma 13. *In S, the following are true.*

(i) $A \subseteq (A]$ *for all* $A \subseteq S$.

(ii) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$.

(iii) $(A]\,(B] \subseteq (AB]$ *for all* $A, B \subseteq S$.

(iv) $(A] = ((A]]$ *for all* $A \subseteq S$.

(vi) $((A]\,(B]] = (AB]$ *for all* $A, B \subseteq S$.

Lemma 14. *Let $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ be a BVF-subset of an intra-regular ordered $\mathcal{AG}$-groupoid S with left identity, then $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is a bipolar-valued fuzzy left ideal of S if and only if $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is a bipolar-valued fuzzy right ideal of S.*

Proof. Assume that S is an intra-regular ordered $\mathcal{AG}$-groupoid with left identity and let $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ be a bipolar-valued fuzzy left ideal of S. Now for $a, b \in S$ there exist $x, y, x', y' \in S$ such that $a \leq (xa^2)y$ and $b \leq (x'b^2)y'$.

Now by using (1), (3) and (4), we have

$$\begin{aligned}\mu_{\mathcal{A}}^{+}(ab) &\geq \mu_{\mathcal{A}}^{+}(((xa^2)y)b) = \mu_{\mathcal{A}}^{+}((by)(x(aa))) = \mu_{\mathcal{A}}^{+}(((aa)x)(yb)) \\ &= \mu_{\mathcal{A}}^{+}(((xa)a)(yb)) = \mu_{\mathcal{A}}^{+}(((xa)(ea))(yb)) = \mu_{\mathcal{A}}^{+}(((ae)(ax))(yb)) \\ &= \mu_{\mathcal{A}}^{+}((a((ae)x))(yb)) = \mu_{\mathcal{A}}^{+}(((yb)((ae)x))a) \geq \mu_{\mathcal{A}}^{+}(a).\end{aligned}$$

Similarly we can get $\mu_{\mathcal{A}}^{-}(ab) \leq \mu_{\mathcal{A}}^{-}(a)$, which implies that $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is a bipolar-valued fuzzy right ideal of S.

Conversely let $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ be a bipolar-valued fuzzy right ideal of S. Now by using (4) and (3), we have

$$\begin{aligned}\mu_{\mathcal{A}}^{+}(ab) &\geq \mu_{\mathcal{A}}^{+}(a((x^{'}b^2)y^{'}) = \mu_{\mathcal{A}}^{+}((x^{'}b^2)(ay^{'})) = \mu_{\mathcal{A}}^{+}((y^{'}a)(b^2x^{'})) \\ &= \mu_{\mathcal{A}}^{+}(b^2((y^{'}a)x)) \geq \mu_{\mathcal{A}}^{+}(b).\end{aligned}$$

In the similar way we can get $\mu_{\mathcal{A}}^{-}(ab) \leq \mu_{\mathcal{A}}^{-}(b)$, which implies that $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is a bipolar-valued fuzzy left ideal of S. ■

Note that a bipolar-valued fuzzy left ideal and a bipolar-valued fuzzy right ideal coincide in an intra-regular ordered $\mathcal{AG}$-groupoid S with left identity.

Lemma 15. *Every bipolar-valued fuzzy two-sided ideal of an intra-regular ordered $\mathcal{AG}$-groupoid S with left identity is a bipolar-valued fuzzy semiprime.*

Proof. Assume that $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is a bipolar-valued fuzzy two-sided ideal of an intra-regular ordered $\mathcal{AG}$-groupoid S with left identity and let $a \in S$, then there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now by using (3) and (4), we have

$$\mu_{\mathcal{A}}^{+}(a) \geq \mu_{\mathcal{A}}^{+}((xa^2)y) = \mu_{\mathcal{A}}^{+}((xa^2)(ey)) = \mu_{\mathcal{A}}^{+}((ye)(a^2x)) = \mu_{\mathcal{A}}^{+}(a^2((ye)x)) \geq \mu_{\mathcal{A}}^{+}(a^2),$$

and similarly

$$\mu_{\mathcal{A}}^{-}(a) \leq \mu_{\mathcal{A}}^{-}((xa^2)y) = \mu_{\mathcal{A}}^{-}((xa^2)(ey)) = \mu_{\mathcal{A}}^{-}((ye)(a^2x)) = \mu_{\mathcal{A}}^{-}(a^2((ye)x)) \leq \mu_{\mathcal{A}}^{-}(a^2).$$

