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STOCHASTIC DIFFRENTIAL EQUATIONS ON BANACH SPACES AND THEIR OPTIMAL FEEDBACK CONTROL

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Abstract

In this paper we consider stochastic differential equations on Banach spaces (not Hilbert). The system is semilinear and the principal operator generating a C_0-semigroup is perturbed by a class of bounded linear operators considered as feedback operators from an admissible set. We consider the corresponding family of measure valued functions and present sufficient conditions for weak compactness. Then we consider applications of this result to several interesting optimal feedback control problems. We present results on existence of optimal feedback operators.

Keywords: stochastic differential equations, Banach spaces, optimal feedback control, objective functionals, Lévy-Prohorov metric, Hausdorff dimension, time-optimal problems.

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1. Introduction

This work is inspired by the fact that most of the available literature Da Prato and Zabczyk [7], Gozzi, Rouy and Swiech [11], Goldys and Maslowski [12], Ahmed [2–4] invokes Hilbert spaces whenever stochastic differential equations are considered in infinite dimensional spaces. This is done both for the state space and the space where the Brownian motion takes values from. Though there is an extensive literature on open loop controls for deterministic systems on infinite dimensional Banach spaces Ahmed [5], Cesari [6], Fattorini [10], to the best of knowledge of the author, there is hardly any on feedback control of stochastic systems on infinite dimensional Banach spaces. It is well known that in the case of reaction diffusion equations, in particular the heat equation, the natural state space is $X = C(D)$, where D is an open bounded domain in R^n, and not $L_2(D)$. Thus study of stochastic differential equations(SDEs) on general Banach spaces is important and this is what we wish to do here.

Another motivation comes from the following facts. In the study of optimal control problems of stochastic differential equations, the standard approach is to use the Bellman's principle of optimality and construct an HJB equation which is, in general, a nonlinear partial differential equation on R^n for finite dimensional SDEs. For infinite dimensional SDEs [2–4, 11, 12], the HJB equation is a nonlinear Partial differential equation on an infinite dimensional Hilbert space. One generally uses the viscosity technique or an abstract technique [2, 4] based on invariant measures and Hilbert spaces like $L_2(H, \mu)$ and the associated Sobolev spaces to prove existence of solutions. To determine the optimal feedback control law, one is required to solve the HJB equation and then construct the feedback control law which turns out to be a function of the solution (the value function) and it's Fréchet derivative. The question of determining the solution of the HJB equation on infinite dimensional Hilbert space is certainly a highly nontrivial task, and then constructing the optimal feedback control law from this is yet another formidable task. Also, it is well known that the value function may not posses Frećhet derivative which is required in the construction.

Here, our approach is direct which avoids the above practical difficulties associated with the HJB approach. We assume the structure of the feedback control law and study the question of existence of an optimal control law from the admissible class of operators. In particular, for linear feedback control we determine the optimal linear operator satisfying certain specified topological constraints.

There are two novelties of this paper. The first is the study of infinite dimensional SDEs on general Banach spaces (leaving the traditional practice of using Hilbert spaces), and the second is the direct study of optimal state feedback control law in the space of bounded linear operators endowed with the strong operator topology (avoiding HJB approach). Extension to weak operator topology is given in Theorem 4.5. Use of weak operator topology and the accompanying merits and demerits are also discussed in Remarks 4.4 and 4.6. Another interesting problem not considered in this paper is to extend our results to differential inclusions along the line of J. Motyl and M. Michta *et al.* using the new concept of upper separated multi functions in Banach lattice [17, 18].

The rest of the paper is organized as follows. In section 2, we introduce the system dynamics considered. In section 3, questions of existence and regularity of solutions are discussed. In section 4, we consider the question of continuous dependence of solutions on the control operators with respect to strong as well as weak operator topologies. In section 5, several interesting standard and nonstandard control problems are presented and the question of existence of optimal control operators are studied.

2. Background Materials

Let X,E be a pair of separable Banach spaces. We assume throughout the rest of the paper that both X and E posses Schauder basis $\{x_i\}$ and $\{e_i\}$ respectively with the associated dual basis $\{x_i^*\} \in X^*$ and $\{e_i^*\} \in E^*$ respectively. Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ denote a complete filtered probability space where $\mathcal{F}_t, t \geq 0$, is an increasing family of sub σ-algebras of the σ-algebra $\mathcal{F}$. The system we consider is governed by a semilinear stochastic differential equation given by

$$dx(t) = Ax(t)dt + Bx(t)dt + f(x(t))dt + CdW(t), x(0) = x_0, \tag{1}$$

for $t \in I \equiv [0, T]$, where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{S(t), t \geq 0\} \subset \mathcal{L}(X)$, $B \in \Gamma \subset \mathcal{L}(X)$, $f : X \longrightarrow X$ is a continuous map, $C \in \mathcal{L}(E, X)$ and $W(t), t \geq 0$, is an E valued Brownian motion. For $B \in \Gamma$, $t \geq 0$, let μ_t^B denote the probability measure induced by the solution process $\{x^B(t), t \geq 0\}$. In other words, for each $t \geq 0$, and $D \in \mathcal{B}(X)$, $\mu_t^B(D) := Prob.\{x^B(t) \in D\}$.

Let $\mathcal{M}_1(X)$ denote the space of regular probability measures defined on $\mathcal{B}(X)$. One of our primary objectives is to find sufficient conditions on Γ and other parameters under which, for each $t \geq 0$, the reachable set of measures given by

$$\mathcal{R}(t) := \Big\{\mu \in \mathcal{M}_1(X) : \mu = \mu_t^B, \text{for } B \in \Gamma\Big\} \tag{2}$$

is tight or weakly relatively compact. In fact we prove that it is weakly compact. Then we use this result to solve several optimal control problems considering $B \in \Gamma$ as the linear feedback operator.

For each $M \geq 1$ and $\omega \in R$, let $\mathcal{G}_0(M, \omega)$ denote the class of infinitesimal generators of C_0-semigroups of operators $\{S(t), t \geq 0\} \subset \mathcal{L}(X)$ with stability parameters $\{M, \omega\}$. It is well known from perturbation theory of semigroups [Ahmed, [1]] that if $A \in \mathcal{G}_0(M, \omega)$ then for any $B \in \mathcal{L}(X)$, we have $A + B \in \mathcal{G}_0(M, M \parallel B \parallel + \omega)$. Thus for every $B \in \mathcal{L}(X)$, the sum $A + B$ generates a C_0 semigroup of operators on X which we shall denote by $S_B(t), t \geq 0$, satisfying

$$\parallel S_B(t) \parallel_{\mathcal{L}(X)} \leq Me^{(M\|B\|+\omega)t} \quad \forall\, t \geq 0. \tag{3}$$

In view of this the mild solution (if one exists) of the evolution equation (1) is given by the solution of the following integral equation

$$x_B(t) = S_B(t)x_0 + \int_0^t S_B(t-s)f(x_B(s))ds + z_B(t), t \geq 0, \tag{4}$$

where the process z_B is the mild solution of the SDE

$$dz(t) = (A+B)z(t)dt + CdW, t \geq 0, z(0) = 0, \tag{5}$$

given by

$$z_B(t) = \int_0^t S_B(t-s)CdW(s), t \geq 0. \tag{6}$$

Now defining $y \equiv y_B = x_B - z_B$, the reader can easily verify that y satisfies the evolution equation

$$dy(t) = (A+B)y(t)dt + f(y(t) + z_B(t))dt, y(0) = x_0. \tag{7}$$

3. Existence of Mild Solution

Before we can prove the existence we need an a-priori bound. This is given in the following lemma.

