FRACTIONAL DISTANCE DOMINATION IN GRAPHS

S. Arumugam\textsuperscript{1,2}, Varughese Mathew\textsuperscript{3} and K. Karuppasamy\textsuperscript{1}

\textsuperscript{1}National Centre for Advanced Research in Discrete Mathematics (n-CARDMATH)
Kalasalingam University, Anand Nagar, Krishnankoil-626 126, India
\textsuperscript{2}School of Electrical Engineering and Computer Science
The University of Newcastle, NSW 2308, Australia
\textsuperscript{3}Department of Mathematics, Mar Thoma College, Tiruvalla-689 103, India

\textbf{e-mail:} s.arumugam.klu@gmail.com
{k_karuppasamy,varughese_m1}@yahoo.co.in

\textbf{Abstract}

Let $G = (V, E)$ be a connected graph and let $k$ be a positive integer with $k \leq \text{rad}(G)$. A subset $D \subseteq V$ is called a distance $k$-dominating set of $G$ if for every $v \in V - D$, there exists a vertex $u \in D$ such that $d(u, v) \leq k$. In this paper we study the fractional version of distance $k$-domination and related parameters.

\textbf{Keywords:} domination, distance $k$-domination, distance $k$-dominating function, $k$-packing, fractional distance $k$-domination .

\textbf{2010 Mathematics Subject Classification:} 05C69, 05C72.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For basic terminology in graphs we refer to Chartrand and Lesniak [3]. For basic terminology in domination related concepts we refer to Haynes \textit{et al.} [9].

Let $G = (V, E)$ be a graph. A subset $D$ of $V$ is called a \textit{dominating set} of $G$ if every vertex in $V - D$ is adjacent to at least one vertex in $D$. A dominating set $D$ is called a \textit{minimal dominating set} if no proper subset of $D$ is a dominating set of $G$. The minimum (maximum) cardinality of a minimal dominating set of $G$ is called the \textit{domination number} (upper domination number) of $G$ and is denoted by $\gamma(G)$ ($\Upsilon(G)$). Let $A$ and $B$ be two subsets of $V$. We say that $B$ \textit{dominates} $A$ if
every vertex in $A - B$ is adjacent to at least one vertex in $B$. If $B$ dominates $A$, then we write $B \rightarrow A$. Meir and Moon [12] introduced the concept of a $k$-packing and distance $k$-domination in a graph as a natural generalisation of the concept of domination. Let $G = (V, E)$ be a graph and $v \in V$. For any positive integer $k$, let $N_k(v) = \{u \in V : d(u, v) \leq k\}$ and $N_k[v] = N_k(v) \cup \{v\}$. A set $S \subseteq V$ is a distance $k$-dominating set of $G$ if $N_k[v] \cap S \neq \emptyset$ for every vertex $v \in V - S$. The minimum (maximum) cardinality among all minimal distance $k$-dominating sets of $G$ is called the distance $k$-domination number (upper distance $k$-domination number) of $G$ and is denoted by $\gamma_k(G)$ ($\Gamma_k(G)$). A set $S \subseteq V$ is said to be an efficient distance $k$-dominating set of $G$ if $|N_k[v] \cap S| = 1$ for all $v \in V - S$. Clearly, $\gamma(G) = \gamma_1(G)$. A distance $k$-dominating set of cardinality $\gamma_k(G)$ ($\Gamma_k(G)$) is called a $\gamma_k$ ($\Gamma_k$)-set. Hereafter, we shall use the term $k$-domination for distance $k$-domination.

Note that, $\gamma_k(G) = \gamma(G^k)$, where $G^k$ is the $k^{th}$ power of $G$, which is obtained from $G$ by joining all pairs of distinct vertices $u, v$ with $d(u, v) \leq k$. A subset $S \subseteq V(G)$ of a graph $G = (V, E)$ is said to be a $k$-packing ([12]) of $G$, if $d(u, v) > k$ for all pairs of distinct vertices $u$ and $v$ in $S$. The $k$-packing number $\rho_k(G)$ is defined to be the maximum cardinality of a $k$-packing set in $G$. The corona of a graph $G$, denoted by $G \circ K_1$, is the graph formed from a copy of $G$ by attaching to each vertex $v$ a new vertex $v'$ and an edge $\{v, v'\}$. The Cartesian product of graphs $G$ and $H$, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G \Box H$ if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. For a survey of results on distance domination we refer to Chapter 12 of Haynes et al. [10].

Hedetniemi et al. [11] introduced the concept of fractional domination in graphs. Grinstead and Slater [6] and Domke et al. [5] have presented several results on fractional domination and related parameters in graphs. Arumugam et al. [1] have investigated the fractional version of global domination in graphs.