Thus $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is a bipolar-valued fuzzy semiprime. ■

Theorem 16. *Let S be an ordered $\mathcal{AG}$-groupoid with left identity, then the following statements are equivalent.*

(i) *S is an intra-regular.*

(ii) *Every bipolar-valued fuzzy two-sided ideal of S is a bipolar-valued fuzzy semiprime.*

Proof. (i)$\rightarrow$(ii) can be followed by Lemma 15.

(ii)$\rightarrow$(i) Let S be an ordered $\mathcal{AG}$-groupoid with left identity and let every bipolar-valued fuzzy two-sided ideal of S is a bipolar-valued fuzzy semiprime. Since $(a^2S]$ is a two-sided ideal of S [9], therefore by using Corollary 10, $(a^2S]$ is semiprime. Clearly $a^2 \in (a^2S]$ [9], therefore $a \in (a^2S]$. Now by using (1), we have

$$a \in (a^2S] = ((aa)S] = ((Sa)a] \subseteq ((Sa)(a^2S)] = (((a^2S)a)S]$$
$$= (((aS)a^2)S] = ((Sa^2)(aS)] \subseteq ((Sa^2)S].$$

Which shows that S is an intra-regular. ■

Lemma 17. *Let S be an ordered $\mathcal{AG}$-groupoid, then the following holds.*

(i) *A BVF-subset $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ is a bipolar-valued fuzzy ordered AG-subgroupoid of S if and only if $\mu^+_{\mathcal{A}} \circ \mu^+_{\mathcal{A}} \subseteq \mu^+_{\mathcal{A}}$ and $\mu^-_{\mathcal{A}} \circ \mu^-_{\mathcal{A}} \supseteq \mu^-_{\mathcal{A}}$.*

(ii) *A BVF-subset $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ is bipolar-valued fuzzy left (right) ideal of S if and only if $\mathcal{S}^+_\Gamma \circ \mu^+_{\mathcal{A}} \subseteq \mu^+_{\mathcal{A}}$ and $\mathcal{S}^-_\Gamma \circ \mu^-_{\mathcal{A}} \supseteq \mu^-_{\mathcal{A}}$ ($\mu^+_{\mathcal{A}} \circ \mathcal{S}^+_\Gamma \subseteq \mu^+_{\mathcal{A}}$ and $\mu^-_{\mathcal{A}} \circ \mathcal{S}^-_\Gamma \supseteq \mu^-_{\mathcal{A}}$).*

Proof. The proof is straightforward. ■

Theorem 18. *For an ordered $\mathcal{AG}$-groupoid S with left identity, the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *$R \cap L = (RL]$, R is any right ideal and L is any left ideal of S such that R is semiprime.*

(iii) *$\mathcal{A} \cap \mathcal{B} = \mathcal{A} \circ \mathcal{B}$, $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ is any bipolar-valued fuzzy right ideal and $\mathcal{B} = (\mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}})$ is any bipolar-valued fuzzy left ideal of S such that $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ is a bipolar-valued fuzzy semiprime.*

Proof. (i)$\rightarrow$(iii) Assume that S is an intra-regular ordered $\mathcal{AG}$-groupoid. Let $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ be any bipolar-valued fuzzy right ideal and $\mathcal{B} = (\mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}})$ be any bipolar-valued fuzzy left ideal of S. Now for $a \in S$ there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now by using (4), (1) and (3), we have

$$a \leq (x(aa))y = (a(xa))y = (y(xa))a \leq (y(x((xa^2)y)))a = (y((xa^2)(xy)))a$$
$$= (y((yx)(a^2x)))a = (y(a^2((yx)x)))a = (a^2(y((yx)x)))a.$$