Lemma 3.1. *Suppose $A \in \mathcal{G}_0(M,\omega)$ and $B \in \mathcal{L}(X)$ and $f : X \longrightarrow X$ is uniformly Lipschitz with Lipschitz constant K. Further suppose the following assumptions hold.*

(A1): $P\{|x_0|_X < \infty\} = 1$,

(A2): $P\{\sup\{|z_B(t)|_X, t \in I\} < \infty\} = 1$.

Then the solution of equation (4), *if one exists, must satisfy*

$$P\{\sup\{|x_B(t)|_X, t \in I\} < \infty\} = 1.$$

Proof. Define $M_B \equiv M \exp\{(M \parallel B \parallel + |\omega|)T\}$, where $I \equiv [0,T], T < \infty$. Suppose the integral equation (4) has a solution. Then using (4), it is easy to verify that

$$|x_B(t)|_X \leq C + M_B K \int_0^t |x_B(s)|_X ds, t \in I,$$

where $C \equiv \big(M_B|f(0)|_X T + M_B|x_0|_X + \sup\{|z_B(t)|_X, t \in I\}\big)$. By virtue of assumptions (A1) and (A2), it is clear that $P\{C < \infty\} = 1$. Thus it follows from the Gronwall inequality that

$$|x_B(t)|_X \leq C \exp\{M_B K T\}, t \in I, P - a.s. \tag{8}$$

From this we may conclude that $P\{\sup\{|x_B(t)|_X, t \in I\} < \infty\} = P\{C < \infty\} = 1$. This completes the proof. ■

Now we are ready to give a proof of existence of a solution of equation (4).

Let $B_\infty(I, X)$ denote the space of strongly measurable functions on I with values in the Banach space X. Furnished with the norm topology,

$$\| x \|_{B_\infty(I,X)} := \sup\{|x(t)|_X, t \in I\},$$

this is a Banach space. For convenience of notation, we write Ω for $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$.

Let $M_o(\Omega, B_\infty(I, X))$ denote the space of $\mathcal{F}_t$-adapted X-valued random processes with trajectories or paths in the Banach space $B_\infty(I, X)$ with probability one. This is a linear topological vector space and it is metrizable with the metric

$$\rho(x, y) := P\{\| x - y \|_{B_\infty(I,X)} \neq 0\}.$$

With respect to this topology, two elements z_1, z_2 are considered identical if and only if $\rho(z_1, z_2) = 0$. Since $B_\infty(I, X)$ is a Banach space, it is clear that $(M_o(\Omega, B_\infty(I, X)), \rho)$ is a complete linear metric space. We write $M_o(\Omega, B_\infty(I, X))$ for $(M_o(\Omega, B_\infty(I, X)), \rho)$.

Theorem 3.2. *Under the assumptions of Lemma* 3.1, *the integral equation* (4) *has a unique solution in the metric space* $M_o(\Omega, B_\infty(I, X))$.

Proof. Since $x_B = y_B + z_B := y + z_B$, it suffices to prove the existence of a mild solution of the evolution equation (7). Define the operator G given by

$$(Gy)(t) \equiv S_B(t)x_0 + \int_0^t S_B(t-s)f(y(s) + z_B(s))ds, t \in I. \tag{9}$$

Clearly, it suffices to prove that G has a fixed point that belongs to the Banach space $B_\infty(I, X)$ with probability one. It follows from Lemma 3.1 that $x_B \in B_\infty(I, X)$ with probability one. By assumption, $z_B \in B_\infty(I, X)$ with probability one. Thus, $y \in B_\infty(I, X)$ also with probability one. We show that $Gy \in B_\infty(I, X)$ with probability one. It follows readily from the expression on the right hand side of (9) that

$$\| Gy \|_{B_\infty(I,X)} \leq c_1 + M_B\Big\{|x_0|_X + KT\big(\| z_B \|_{B_\infty(I,X)} + \| y \|_{B_\infty(I,X)} \big)\Big\}, \tag{10}$$

with probability one where $c_1 \equiv M_B|f(0)|_X T$. Hence $Gy \in B_\infty(I, X)$ with probability one whenever $y \in B_\infty(I, X)$ with probability one. Thus, with probability one, G maps $B_\infty(I, X)$ into itself. We show that G has a fixed point in $M_o(\Omega, B_\infty(I, X))$. Define

$$d_t(x, y) := \sup_{0\leq s\leq t} |x(s) - y(s)|_X,$$

for $x, y \in B_\infty(I,X)$. Let $\{y_1, y_2\} \in M_o(\Omega, B_\infty(I,X))$ be any pair. Then it is easy to verify that

$$|(Gy_1)(t) - (Gy_2)(t)|_X \leq M_B K \int_0^t |y_1(s) - y_2(s)|_X \, ds \ \text{ for all } t \in I,\ P - a.s,$$

and hence it follows from the definition of $d_t(\cdot,\cdot)$ that

$$d_t(Gy_1, Gy_2)| \leq M_B K \int_0^t d_s(y_1, y_2) \, ds \ \text{ for all } t \in I,\ P - a.s. \tag{11}$$

Since, by Lemma 3.1, $Gy_1, Gy_2 \in M_o(\Omega, B_\infty(I,X))$ for every $y_1, y_2 \in M_o(\Omega, B_\infty(I,X))$, we can use the above inequality to generate the second iterate yielding

$$d_t(G^2 y_1, G^2 y_2)| \leq M_B K \int_0^t d_s(Gy_1, Gy_2) \, ds \ \text{ for all } t \in I,\ P - a.s.$$

where $G^2 := GoG$ denotes the second iterate of G (composition of G with itself). Now substituting (11) into the above inequality, and noting that $d_t(x,y)$ is a nondecreasing function of $t \geq 0$, it is easy to verify that

$$d_t(G^2 y_1, G^2 y_2) \leq (M_B K)^2 (t^2/2) \ d_t(y_1, y_2), t \in I, P - a.s.$$

Continuing this process for the third iterate, we have

$$d_t(G^3 y_1, G^3 y_2) \leq (M_B K)^3 (t^3/3!) \ d_t(y_1, y_2), t \in I, P - a.s.$$

Thus carrying out n iterations, we obtain the following inequality

$$d_t(G^n y_1, G^n y_2) \leq \frac{(M_B K t)^n}{n!} \ d_t(y_1, y_2), t \in I, P - a.s, \tag{12}$$

and hence

$$\| G^n y_1 - G^n y_2 \|_{B_\infty(I,X)} \leq \alpha_n \| y_1 - y_2 \|_{B_\infty(I,X)} \ \ P - a.s, \tag{13}$$

where $\alpha_n = (M_B K T)^n / n!$. Clearly, for sufficiently large n, $0 < \alpha_n < 1$ and G^n is a contraction on the metric space $M_o(\Omega, B_\infty(I,X))$. Thus by the Banach fixed point theorem, G^n and hence G has a unique fixed point in $M_o(\Omega, B_\infty(I,X))$. This completes the proof. ■