Let $G = (V, E)$ be a graph. Let $g: V \rightarrow \mathbb{R}$ be any function. For any subset $S$ of $V$, let $g(S) = \sum_{v \in S} g(v)$. The weight of $g$ is defined by $|g| = g(V) = \sum_{v \in V} g(v)$. For a subset $S$ of $V$, the function $\chi_S: V \rightarrow \{0, 1\}$ defined by

$$\chi_S(v) = \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{if } v \notin S, \end{cases}$$

is called the characteristic function of $S$.

A function $g: V \rightarrow [0,1]$ is called a dominating function (DF) of the graph $G = (V, E)$ if $g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ for all $v \in V$. For functions $f, g$ from $V \rightarrow [0,1]$ we write $f \leq g$ if $f(v) \leq g(v)$ for all $v \in V$. Further, we write $f < g$ if $f \leq g$ and $f(v) < g(v)$ for some $v \in V$. A DF $g$ of $G$ is minimal (MDF) if $f$ is not a DF for all functions $f: V \rightarrow [0,1]$ with $f < g$. 


The fractional domination number $\gamma_f(G)$ and the upper fractional domination number $\Gamma_f(G)$ are defined as follows:
$$\gamma_f(G) = \min \{|g| : g \text{ is a minimal dominating function of } G\},$$
$$\Gamma_f(G) = \max \{|g| : g \text{ is a minimal dominating function of } G\}.$$  
For a dominating function $f$ of $G$, the boundary set $B_f$ and the positive set $P_f$ are defined by
$$B_f = \{u \in V(G) : f(N[u]) = 1\} \text{ and } P_f = \{u \in V(G) : f(u) > 0\}.$$  
A function $g : V \to [0,1]$ is called a packing function (PF) of the graph $G = (V,E)$ if $g(N[v]) = \sum_{u \in N[v]} g(u) \leq 1$ for all $v \in V$. The lower fractional packing number $p_f(G)$ and the fractional packing number $P_f(G)$ are defined as follows:
$$p_f(G) = \min \{|g| : g \text{ is a maximal packing function of } G\},$$
$$P_f(G) = \max \{|g| : g \text{ is a maximal packing function of } G\}.$$  
It was observed in Chapter 3 of [10] that for every graph $G$, $1 \leq \gamma_f(G) = P_f(G) \leq \frac{n}{r+1}$. We need the following theorems:

**Theorem 1.1** [5]. For a graph $G$, $p_f(G) \leq \rho_2(G) \leq P_f(G)$.

**Theorem 1.2** [2]. A DF $f$ of $G$ is an MDF if and only if $B_f \to P_f$.

**Theorem 1.3** [2]. If $f$ and $g$ are MDFs of $G$ and $0 < \lambda < 1$ then $h_\lambda = \lambda f + (1 - \lambda)g$ is an MDF of $G$ if and only if $B_f \cap B_g \to P_f \cup P_g$.

**Theorem 1.4** [5]. If $G$ is an $r$-regular graph of order $n$, then $\gamma_f(G) = \frac{n}{r+1}$.

**Theorem 1.5** [4]. Let $G$ be a block graph. Then for any integer $k \geq 1$, we have $\rho_{2k}(G) = \gamma_k(G)$.

For other families of graphs satisfying $\rho_2(G) = \gamma(G)$, we refer to Rubalcaba et al. [13].

**Definition 1.6** [15]. A linear Benzenoid chain $B(h)$ of length $h$ is the graph obtained from $P_2 \Box P_{h+1}$ by subdividing exactly once each edge of the two copies of $P_{h+1}$. Hence $B(h)$ is a subgraph of $P_2 \Box P_{2h+1}$. The graph $B(4)$ is given in Figure 1.

![Figure 1. B(4).](image)

**Theorem 1.7** [15]. For the linear benzenoid chain $B(h)$, we have
$$\gamma_k(B(h)) = \begin{cases} \left\lceil \frac{h+1}{k} \right\rceil & \text{if } k \neq 2, \\ \left\lceil \frac{h+2}{k} \right\rceil & \text{if } k = 2. \end{cases}$$
We refer to Scheinerman and Ullman [14] for fractionalization techniques of various graph parameters. Hattingh et al. [8] introduced the distance \( k \)-dominating function and proved that the problem of computing the upper distance fractional domination number is NP-complete. In this paper we present further results on fractional distance \( k \)-domination.

### 2. Distance \( k \)-dominating Function

Hattingh et al. [8] introduced the following concept of fractional distance \( k \)-domination.

**Definition 2.1.** A function \( g : V \rightarrow [0, 1] \) is called a distance \( k \)-dominating function or simply a \( k \)-dominating function \((kDF)\) of a graph \( G = (V, E) \), if for every \( v \in V \), \( g(N_k[v]) = \sum_{u \in N_k[v]} g(u) \geq 1 \). A \( k \)-dominating function \((kDF)\) \( g \) of a graph \( G \) is called a minimal \( k \)-dominating function \((MkDF)\) if \( f \) is not a \( k \)-dominating function of \( G \) for all functions \( f : V \rightarrow [0, 1] \) with \( f < g \). The fractional \( k \)-domination number \( \gamma_{kf}(G) \) and the upper fractional \( k \)-domination number \( \Gamma_{kf}(G) \) are defined as follows:

\[
\gamma_{kf}(G) = \min\{|g| : g \text{ is an } MkDF \text{ of } G\}, \\
\Gamma_{kf}(G) = \max\{|g| : g \text{ is an } MkDF \text{ of } G\}.
\]

We observe that if \( k \geq \text{rad}(G) \), then \( \Delta(G^k) = n - 1 \) and \( \gamma_{kf}(G) = 1 \). Hence throughout this paper, we assume that \( k < \text{rad}(G) \).