Therefore

$$\begin{aligned}(\mu_{\mathcal{A}}^{+} \circ \mu_{\mathcal{B}}^{+})(a) &= \bigvee_{a \leq (a^2(y((yx)x)))a} \{\mu_{\mathcal{A}}^{+}(a^2(y((yx)x))) \wedge \mu_{\mathcal{B}}^{+}(a)\} \\ &\geq \mu_{\mathcal{A}}^{+}(a) \wedge \mu_{\mathcal{B}}^{+}(a) = (\mu_{\mathcal{A}}^{+} \cap \mu_{\mathcal{B}}^{+})(a)\end{aligned}$$

and

$$\begin{aligned}(\mu_{\mathcal{A}}^{-} \circ \mu_{\mathcal{B}}^{-})(a) &= \bigwedge_{a \leq (a^2(y((yx)x)))a} \{\mu_{\mathcal{A}}^{-}(a^2(y((yx)x))) \vee \mu_{\mathcal{B}}^{-}(a)\} \\ &\leq \mu_{\mathcal{A}}^{-}(a) \wedge \mu_{\mathcal{B}}^{-}(a) = (\mu_{\mathcal{A}}^{-} \cup \mu_{\mathcal{B}}^{-})(a).\end{aligned}$$

Which imply that $\mathcal{A} \circ \mathcal{B} \supseteq \mathcal{A} \cap \mathcal{B}$ and by using Lemma 17, $\mathcal{A} \circ \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}$, therefore $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \circ \mathcal{B}$.

(iii)→(ii) Let R be any right ideal and L be any left ideal of an ordered $\mathcal{AG}$-groupoid S, then by Lemma 7, $\Omega_R = (\mu_{\Omega_R}^{+}, \mu_{\Omega_R}^{-})$ and $\Omega_L = (\mu_{\Omega_L}^{+}, \mu_{\Omega_L}^{-})$ are bipolar-valued fuzzy right and bipolar-valued fuzzy left ideals of S respectively. As $(RL] \subseteq R \cap L$ is obvious [9]. Let $a \in R \cap L$, then $a \in R$ and $a \in L$. Now by using Lemma 7 and given assumption, we have

$$\mu_{\Omega_{(RL]}}^{+}(a) = (\mu_{\Omega_R}^{+} \circ \mu_{\Omega_L}^{+})(a) = (\mu_{\Omega_R}^{+} \cap \mu_{\Omega_L}^{+})(a) = \mu_{\Omega_R}^{+}(a) \wedge \mu_{\Omega_L}^{+}(a) = 1,$$

and similarly

$$\mu_{\Omega_{(RL]}}^{-}(a) = (\mu_{\Omega_R}^{-} \circ \mu_{\Omega_L}^{-})(a) = (\mu_{\Omega_R}^{-} \cup \mu_{\Omega_L}^{-})(a) = \mu_{\Omega_R}^{-}(a) \vee \mu_{\Omega_L}^{-}(a) = -1.$$

Which imply that $a \in (RL]$ and therefore $R \cap L = (RL]$. Now by using Corollary 10, R is semiprime.