Remark 3.3. In Lemma 3.1, we assumed that the process z_B given by the stochastic integral

$$z_B(t) = \int_0^t S_B(t-s)CdW(s), t \in I, \tag{14}$$

belongs to the Banach space $B_\infty(I, X)$ with probability one. Here, we give a sufficient condition that guarantees this property. First let $\{e_i\} \subset E$ be a Schauder basis with $\{e_i^*\} \subset E^*$ the corresponding dual basis so that they form a biorthogonal system. Let $\{W(t), t \geq 0\}$ be an E valued Wiener process with $P\{W(0) = 0\} = 1$ and $\mathbb{E}(e^*, W(t))_{E^*,E} = 0$ for every $e^* \in E^*$ and every $t \geq 0$. Further, assuming that W has independent increments over disjoint intervals of time, the incremental covariance operator of the process is given by

$$\mathbb{E}(e^*, W(t))^2 := (Q_W(t)e^*, e^*) = t(Qe^*, e^*)_{E,E^*}$$

where Q denotes the incremental covariance of the Wiener process W. If W is a weak second order Wiener process, it follows from a result of [Weron [16], Proposition 1] that $Q \in \mathcal{L}(E^*, E) \subset \mathcal{L}(E^*, E^{**})$. Clearly, Q is positive and symmetric. We assume that $Q \in \mathcal{L}_1^+(E^*, E) \subset \mathcal{L}_1(E^*, E)$ where the later space is the Banach space of symmetric nuclear operators from E^* to E. In this case the covariance of the random element $z_B(t)$ given by

$$Q_{z_B}(t) = \int_0^t (S_B(r)CQC^*S_B^*(r)dr, t \in I, \tag{15}$$

is also positive nuclear. This follows from the facts that the composition of a nuclear operator with any bounded linear operator is nuclear and that $S_B(r)C$ is a family of bounded operators. Thus

$$\sup\{Tr(Q_{z_B}(t)), t \in I\} = TrQ_{z_B}(T) = \int_0^T Tr((S_B(r)CQC^*S_B^*(r))dr < \infty. \tag{16}$$

Then it follows from Tchebychev inequality that

$$P\{\sup\{|z_B(t)|_X, t \in I\} \geq r\} \leq (1/r^2)TrQ_{z_B}(T),$$

and hence letting $r \to \infty$ we conclude that $P\{\| z_B \|_{B_\infty(I,X)} < \infty\} = 1$.

4. CONTINUOUS DEPENDENCE OF SOLUTIONS

Here we are interested in the regularity of the map

$$B \longrightarrow x_B$$

from $\mathcal{L}(X)$ to $M_o(\Omega, B_\infty(I,X))$. Continuity being a topological property, it is necessary to identify appropriate topologies on the domain and the range spaces before any regularity property can be determined. We assume that $\mathcal{L}(X)$ is given the strong operator topology, τ_{so}. It is well known that, with respect to this topology, $(\mathcal{L}(X), \tau_{so})$ is a locally convex sequentially complete topological vector space [6]. For the range space $M_o(\Omega, B_\infty(I,X))$ we have already the metric topology as discussed in the preceding section.

Theorem 4.1. *Consider the system* (1) *with* $\Gamma \subset (\mathcal{L}(X), \tau_{so})$ *norm bounded by a finite number* $\gamma > 0$ *and suppose the assumptions of Lemma* 3.1 (*Theorem* 3.2) *hold. Then the map* $B \longrightarrow x_B$ *is continuous with respect to the strong operator topology on* $\mathcal{L}(X)$ *and metric topology on* $M_o(\Omega, B_\infty(I,X))$.

Proof. By assumption $\Gamma \subset (\mathcal{L}(X), \tau_{so})$ is norm bounded by $\gamma > 0$ in the sense that

$$\sup\{\| B \|_{\mathcal{L}(X)}, B \in \Gamma\} \leq \gamma.$$

Let $\{B_n\} \subset \Gamma$ and suppose $B_n \xrightarrow{\tau_{so}} B_o$ and let $x_n \in M_o(\Omega, B_\infty(I,X))$ denote the (mild) solution corresponding to B_n, and x_o the mild solution corresponding to the operator B_o respectively. This statement follows from Theorem 3.2. We must verify that $x_n \xrightarrow{\rho} x_o$ in the metric topology. Considering the integral equation associated with the system (1) and subtracting the mild solution x_n from the mild solution x_o, we arrive at the following expression

$$x_o(t) - x_n(t) = \int_0^t S(t-s)(B_o - B_n)x_o(s)ds + \int_0^t S(t-s)B_n(x_o(s) - x_n(s))ds$$
$$+ \int_0^t S(t-s)(f(x_o(s)) - f(x_n(s)))ds, t \in I. \tag{17}$$

Taking the norm (in X) of either side of the identity (17) and using standard triangle inequality and recalling that $\{B_o, B_n\} \subset \Gamma$ where Γ is a norm bounded subset of $\mathcal{L}(X)$ with the bound $\gamma > 0$, f is uniformly Lipschitz with Lipschitz constant K, and $M = \sup\{\| S(t) \|_{\mathcal{L}(X)}, \in I\}$, we obtain the following inequality

$$|x_o(t) - x_n(t)|_X \leq |e_n(t)|_X + \tilde{M} \int_0^t |x_o(s) - x_n(s)|_X \, ds, t \in I, P-a.s, \tag{18}$$

where $\tilde{M} = M(\gamma + K)$ and e_n is given by

$$e_n(t) \equiv \int_0^t S(t-r)(B_o - B_n)x_o(r) \, dr, t \in I. \tag{19}$$

Evaluating the norm of e_n, it follows from (19) that

$$|e_n(t)|_X \leq M \int_0^t |(B_o - B_n)x_o(r)|_X \, dr, t \in I, P - a.s. \tag{20}$$

Since $B_n \xrightarrow{\tau_{so}} B_o$ and $x_o \in B_\infty(I, X)$ with probability one, implying that $x_o(r) \in X$ for all $r \in I$, P-a.s, we conclude that with probability one,

$$\lim_{n\to\infty} |(B_o - B_n)x_o(r)|_X = 0 \;\; \forall \;\; r \in I. \tag{21}$$

By assumption, the set $\Gamma \subset \mathcal{L}(X)$ is norm bounded by γ and $B_n, B_o \in \Gamma$. Thus we have

$$|(B_o - B_n)x_o(t)|_X \leq 2\gamma|x_o(t)|_X, t \in I, P - a.s. \tag{22}$$

By use of Lemma 3.1, corresponding to $B = B_o$, one can easily derive the following inequality

$$|x_o(t)|_X \leq C_o + M_o \int_0^t |x_o(s)|_X ds, t \in I, P - a.s, \tag{23}$$

where $C_o = M_o|f(0)|_X T + M_o|x(0)|_X + \sup\{|z_o(t)|_X, t \in I\}$. Here M_o corresponds to $M_{B_o} = \sup\{\| \; S_{B_o}(t) \; \|_{\mathcal{L}(X)}, t \in I\}$ and z_o corresponds to z_{B_o}. By virtue of assumptions (A1) and (A2), we have $P\{C_o < \infty\} = 1$. Thus by Gronwall inequality it follows from (23) that

$$\sup\{|x_o(t)|_X, t \in I\} \leq C_o \exp(M_o T), P - a.s.$$

Further, it follows from the same inequality that

$$\int_0^t |x_o(s)|_X ds \leq TC_o + M_o t \int_0^t |x_o(s)|_X ds, t \in I, P - a.s. \tag{24}$$

Choosing $t^* > 0$ sufficiently small, so that $M_o t^* < 1$, it follows from (24) that

$$\int_0^{t^*} |x_o(s)|_X ds \leq \frac{TC_o}{(1 - M_o t^*)}, \quad P - a.s. \tag{25}$$

Since I is a compact interval, it can be covered by a finite number of subintervals of the form $\{[kt^*, (k+1)t^*], k = 0, 1, \dots\}$. Thus it follows from (25) and the fact that $P\{C_o < \infty\} = 1$, that x_o is Bochner integrable with probability one, that is,

$$P\bigg\{ \int_I |x_o(s)|_X ds < \infty \bigg\} = 1.$$

Hence it follows from (20)-(22) and Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \sup\{|e_n(t)|_X, t \in I\} = 0 \quad P - a.s. \tag{26}$$

On the other hand, by virtue of Gronwall inequality, it follows from (18) that

$$|x_o(t) - x_n(t)|_X \leq |e_n(t)|_X + (\tilde{M} \exp \tilde{M}T) \int_0^t |e_n(s)|_X ds, t \in I, \quad P - a.s. \tag{27}$$

Using (26) and (27) we conclude that

$$\lim_{n\to\infty} \parallel x_n - x_o \parallel_{B_\infty(I,X)} = 0, \quad P - a.s.$$

In other words, $x_n \xrightarrow{\rho} x_o$ in the metric topology of $M_o(\Omega, B_\infty(I, X))$. Thus we have proved that the map $B \longrightarrow x_B$ is continuous (on Γ) with respect to the strong operator topology on $\mathcal{L}(X)$ and metric topology on $M_o(\Omega, B_\infty(I, X))$. ■

Next we present sufficient conditions under which the reachable set $\mathcal{R}(t)$ given by the expression (2) is weakly compact.

Theorem 4.2. *Consider the system* (SDE) (1) *and suppose the assumptions of Theorem* 4.1 *hold. Let $\Gamma \subset \mathcal{L}(X)$ denote the set of admissible feedback operators and suppose that it is compact in the strong operator topology τ_{so} on $\mathcal{L}(X)$. Then for each $t \in I$, the reachable set $\mathcal{R}(t)$ is a weakly sequentially compact subset of $\mathcal{M}_1(X)$.*

Proof. Let $\{\mu^n\} \in \mathcal{R}(t)$ be any sequence. Then by definition there exists a sequence $\{B_n\} \subset \Gamma$ and a corresponding sequence $\{x_n\} \subset M_o(\Omega, B_\infty(I, X))$ of mild solutions of the evolution equation (1) with $\mu_t^n(\cdot) := Px_n^{-1}(t)(\cdot) = P\{x_n(t) \in (\cdot)\}$. Since $(\mathcal{L}(X), \tau_{so})$ is a locally convex sequentially complete topological vector space and Γ is compact in the strong operator topology, there exists a subsequence of the sequence $\{B_n\}$, relabeled as the original sequence, and a $B_o \in \Gamma$ such that

$$B_n \xrightarrow{\tau_{so}} B_o.$$

Considering that the sequence $\{x_n\}$ has been also relabeled accordingly, it follows from Theorem 4.1 that there exists an $x_o \in M_o(\Omega, B_\infty(I, X))$, the mild solution of equation (1) corresponding to B_o, such that, for each $t \in I$, $x_n(t) \xrightarrow{s} x_o(t)$ in X with probability one (P-a.s). Let $BC(X) = C_b(X)$ denote the Banach space of bounded continuous functions on X endowed with the topology induced by the supnorm. Then for any $\varphi \in BC(X)$

$$\varphi(x_n(t)) \longrightarrow \varphi(x_o(t)) \quad P - a.s.$$

This is equivalent to

$$\int_X \varphi(\xi)\mu_t^n(d\xi) \longrightarrow \int_X \varphi(\xi)\mu_t^o(d\xi)$$

where μ_t^o is the measure induced by the random element $x_o(t)$, that is, $\mu_t^o(S) := P\{x_o(t) \in S\}$ for $S \in \mathcal{B}(X)$. Since Γ is compact in the strong operator topology τ_{so} and hence closed in this topology, $B_o \in \Gamma$ as stated above, and therefore $\mu_t^o \in \mathcal{R}(t)$. Thus, we have proved that every sequence in $\mathcal{R}(t)$ has a subsequence that converges weakly to an element of $\mathcal{R}(t)$. Clearly, it follows from this result that the reachable set $\mathcal{R}(t)$ is a weakly sequentially compact subset of $\mathcal{M}_1(X)$ for every $t \in I$. ■

As a corollary of the above theorem we have the following result.

Corollary 4.3. *Consider the feedback system SDE* (1) *or equivalently* (4) *and suppose the assumptions of Theorem* 4.2 *hold. Then, for each* $t \in I$ *the reachable set* $\mathcal{R}(t) \subset \mathcal{M}_1(X)$ *is tight in the sense that for every* $\varepsilon > 0$ *there exists a compact set* $K_\varepsilon \subset X$ *such that* $\mu(K_\varepsilon^{'}) = \mu(X \setminus K_\varepsilon) < \varepsilon$ *uniformly with respect to* $\mu \in \mathcal{R}(t)$.

Let $C_b(X)$ denote the Banach space of real valued bounded continuous functions on X furnished with the standard sup norm topology and $\mathcal{M}_b(X)$ the space of regular bounded finitely additive Borel measures on $\mathcal{B}(X)$ with the standard total variation norm. It is well known (see Dunford and Schwartz [8]) that the topological dual of $C_b(X)$ is $\mathcal{M}_b(X)$. Since the spaces $\{C_b(X), \mathcal{M}_b(X)\}$ do not satisfy the RNP (Radon-Nikodym Property), the topological dual of $L_1(I, C_b(X))$ is not $L_\infty(I, \mathcal{M}_b(X))$ [Diestel and Uhl., Jr, [9], Theorem 1, p.98]. However, it follows from the theory of "Lifting" [Tulcea and Tulcea [15], Theorem 7, p.92] that

$$(L_1(I, C_b(X)))^* \cong L_\infty^w(I, \mathcal{M}_b(X)).$$

The elements of $L_\infty^w(I, \mathcal{M}_b(X))$ are merely weakly measurable functions on I with values in $\mathcal{M}_b(X)$ endowed with the natural w^* (weak star) topology. We are interested in the class of probability measure valued functions $M_w(I, \mathcal{M}_1(X))$ which is a subset of the space $L_\infty^w(I, \mathcal{M}_b(X))$.

Note that $M_w(I, \mathcal{M}_1(X))$ denotes the topological space of weakly measurable functions from I to the space of Borel probability measures $\mathcal{M}_1(X)$ endowed with standard weak topology. For convenience of notation, we introduce the set

$$\mathcal{R} := \{\mu \in M_w(I, \mathcal{M}_1(X)) : \mu_t = \mu|_t \in \mathcal{R}(t), t \in I\}$$

where $\mathcal{R}(t)$ denotes the reachable set as defined by the expression (2).

Remark 4.4 The continuity result given by Theorem 4.1 is crucial for later applications to control. In Theorem 4.2 (and its Corollary 4.3), we assumed that the set Γ is compact in the strong operator topology on $\mathcal{L}(X)$. This is certainly a weaker condition than compactness in the uniform operator topology, while it is stronger than the weak operator topology. If one wishes to use the weak operator topology, one must sacrifice the generality of the semigroup $S(t), t \geq 0$. We need compactness of the semigroup as stated in the following theorem. The compactness assumption, however, limits the class of systems that can be covered. So it is a matter of tradeoff between general C_0-semigroups partnered with strong operator topology for $\mathcal{L}(X)$ on one hand and compact semigroups partnered with weak operator topology on the other.