(ii)→(i) Let S be an ordered $\mathcal{AG}$-groupoid with left identity, then clearly $(Sa]$ is a left ideal of S [9] such that $a \in (Sa]$ and $(a^2S]$ is a right ideal of S such that $a^2 \in (a^2S]$. Since by assumption, $(a^2S]$ is semiprime, therefore $a \in (a^2S]$. Now by using Lemma 13, (3), (1) and (4), we have

$$\begin{aligned}a &\in (a^2S] \cap (Sa] = ((a^2S](Sa]] = (a^2S](Sa] \subseteq ((a^2S)(Sa)] = ((aS)(Sa^2)] \\ &= (((Sa^2)S)a] = (((Sa^2)(eS))a] \subseteq (((Sa^2)(SS))a] = (((SS)(a^2S))a] \\ &= ((a^2((SS)S))a] \subseteq ((a^2S)S] = ((SS)(aa)] = ((aa)(SS)] \subseteq ((aa)S] \\ &= ((Sa)a] \subseteq ((Sa)(a^2S)] = (((a^2S)a)S] = (((aS)a^2)S] \subseteq ((Sa^2)S].\end{aligned}$$

Which shows that S is intra-regular. ■

Theorem 19. *An ordered $\mathcal{AG}$-groupoid S with left identity is intra-regular if and only if for each bipolar-valued fuzzy two-sided ideal $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ of S, $\mathcal{A}(a) = \mathcal{A}(a^2)$ for all a in S.*

Proof. Assume that S is an intra-regular ordered $\mathcal{AG}$-groupoid with left identity and let $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ be a bipolar-valued fuzzy two-sided ideal of S. Let $a \in S$, then there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now by using (3) and (4), we have

$$\begin{aligned}\mu^+_{\mathcal{A}}(a) &\geq \mu^+_{\mathcal{A}}((xa^2)y) = \mu^+_{\mathcal{A}}((xa^2)(ey)) = \mu^+_{\mathcal{A}}((ye)(a^2x)) = \mu^+_{\mathcal{A}}(a^2((ye)x)) \\ &\geq \mu^+_{\mathcal{A}}(a^2) = \mu^+_{\mathcal{A}}(aa) \geq \mu^+_{\mathcal{A}}(a) \Longrightarrow \mu^+_{\mathcal{A}}(a) = \mu^+_{\mathcal{A}}(a^2).\end{aligned}$$

Similarly we can show that $\mu^-_{\mathcal{A}}(a) = \mu^-_{\mathcal{A}}(a^2)$ and therefore $\mathcal{A}(a) = \mathcal{A}(a^2)$ holds for all a in S.

Conversely, assume that for any bipolar-valued fuzzy two-sided ideal $\mathcal{A} = (\mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}})$ of S, $\mathcal{A}(a) = \mathcal{A}(a^2)$ holds for all a in S. As $(a^2S]$ is a two-sided ideal of S with left identity, then by Lemma 7, $\Omega_{(a^2S]} = (\mu^+_{\Omega_{(a^2S]}}, \mu^-_{\Omega_{(a^2S]}})$ is a bipolar-valued fuzzy two-sided ideal of S. Therefore by given assumption and using the fact that $a^2 \in (a^2S]$, we have

$$\mu^+_{\Omega_{(a^2S]}}(a) = \mu^+_{\Omega_{(a^2S]}}(a^2) = 1 \quad \text{and} \quad \mu^-_{\Omega_{(a^2S]}}(a) = \mu^-_{\Omega_{(a^2S]}}(a^2) = -1,$$

which implies that $a \in (a^2S]$. Now by using (4) and (2), we have $a \in ((Sa^2)S]$ and therefore S is intra-regular. ■

Lemma 20. *Every two-sided ideal of an ordered $\mathcal{AG}$-groupoid S with left identity is semiprime.*

Proof. The proof is straightforward. ■

Theorem 21. *Let S be an ordered $\mathcal{AG}$-groupoid with left identity, then the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *Every bipolar-valued fuzzy two-sided ideal of S is idempotent.*

Proof. (i)→(ii) Assume that S is an intra-regular ordered $\mathcal{AG}$-groupoid with left identity and let $a \in S$, then there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now by using (4), (1) and (3), we have

$$a \leq (x(aa))y = (a(xa))y = (y(xa))a = ((ex)(ya))a = ((ay)(xe))a = (((xe)y)a)a.$$