In any case we present below a result involving weak operator topology. Let $(\mathcal{L}(X), \tau_{wo})$ denote the space of bounded linear operators in X endowed with the weak operator topology τ_{wo}.

Theorem 4.5. *Consider the system* (1) *with* $\Gamma \subset (\mathcal{L}(X), \tau_{wo})$ *norm bounded by a finite number* $\gamma > 0$ *and suppose the assumptions of Lemma* 3.1 *hold and further assume that the semigroup* $\{S(t), t \geq 0\}$ *is compact for* $t > 0$*. Then the map* $B \longrightarrow x_B$ *is continuous with respect to the weak operator topology on* $\mathcal{L}(X)$ *and metric topology on* $M_o(\Omega, B_\infty(I, X))$*.*

Proof. (outline) Since the major part of the proof is quite similar to that of Theorem 4.1, we indicate only important differences and the critical steps to address them. The major difference arises only from the first term on the right hand side of the expression (17) or equivalently (19) which is reproduced below for convenience of the reader:

$$e_n(t) := \int_0^t S(t-s)(B_o - B_n)x_o(s)ds, t \in I.$$

Since, now we are dealing with the weak operator topology we can only state that

$$(B_o - B_n)x_s(s) \xrightarrow{w} 0 \quad \text{in} \quad X \text{ for each } s \in I,\ P - a.s.$$

Here, we use the assumption on compactness of the semigroup $S(t), t > 0$. From this we verify that $e_n(t) \xrightarrow{s} 0$ in X uniformly on the interval I with probability one. First note that, since x_o is almost surely Bochner integrable, $\lim_{t \downarrow 0} e_n(t) = 0$. For any $\varepsilon \in (0, T]$, we can split the expression for $e_n(t)$ as follows

$$e_n(t) = S(\varepsilon)\left(\int_0^{t-\varepsilon} S(t-\varepsilon-s)(B_o - B_n)x_o(s)\ ds\right) + \int_{t-\varepsilon}^t S(t-s)(B_o - B_n)x_0(s)\ ds.$$

It is well known that a linear operator between any two Banach spaces is strongly continuous if and only if it is weakly continuous [6, V.3.15, p.422]. This implies that the integral within the parenthesis in the above expression converges to zero weakly in X. Since $S(\varepsilon), \varepsilon > 0$, is a compact operator, and the term within the parenthesis is weakly convergent to zero, it is clear that as $n \to \infty$ the first term converges to zero strongly in X P-a.s uniformly on the interval $(\varepsilon, T]$. Considering the second term and recalling that $\{B_o, B_n\} \subset \Gamma$ we have the following estimate for the second term

$$\left| \int_{t-\varepsilon}^{t} S(t-s)(B_o - B_n)x_o(s)ds \right|_X \leq \left(2\gamma M \int_{t-\varepsilon}^{t} |x_o(s)|_X ds \right).$$

It follows from this estimate, and P-a.s Bochner integrability of x_o on the interval I, that the second term of e_n converges to zero P-a.s as $\varepsilon \to 0$. Thus under the given assumptions we have proved that $e_n(t) \to 0$ strongly in X uniformly on I P-a.s. The rest of the materials in the proof of Theorem 4.1 remains unchanged. This proves the continuity of the map $B \longrightarrow x_B$ in the weak operator topology on Γ and metric topology on $M_o(\Omega, B_\infty(I, X))$. This completes the outline of our proof. ■

Remark 4.6. It is interesting to note that if the state space X is a reflexive Banach space, any closed ball $B_\gamma(\mathcal{L}(X))$ of radius γ (of the space $\mathcal{L}(X)$) centered at the origin is compact in the weak operator topology [see Dunford [6]]. Thus according to the above theorem we can choose $\Gamma = B_\gamma(\mathcal{L}(X))$. This is certainly a great advantage, but at the cost of generality of the state space and generality of the semigroup $S(t)$.

5. Existence of Optimal Linear State Feedback Controls

Here we consider the system (1) with $B \in \Gamma \subset \mathcal{L}(X)$, considered as the linear state feedback control operator. We consider several control problems. Let

$$d_\pi : \mathcal{M}_1(X) \times \mathcal{M}_1(X) \longrightarrow [0, 1]$$

denote the Lévy-Prohorov metric on the space of Probability measures $\mathcal{M}_1(X)$. Since, throughout the paper, X is assumed to be a separable Banach space, the metric space $(\mathcal{M}_1(X), d_\pi)$ is a separable metric space.

Problem 1 (P1). A classical control problem is given by a cost functional of the form

$$J(B) = \int_0^T \left\{ \int_X \ell(t, x)\mu_t^B(dx) + d_\pi(\mu_t^B, \nu_t) \right\} dt + d_\pi(\mu_T^B, \varpi) + g(Tr(B\mathbf{\Sigma})), \tag{28}$$

where ν, ϖ and $\mathbf{\Sigma}$ are given with $\nu \in M_w(I, \mathcal{M}_1(X))$, $\varpi \in \mathcal{M}_1(X)$ and $\mathbf{\Sigma} \in \mathcal{L}_1(X)$, the space of nuclear operators in X. The cost of the size and complexity of feedback operators used is measured through the last term. Physical interpretation of the remaining terms is transparent. The objective is to find, for the system (1), an operator $B \in \Gamma \subset \mathcal{L}(X)$ that minimizes the cost functional (28).

We introduce the following assumptions:

(a1): There exist $\ell_0, \ell_1 \in L_1^+(I)$ and $\psi \in C_b^+(X)$ such that

$$|\ell(t,x)| \leq \ell_0(t) + \ell_1(t)\psi(x) \quad \forall \quad t \in I, x \in X.$$

(a2): The operator $\mathbf{\Sigma} \in \mathcal{L}_1(X)$, and the function $g : R \longrightarrow R$ is continuous and bounded on bounded sets.

Then we have the following result.

Theorem 5.1. *Consider the system given by the SDE* (1) *subject to the state feedback control law $B \in \Gamma \subset (\mathcal{L}(X), \tau_{so})$ and the cost functional given by* (28). *Suppose the assumptions* (a1) *and* (a2) *hold and that $\nu \in M_w(I, \mathcal{M}_1(X))$, $\varpi \in \mathcal{M}_1(X)$ and $\mathbf{\Sigma} \in \mathcal{L}_1(X)$ and Γ is compact in the strong operator topology. Then, there exists an optimal control law minimizing the cost functional J.*

Proof. Since weak convergence is equivalent to convergence in the Lévy-Prohorov metric, and both $\nu, \mu^B \in M_w(I, \mathcal{M}_1(X))$, the function $t \longrightarrow d_\pi(\mu_t^B, \nu_t)$ is measurable. And under the assumption (a1) the expression within the bracket of the functional (28) is integrable and therefore by Theorem 4.2,

$$B \longrightarrow \int_0^T \left\{ \int_X \ell(t,x)\mu_t^B(dx) + d_\pi(\mu_t^B, \nu_t) \right\} dt$$

is continuous with respect to the strong operator topology τ_{so}. Again, by Theorem 4.2 and the equivalence of Prohorov metric and the topology of weak convergence on $\mathcal{M}_1(X)$, $B \longrightarrow d_\pi(\mu_T^B, \varpi)$ is continuous in the strong operator topology. Since $\mathbf{\Sigma}$ is nuclear, it is easy to verify that the last term is lower semicontinuous with respect to the strong operator topology. Hence the map $B \longrightarrow J(B)$ (given by (28)) is lower semicontinuous with respect to the strong operator topology. Thus the conclusion follows from τ_{so} compactness of Γ. ■

Problem 2 (P2) (Target Seeking). Let C be a closed subset of X considered to be a friendly zone. The designer wants a control law that forces the system to seek for this site and maximize the probability of residence there. In other words, the region C is the most desirable site in X. Let λ be a countably additive nonnegative measure on the sigma algebra of subsets of the set $I = [0,T]$. The objective is to find a feedback operator $B \in \Gamma \subset \mathcal{L}(X)$ that maximizes the functional

$$J(B) = \int_I \mu_t^B(C)\lambda(dt). \tag{29}$$

In regards to this problem we have the following result.

Theorem 5.2. *Consider the system* (1) *with the admissible set of feedback operators* $\Gamma \subset (\mathcal{L}(X), \tau_{so})$ *and the objective functional* (29) *with* λ *being a nonnegative countably additive measure having bounded total variation and suppose the assumptions of Theorem* 4.2 *hold. Then, the problem* **(P2)** *has a solution, that is, there exists a* $B_o \in \Gamma$ *such that* $J(B_o) \geq J(B)$ *for all* $B \in \Gamma$.

Proof. We show that the functional (29) is upper semi continuous with respect to the strong operator topology on $\mathcal{L}(X)$. Let $\{B_n, B_o\} \subset \Gamma$ and $\{\mu_t^n, \mu_t^o, t \in I\} \subset \mathcal{M}_1(X)$ a family of measure valued functions associated with the mild solutions of the system (1) corresponding to the sequence $\{B_n, B_o\}$ respectively. Suppose $B_n \xrightarrow{\tau_{so}} B_o$; then by Theorem 4.2, along a subsequence (if necessary), $\mu_t^n \xrightarrow{w} \mu_t^o$ in $\mathcal{M}_1(X)$ for each $t \in I$. Thus, for any closed set $C \subset X$, it follows from a well known result [Parthasarathy, [14], Theorem 6.1, p.40] that

$$\overline{\lim}\, \mu_t^n(C) \leq \mu_t^o(C) \quad \text{for each} \quad t \in I. \tag{30}$$

Since λ is a positive measure having bounded variation, it follows readily from this inequality that

$$\int_I \overline{\lim}\, \mu_t^n(C)\lambda(dt) \leq \int_I \mu_t^o(C)\lambda(dt). \tag{31}$$

Clearly, $\overline{\lim} \int_I \mu_t^n(C)\lambda(dt) \leq \int_I \overline{\lim}\, \mu_t^n(C)\lambda(dt)$. Thus

$$\overline{\lim}\, J(B_n) \equiv \overline{\lim} \int_I \mu_t^n(C)\lambda(dt) \leq \int_I \mu_t^o(C)\lambda(dt) \equiv J(B_o) \tag{32}$$

proving that J is upper semicontinuous with respect to the strong operator topology. Since by hypothesis Γ is compact in the strong operator topology, we conclude that J attains its supremum on Γ. Hence there exists (at least one) optimal control law $B_o \in \Gamma$. This completes the proof. ■

Remark 5.3. This result can be readily extended to the case of a moving target $C(t), t \geq 0$. It is only necessary that $t \to C(t)$ be a measurable multifunction with values which are closed subsets of X. The reader may try to give the details.

Problem 3 (P3) (Obstacle Evasion). The concern here is to avoid an obstacle (danger zone) described by an open set $D \subset X$. In contrast with the problem 2, the objective here is to find a feedback operator that minimizes the functional

$$J(B) = \int_I \mu_t^B(D)\vartheta(dt) \tag{33}$$

where ϑ is again a countably additive nonnegative measure defined on the sigma algebra of subsets of the set I.

Theorem 5.4. *Consider the system* (1) *with the objective functional* (33) *and the admissible set of feedback operators* $\Gamma \subset (\mathcal{L}(X), \tau_{so})$ *and suppose the assumptions of Theorem* 4.2 *hold and that* ϑ *is a countably additive nonnegative measure having bounded total variation. Then, the problem* **(P3)** *has a solution, that is, there exists a* $B_o \in \Gamma$ *such that* $J(B_o) \leq J(B)$ *for all* $B \in \Gamma$.

Proof. The proof is similar to that of Theorem 5.2 and this is based on the inequality

$$\underline{\lim}\, \mu_t^n(D) \geq \mu_t^o(D) \quad \text{for each} \quad t \in I, \tag{34}$$

in place of the inequality (30). ■

Remark 5.5. This result can be extended also to the case of a moving target $D(t), t \geq 0$. It is only necessary that $t \to D(t)$ be a measurable multifunction with values which are open subsets of X.

Problem 4 (P4). Another interesting problem is:

$$J(B) = \int_0^T F(\mu_t^B(\varphi_1), \ldots, \mu_t^B(\varphi_n))dt \longrightarrow \inf, \tag{35}$$

where $\mu_t(\varphi) \equiv \int_X \varphi(\xi)\mu_t(d\xi), \varphi \in BC(X)$. The functions $\varphi_i \in BC(X)$, $i = 1, 2, \ldots, n$.

Theorem 5.6. *Consider the system* (1) *with the cost functional* (35) *and admissible set of (feedback) operators* $\Gamma \subset (\mathcal{L}(X), \tau_{so})$. *Suppose the assumptions of Theorem* 4.2 *hold and* $F : R^n \longrightarrow R$ *is a lower semicontinuous function bounded on bounded sets and bounded away from* $-\infty$. *Then the problem* **(P4)** *has a solution, that is, there exists a* $B_0 \in \Gamma$ *at which* J *attains its minimum.*

Proof. Using Fatou's Lemma, it is easy to verify that under the given assumptions $B \to J(B)$ is lower semicontinuous with respect to the strong operator topology. Thus the conclusion follows from τ_{so} compactness of the set Γ. ■

Problem 5 (P5) (Time Optimal Problem). Suppose the initial measure $\mu_0 := Px_0^{-1}$ is supported on a bounded (norm) closed set $V \subset X$ and $B_r(X)$ is a closed ball of radius $r > 0$ containing V in its interior. Define the objective functional

$$J(B) := \inf\{t \geq 0 : \mu_t^B(B_r(X)) < 1 - \rho\} \tag{36}$$

for some $\rho \in (0,1)$. If the underlying set $\{t \geq 0 : \mu_t^B(B_r(X)) < 1 - \rho\}$ is nonempty, it means that some mass has leaked outside of V. If the set is empty, we set $\inf(\emptyset) = \infty$. The problem is to find an operator from the admissible class that maximizes the functional J given by (36). This is equivalent to maximizing the first time leakage of probability mass from the set V (support of the initial measure μ_0) exceeds ρ. For example, if $\rho = 0.1$, the leakage is ten percent. Clearly, the larger the ball $B_r(X)$ is the longer is the escape time.

Theorem 5.7. *Consider the system* (1) *with the admissible set of feedback operators* $\Gamma \subset (\mathcal{L}(X), \tau_{so})$ *and objective functional* (36). *Suppose the assumptions of Theorem* 4.2 *hold and that* Γ *is compact in the strong operator topology. Then, there exists an operator in* Γ *that maximizes the functional* (36).