Let $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ be a bipolar-valued fuzzy two-sided ideal of S, then by using Lemma 7, we have $\mu_{\mathcal{A}}^{+} \circ \mu_{\mathcal{A}}^{+} \subseteq \mu_{\mathcal{A}}^{+}$ and also we have

$$(\mu_{\mathcal{A}}^{+} \circ \mu_{\mathcal{A}}^{+})(a) = \bigvee_{a \leq (((xe)y)a)a} \{\mu_{\mathcal{A}}^{+}(((xe)y)a) \wedge \mu_{\mathcal{A}}^{+}(a)\} \geq \mu_{\mathcal{A}}^{+}(a) \wedge \mu_{\mathcal{A}}^{+}(a) = \mu_{\mathcal{A}}^{+}(a).$$

This implies that $\mu_{\mathcal{A}}^{+} \circ \mu_{\mathcal{A}}^{+} \supseteq \mu_{\mathcal{A}}^{+}$ and similarly we can get $\mu_{\mathcal{A}}^{-} \circ \mu_{\mathcal{A}}^{-} \subseteq \mu_{\mathcal{A}}^{-}$. Now by using Lemma 7, $\mu_{\mathcal{A}}^{+} \circ \mu_{\mathcal{A}}^{+} \subseteq \mu_{\mathcal{A}}^{+}$ and $\mu_{\mathcal{A}}^{-} \circ \mu_{\mathcal{A}}^{-} \supseteq \mu_{\mathcal{A}}^{-}$. Thus $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is idempotent.

(ii)→(i) Assume that every two-sided ideal of an ordered $\mathcal{AG}$-groupoid S with left identity is idempotent and let $a \in S$. Since $(a^2S]$ is a two-sided ideal of S, therefore by Lemma 7, its characteristic function $\Omega_{(a^2S]} = (\mu_{\Omega_{(a^2S]}}^{+}, \mu_{\Omega_{(a^2S]}}^{-})$ is a bipolar-valued fuzzy two-sided ideal of S. Since $a^2 \in (a^2S]$ so by Lemma 20 $a \in (a^2S]$ and therefore $\mu_{\Omega_{(a^2S]}}^{+}(a) = 1$ and $\mu_{\Omega_{(a^2S]}}^{-}(a) = -1$. Now by using the given assumption and Lemma 7, we have

$$\mu_{\Omega_{(a^2S]}}^{+} \circ \mu_{\Omega_{(a^2S]}}^{+} = \mu_{\Omega_{(a^2S]}}^{+} \quad \text{and} \quad \mu_{\Omega_{(a^2S]}}^{+} \circ \mu_{\Omega_{(a^2S]}}^{+} = \mu_{\Omega_{((a^2S]]^2}}^{+}.$$

Thus, we have

$$\left(\mu_{\Omega_{((a^2S]]^2}}^{+}\right)(a) = \left(\mu_{\Omega_{(a^2S]}}^{+}\right)(a) = 1$$

and similarly we can get,

$$\left(\mu_{\Omega_{((a^2S]]^2}}^{-}\right)(a) = \left(\mu_{\Omega_{(a^2S]}}^{-}\right)(a) = -1,$$

which imply that $a \in ((a^2S]]^2$. Now by using Lemma 13 and (3), we have

$$a \in ((a^2S]]^2 = (a^2S]^2 = (a^2S](a^2S] \subseteq ((a^2S)(a^2S)] = ((Sa^2)(Sa^2)] \subseteq ((Sa^2)S].$$

Which shows that S is intra-regular. ■

Theorem 22. *For an ordered $\mathcal{AG}$-groupoid S with left identity, the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *$\mathcal{A} = (\Gamma \circ \mathcal{A})^2$, where $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ is any bipolar-valued fuzzy two-sided ideal of S and $\Gamma = (\mathcal{S}_{\Gamma}^{+}, \mathcal{S}_{\Gamma}^{-})$.*