Proof. If the set $\{t \geq 0 : \mu_t^B(B_r(X)) < 1 - \rho\}$ is empty for any $B \in \Gamma$, there is nothing to prove. So assume the contrary. We show that $J(B)$ given by (36) is upper semicontinuous with respect to the strong operator topology. Let $\{\mu_t^n, \mu_t^o, t \geq 0\}$ denote the measure valued functions corresponding to the operators $\{B_n, B_o\}$ respectively. Let $B_n \xrightarrow{\tau_{so}} B_o$ then, by Theorem 4.2, along a subsequence if necessary, $\mu_t^n \xrightarrow{w} \mu_t^o$ in $\mathcal{M}_1(X)$. Since $B_r(X)$ is a closed set we have

$$\overline{\lim}\, \mu_t^n(B_r(X)) \leq \mu_t^o(B_r(X)). \tag{37}$$

With a little reflection, it follows from this that

$$\{t \geq 0 : \mu_t^o(B_r(X)) < 1 - \rho\} \subseteq \{t \geq 0 : \overline{\lim}\, \mu_t^n(B_r(X)) < 1 - \rho\}$$

and hence we have

$$\inf\{t \geq 0 : \mu_t^o(B_r(X)) < 1 - \rho\} \geq \inf\{t \geq 0 : \overline{\lim}\, \mu_t^n(B_r(X)) < 1 - \rho\}. \tag{38}$$

Clearly, it follows from the definition of limsup that for every $\varepsilon \in (0,1)$ sufficiently small there exists an integer n_ε such that for all $n > n_\varepsilon$ we have

$$\mu_t^n(B_r(X)) < \overline{\lim}\, \mu_t^k(B_r(X)) + \varepsilon.$$

Again it follows from a little reflection that

$$\begin{aligned}&\left\{t \geq 0 : \overline{\lim}\, \mu_t^k(B_r(X)) + \varepsilon < 1 - \rho\right\}\\ &\subseteq \left\{t \geq 0 : \mu_t^n(B_r(X)) < 1 - \rho\right\} \text{ for all } n > n_\varepsilon,\end{aligned}$$

and hence

$$\begin{aligned}&\inf\left\{t \geq 0 : \mu_t^n(B_r(X)) < 1 - \rho\right\}\\ &\leq \inf\left\{t \geq 0 : \overline{\lim}\, \mu_t^k(B_r(X)) + \varepsilon < 1 - \rho\right\} \text{ for all } n > n_\varepsilon.\end{aligned} \tag{39}$$

It follows from (39 and (38) that for every $\varepsilon \in (0, 1)$ sufficiently small

$$\overline{\lim}\{\inf\{t \geq 0 : \mu_t^n(B_r(X)) < 1 - \rho\}\} \leq \inf\{t \geq 0 : \mu_t^o(B_r(X)) + \varepsilon < 1 - \rho\}. \tag{40}$$

Since $\varepsilon \in (0, 1)$ can be chosen arbitrarily small, it follows from (40) and the definition of the functional (36) that

$$\overline{\lim}\, J(B_n) \leq J(B_0). \tag{41}$$

This proves upper semicontinuity of the map $B \to J(B)$ (see (36)) with respect to the strong operator topology. Since Γ is compact in this topology, J attains its supremum on it. This completes the proof. ∎

Problem 6 (P6) (Complexity Control). A problem of significant interest is to reduce the complexity of the (approximate) support of the measure induced by the solution process while minimizing the migration of its mass outside a compact set (not a-priori specified). Complexity may be quantified by use of Hausdorff dimension which is also a good measure of the degrees of freedom.

For convenience of the reader we make a brief digression to recall the notion of Hausdorff dimension. First, let us consider R^n and let K be a bounded subset of it and $r > 0$ and let $N_K(r)$ denote the minimum number of balls of radius r required to cover K. Then define the Hausdorff dimension d_H of K to be

$$d_H(K) := -\lim_{r \to 0} \frac{\log N_K(r)}{\log r}.$$

For example, if K is the unit cube in R^n, it is clear that the number of r cubes (cubes of side r) required to cover it is $N(r) = (1/r^n)$. Thus it follows from the above expression that $d_H(K) = n$. In fact the Hausdorff dimension is well defined in any metric space. Let (X, ϱ) be a metric space with the distance function ϱ. Define the diameter of any bounded set $C \subset X$ by $D(C) := \sup\{\varrho(x, y), x, y \in C\}$.

For any $\delta > 0$, let Π_δ denote the class of all countable δ covers of C, that is, a collection of sets $\{C_i(\delta), i \in N\}$ each of diameter less than δ such that

$$C \subseteq \bigcup_{i \in N} C_i(\delta).$$

For each number $d > 0$, define the function

$$H^d_\delta(C) := \inf_{\Pi_\delta} \sum_{i \in N} \big[D(C_i(\delta))\big]^d$$

where the infimum is taken over all countable δ-covers Π_δ of the set C. Note that $\delta \longrightarrow H^d_\delta(C)$ is a monotonically decreasing function and hence the limit

$$\lim_{\delta \downarrow 0} H^d_\delta(C) = H^d(C)$$

exists with values in $R_+ \equiv [0, \infty]$. This is called the d-dimensional Hausdorff measure of the set C and it is well defined for the Borel sets of (X, ϱ). If X is R^n with any of the equivalent metrics, and $d < n$ it is easy to see that $H^d(C) = +\infty$. On the other hand, if $d > n$ then $H^d(C) = 0$. In the general case, it was shown by Hausdorff that there exists a critical number d^*, not necessarily an integer, for which $H^d(C) = +\infty$ for all $d < d^*$ and that $H^d(C) = 0$ for all $d > d^*$. This very critical number is called the Hausdorff dimension of the set C and it is formally defined as

$$d_H(C) := \inf\{d \geq 0 : H^d(C) = 0\} = d^*(C).$$

Note that the Hausdorff dimension can be also computed by the expression

$$d_H(C) = \sup\{d \geq 0 : H^d(C) = \infty\} \equiv d^*(C).$$

Now we can continue with the problem stated above. Let

$$\mathcal{K}(X) := \{K \subset X : K \ \text{ compact}\}$$

denote the hyperspace of compact subsets of X. Suppose this is furnished with a metric topology ρ_H, for example, the standard Hausdorff metric, such that $(\mathcal{K}(X), \rho_H)$ is a complete metric space. If X is separable, then $(\mathcal{K}(X), \rho_H)$ is a Polish space. Let $d_H(K)$ denote the Hausdorff dimension of the set $K \subset X$ and β a large positive number weighing the leakage of mass outside K. An appropriate functional incorporating the above concerns is given by

$$J(B) := \inf\left\{ d_H(K) + \frac{\beta}{T} \int_0^T \mu^B_t(X \setminus K)dt, K \in \mathcal{K}(X) \right\}. \quad (42)$$

The objective is to find a feedback operator $B \in \Gamma$ that minimizes this functional. It is known from the work of Mattila and Mauldin [13, Theorem 2.1] that the Hausdorff dimension function $d_H(\cdot)$ is of Baire class 2 not belonging to class 1. The later consists of semicontinuous functions. Thus d_H can not be lower semicontinuous. However, it is known that the Baire class 1 is dense in the Baire class 2 and so it can be approximated by the point wise (here point is any $L \in \mathcal{K}(X)$) limit of a sequence of semicontinuous functions.

In order to avoid these technicalities, we consider a set function $\nu : (\mathcal{K}(X), \rho_H) \longrightarrow [0, \infty]$ satisfying the following properties:

(Q1): $\nu(F) = 0$ for all $F \in \mathcal{N}$ where $\mathcal{N}$ consists of singletons, finite subsets and empty sets from $\mathcal{K}(X)$.

(Q2): $\nu(K_1) \leq \nu(K_2)$ for all $K_1, K_2 \in \mathcal{K}(X)$ whenever $K_1 \subset K_2$.

(Q3): $\overline{\lim}_{d_H(K)\to\infty} \nu(K) = \infty$.

We use this set function to replace the functional (42) by

$$J(B) := \inf\Big\{\nu(K) + \frac{\beta}{T}\int_0^T \mu_t^B(X \setminus K)dt, K \in \mathcal{K}(X)\Big\}. \tag{43}$$

Minimizing this cost functional is equivalent to minimizing the concentration of (time) average mass of the measure outside a compact set while keeping the Hausdorff dimension of this set as small as possible subject to the operator constraint $\Gamma \subset (\mathcal{L}(X), \tau_{so})$.

Theorem 5.8. *Consider the system* (1) *with the cost functional* (43) *and suppose Theorem* 4.2 *holds and that the set function* $\nu : (\mathcal{K}(X), \rho_H) \longrightarrow [0, \infty]$ *is lower semicontinuous with respect to the metric* ρ_H *and satisfies the properties* (Q1)–Q(3). *Then, the problem* **(P6)** *has a solution, that is, there exists an operator minimizing the cost function* $J(B)$ *on* $\Gamma \subset \mathcal{L}(X)$.

Proof. Define the functional $\ell : \Gamma \times \mathcal{K}(X) \longrightarrow [0, \infty]$ as follows

$$\ell(B, K) := \nu(K) + \frac{\beta}{T}\int_0^T \mu_t^B(X \setminus K)dt,$$

and note that $J(B) := \inf\{\ell(B, K), K \in \mathcal{K}(H)\}$. Let $\mathcal{N} \subset \mathcal{K}(X)$ denote the class of finite subsets of the space X including the empty set. It is easy to verify that ℓ satisfies the following properties:

(1) $\ell(B, K) \geq 0$,

(2) $\ell(B, N) = \beta$ for all $N \in \mathcal{N}$ and

(3) $\overline{\lim}_{d_H(K)\to\infty} \ell(B, K) = \infty$.

These properties hold for all $B \in \mathcal{L}(X)$. Property (1) is obvious. Property 2 follows from the property (Q1) of ν and the fact that the measure valued function $\{\mu_t^B, t \in I\}$ induced by the solution of equation (4) corresponding any $B \in \mathcal{L}(X)$ is non atomic. Property (3) follows from the property (Q3) of ν. Hence there exists an $L \in \mathcal{K}(X)$ such that for all $K(\subset L) \in \mathcal{K}(X)$ we have $\ell(B,K) \leq \ell(B,L)$ for all $B \in \Gamma$. Thus, without any loss of generality or rigor, we may consider the optimization on the metric space $(\mathcal{K}(L), \rho_H)$. Since for each fixed $B \in \Gamma$, the map $K \longrightarrow \ell(B,K)$ is lower semicontinuous and $(\mathcal{K}(L), \rho_H)$ is a compact metric space contained in $(\mathcal{K}(X), \rho_H)$, it follows that $K \longrightarrow \ell(B,K)$ attains its minimum (not necessarily unique) on $\mathcal{K}(L) \subset \mathcal{K}(X)$. Thus, for each $B \in \Gamma$, there exists a $K_B \in \mathcal{K}(L) \subset \mathcal{K}(X)$ such that $\ell(B, K_B) \leq \ell(B,K)$ for all $K \in \mathcal{K}(X)$. Now let $\{B_\alpha\} \subset \Gamma, \alpha \in \Lambda$ (a directed set), be a minimizing net for $J(B)$ given by the expression (43). It follows from the preceding analysis that there exists a net $\{K_\alpha\}_{\alpha\in\Lambda} \in \mathcal{K}(L) \subset \mathcal{K}(X)$ such that $J(B_\alpha) = \ell(B_\alpha, K_\alpha)$, and that

$$J(B_\alpha) \to \inf\{J(B), B \in \Gamma\} := m.$$

Since Γ is compact in the strong operator topology, there exists a sub net of the net $\{B_\alpha, K_\alpha\} \in \Gamma \times \mathcal{K}(L)$, relabeled as the original net $\{B_\alpha, K_\alpha\}$, and $\{B_o, K_o\} \in \Gamma \times \mathcal{K}(L)$ such that

$$B_\alpha \xrightarrow{\tau_{so}} B_o, \quad \& \quad K_\alpha \xrightarrow{\rho_H} K_o.$$

Let $\{\mu^\alpha\} \in \mathcal{R}$ denote the measure induced by the solution of equation (1) or equivalently (4) corresponding to $B_\alpha \in \Gamma$. By Theorem 4.2, $\mathcal{R}|_t \equiv \mathcal{R}(t)$ is uniformly tight for a.a $t \in I$ and therefore there exists a $\mu^o \in \mathcal{R}$ such that along a subnet, if necessary,

$$\mu_t^\alpha \xrightarrow{w} \mu_t^o \qquad \text{for } a.a\ t \in I.$$

Since the sets $\{K_\alpha, K_o, \alpha \in \Lambda\} \in \mathcal{K}(L) \subset \mathcal{K}(X)$ are compact subsets of L, and $\{\mu_t^\alpha, \alpha \in \Lambda\}$ is weakly compact (so uniformly tight) for almost all $t \in I$, we have, along a further sub net if necessary,

$$\overline{\lim}\, \mu_t^\alpha(K_\alpha) \leq \mu_t^o(K_o).$$

From this it is easy to verify that

$$\frac{1}{T}\int_0^T \mu_t^o(X \setminus K_o)dt \ \leq \underline{\lim}\, \frac{1}{T}\int_0^T \mu_{t,}^\alpha(X \setminus K_\alpha)dt.$$

This proves that the second component of the cost functional (43) is lower semicontinuous. By our assumption ν is lower semicontinuous and so $\nu(K_o) \leq \underline{\lim}_{\alpha\in\Lambda}\, \nu(K_\alpha)$ whenever $K_\alpha \xrightarrow{\rho_H} K_o$. Using these results we conclude that

$$J(B_o) \leq \underline{\lim}\, J(B_\alpha).$$

Thus $B \longrightarrow J(B)$ (see (43)) is lower semicontinuous on Γ in the strong operator topology of $\mathcal{L}(X)$. Hence J attains its infimum on Γ, proving the existence. ■

Remark 5.9. So far we have considered constant operators as the admissible set for feedback controls. This can be generalized to a class of time varying operators $\{B(t), t \geq 0\}$, measurable in the strong operator topology and taking values in any set $\Gamma \subset \mathcal{L}(X)$ which is compact in the strong operator topology. Let Ξ denote this class of operator valued functions defined on any closed bounded interval $I \subset R$. Endowed with the Tychonoff product topology, the set Ξ is compact. For feedback controls we can choose this set and show that all the results presented above remain valid.

Remark 5.10. It follows from Theorem 4.5, that if the semigroup $S(t), t \geq 0$, is compact for $t > 0$ and the admissible set Γ of feedback operators is compact in the weak operator topology, all the results of section 5 remain valid.

An Open Problem. The question of extensions to nonlinear feedback control operators is interesting. Let $\mathcal{F}(X) := \mathcal{F}(X, X)$ denote the class of functions mapping X to X. An appropriate topology on this function space is needed so that the admissible set of feedback operators $\mathcal{G} \subset \mathcal{F}(X)$ is compact in this topology. The topology that is too strong is not useful and the topology that is too weak can not be applied to differential equations. The compact-open topology is a good possibility. This is left as an open problem.

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