Proof. (i)→(ii) Let S be a an intra-regular ordered $\mathcal{AG}$-groupoid and let $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ be any bipolar-valued fuzzy two-sided ideal of S, then it is easy to see that $\Gamma \circ \mathcal{A}$ is also a bipolar-valued fuzzy two-sided ideal of S. Now by using Theorem 21, $\Gamma \circ \mathcal{A}$ is idempotent and therefore, we have

$$(\Gamma \circ \mathcal{A})^2 = \Gamma \circ \mathcal{A} \subseteq \mathcal{A}.$$

Now let $a \in S$, since S is intra-regular so there exists $x \in S$ such that $a \leq (xa^2)y$. Now by using (4), (3) and (1), we have

$$\begin{aligned} a &\leq (x(aa))y = (a(xa))y \leq (((xa^2)y)(xa))(ey) = (ye)((xa)((xa^2)y)) \\ &= (xa)((ye)((xa^2)y)) = (xa)(((ye)(x(aa)))y) = (xa)(((ye)(a(xa)))y) \\ &= (xa)((a((ye)(xa)))y) = (xa)((y((ye)(xa)))a) \leq (xa)p, \end{aligned}$$

where $p \leq ((y((ye)(xa)))a)$ (by reflexive property) and therefore, we have

$$\begin{aligned} (\mathcal{S}_\Gamma^+ \circ \mu_\mathcal{A}^+)^2(a) &= \bigvee_{a \leq (xa)p} \{(\mathcal{S}_\Gamma^+ \circ \mu_\mathcal{A}^+)(xa) \wedge (\mathcal{S}_\Gamma^+ \circ \mu_\mathcal{A}^+)(p)\} \\ &\geq (\mathcal{S}_\Gamma^+ \circ \mu_\mathcal{A}^+)(xa) \wedge (\mathcal{S}_\Gamma^+ \circ \mu_\mathcal{A}^+)(p) \\ &= \bigvee_{xa \leq xa} \{\mathcal{S}_\Gamma^+(x) \wedge \mu_\mathcal{A}^+(a)\} \wedge \bigvee_{p \leq (y(y(xa)))a} \{\mathcal{S}_\Gamma^+(y((ye)(xa))) \wedge \mu_\mathcal{A}^+(a)\} \\ &\geq \mathcal{S}_\Gamma^+(x) \wedge \mu_\mathcal{A}^+(a) \wedge \mathcal{S}_\Gamma^+(y((ye)(xa))) \wedge \mu_\mathcal{A}^+(a) = \mu_\mathcal{A}^+(a). \end{aligned}$$

Similarly we can get $(\mathcal{S}_\Gamma^- \circ \mu_\mathcal{A}^-)^2(a) \leq \mu_\mathcal{A}^-(a)$, which implies that $(\Gamma \circ \mathcal{A})^2 \supseteq \mathcal{A}$. Thus we get the required $\mathcal{A} = (\Gamma \circ \mathcal{A})^2$.

(ii)$\rightarrow$(i) Let $\mathcal{A} = (\Gamma \circ \mathcal{A})^2$ holds for any bipolar-valued fuzzy two-sided ideal $\mathcal{A} = (\mu_\mathcal{A}^+, \mu_\mathcal{A}^-)$ of S, then by using Lemma 17 and given assumption, we have

$$\mu_\mathcal{A}^+ = (\mathcal{S}_\Gamma^+ \circ \mu_\mathcal{A}^+)^2 \subseteq (\mu_\mathcal{A}^+)^2 = \mu_\mathcal{A}^+ \circ \mu_\mathcal{A}^+ \subseteq S \circ \mu_\mathcal{A}^+ \subseteq \mu_\mathcal{A}^+.$$

Which shows that $\mu_\mathcal{A}^+ = \mu_\mathcal{A}^+ \circ \mu_\mathcal{A}^+$ and similarly $\mu_\mathcal{A}^- = \mu_\mathcal{A}^- \circ \mu_\mathcal{A}^-$, therefore $\mathcal{A} = \mathcal{A} \circ \mathcal{A}$. Thus by using Lemma 21, S is intra-regular. ∎

Lemma 23. *Let S be an intra-regular ordered $\mathcal{AG}$-groupoid with left identity and let $\mathcal{A} = (\mu_\mathcal{A}^+, \mu_\mathcal{A}^-)$ and $\mathcal{B} = (\mu_\mathcal{B}^+, \mu_\mathcal{B}^-)$ be any bipolar-valued fuzzy two-sided ideals of S, then $\mathcal{A} \circ \mathcal{B} = \mathcal{A} \cap \mathcal{B}$.*

Proof. The proof is straightforward. ∎

Lemma 24. *In an intra-regular ordered $\mathcal{AG}$-groupoid S, $\Gamma \circ \mathcal{A} = \mathcal{A}$ and $\mathcal{A} \circ \Gamma = \mathcal{A}$ holds for a BVF-subset $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ of S, where $\Gamma = (S_{\Gamma}^{+}, S_{\Gamma}^{-})$.*

Proof. The proof is straightforward. ■

Corollary 25. *In an intra-regular ordered $\mathcal{AG}$-groupoid S, $\Gamma \circ \mathcal{A} = \mathcal{A}$ and $\mathcal{A} \circ \Gamma = \mathcal{A}$ holds for every bipolar-valued fuzzy two-sided ideal of S.*

Theorem 26. *The set of bipolar-valued fuzzy two-sided ideals of an intra-regular ordered $\mathcal{AG}$-groupoid S with left identity forms a semilattice structure with identity Γ, where $\Gamma = (\mathcal{S}_{\Gamma}^{+}, \mathcal{S}_{\Gamma}^{-})$.*

Proof. Assume that $\mathbb{B}_{+_{\mu^-}}$ is the set of bipolar-valued fuzzy two-sided ideals of an intra-regular ordered $\mathcal{AG}$-groupoid S and let $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$, $\mathcal{B} = (\mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-})$ and $\mathcal{C} = (\mu_{\mathcal{C}}^{+}, \mu_{\mathcal{C}}^{-})$ be in $\mathbb{B}_{+_{\mu^-}}$. Clearly $\mathbb{B}_{+_{\mu^-}}$ is closed and by Theorem 21, $\mathcal{A}^2 = \mathcal{A}$. Now by using Lemma 23, we get $\mathcal{A} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{A}$ and therefore, we have

$$(\mathcal{A} \circ \mathcal{B}) \circ \mathcal{C} = (\mathcal{B} \circ \mathcal{A}) \circ \mathcal{C} = (\mathcal{C} \circ \mathcal{A}) \circ \mathcal{B} = (\mathcal{A} \circ \mathcal{C}) \circ \mathcal{B} = (\mathcal{B} \circ \mathcal{C}) \circ \mathcal{A} = \mathcal{A} \circ (\mathcal{B} \circ \mathcal{C}).$$

It is easy to see from Corollary 25 that Γ is an identity in $\mathbb{B}_{+_{\mu^-}}$. ■

Conclusion. In this paper, we introduced the concept of bipolar-valued fuzzification of an ordered $\mathcal{AG}$-groupoid and discussed some structural properties of bipolar-valued fuzzy two-sided ideals of an intra-regular ordered $\mathcal{AG}$-groupoid. We characterized an intra-regular ordered $\mathcal{AG}$-groupoid in terms of bipolar-valued fuzzy two-sided ideals.

In our future study of bipolar-valued fuzzy structure of ordered $\mathcal{AG}$-groupoids, may be the following topics should be considered:

1. To characterize other classes of ordered $\mathcal{AG}$-groupoids by the properties of these bipolar-valued fuzzy ideals.
2. To characterize regular and intra-regular ordered $\mathcal{AG}$-groupoids by the properties of these bipolar-valued fuzzy ideals.

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