**Lemma 2.2** [8]. Let \( f \) be a \( k \)-dominating function of a graph \( G = (V, E) \). Then \( f \) is minimal \( k \)-dominating if and only if whenever \( f(v) > 0 \) there exists some \( u \in N_k[v] \) such that \( f(N_k[u]) = 1 \).

**Remark 2.3.** The characteristic function of a \( \gamma_k \)-set and that of a \( \Gamma_k \)-set of a graph \( G \) are \( MkDFs \) of \( G \). Hence it follows that \( 1 \leq \gamma_{kf}(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq \Gamma_{kf}(G) \).

**Definition 2.4.** A function \( g : V \rightarrow [0, 1] \) is called a distance \( k \)-packing function or simply a \( k \)-packing function of a graph \( G = (V, E) \), if for every \( v \in V \), \( g(N_k[v]) \leq 1 \). A \( k \)-packing function \( g \) of a graph \( G \) is maximal if \( f \) is not a \( k \)-packing function of \( G \) for all functions \( f : V \rightarrow [0, 1] \) with \( f > g \). The fractional \( k \)-packing number \( p_{kf}(G) \) and the upper fractional \( k \)-packing number \( P_{kf}(G) \) are defined as follows:

\[
p_{kf}(G) = \min\{|g| : g \text{ is a maximal } k \text{-packing function of } G\}, \\
P_{kf}(G) = \max\{|g| : g \text{ is a maximal } k \text{-packing function of } G\}.
\]

**Observation 2.5.** The fractional \( k \)-domination number \( \gamma_{kf}(G) \) is the optimal solution of the following linear programming problem (LPP).
Lemma 2.6. For any graph $f$ defined on $G$ the graph $(V, E)$ we observe that Fractional Distance Domination in Graphs is sharp.

Proof. Since $|N_k[u]| \geq k + 1$ for all $u \in V$, it follows that the constant function $f$ defined on $V$ by $f(v) = \frac{1}{k+1}$ for all $v \in V$ is a $k$-dominating function with $|f| = \frac{n}{k+1}$. Hence $\gamma_k(G) \leq \frac{n}{k+1}$. To prove the sharpness of this bound, consider the graph $G$ consisting of a cycle of length $2k$ with a path of length $k$ attached to each vertex of the cycle. Clearly $n = 2k(k + 1)$. Further the set $S$ of all pendant vertices of $G$ forms an efficient $k$-dominating set of $G$ and hence $\sum_{u \in N_k[v]} f(u) = 1$ for all $v \in V$ where $f$ is the characteristic function of $S$. Hence $\gamma_k(G) = \gamma_k(G) = 2k = \frac{n}{k+1}$.

Observation 2.7. We observe that $\gamma_k(G) = \gamma_f(G^k)$. Hence the following is an immediate consequence of Theorem 1.2.

Let $G$ be a graph and let $A, B \subseteq V$. We say that $A, k$-dominates $B$ if $N_k[v] \cap A \neq \emptyset$ for all $v \in B$ and we write $A \rightarrow_k B$. Now for any $kDF$ $f$ of $G$ let $P_f = \{u \in V(G) : f(u) > 0\}$ and $B_f = \{u \in V(G) : f(N_k[u]) = 1\}$. Then $f$ is an $MkDF$ of $G$ if and only if $B_f \rightarrow_k P_f$.

Observation 2.8. If $f$ and $g$ are $kDF$s of a graph $G = (V, E)$ and $\lambda \in (0, 1)$, then the convex combination of $f$ and $g$ defined by $h_\lambda (v) = \lambda f(v) + (1 - \lambda)g(v)$ for all $v \in V$ is a $kDF$ of $G$. However, the convex combination of two $MkDF$s of a graph $G$ need not be minimal, as shown in the following example.

Consider the cycle $G = C_7 = (u_1 u_2 \ldots u_7 u_1)$ with $k = 2$. The function $f : V(G) \rightarrow [0, 1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_1, u_5\}, \\ 0 & \text{otherwise,} \end{cases}$$

is a minimal 2-dominating function of $G$ with $P_f = \{u_1, u_5\}$, $B_f = \{u_1, u_2, u_4, u_5\}$. Also, the function $g : V(G) \rightarrow [0, 1]$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x \in \{u_3, u_6\}, \\ 0 & \text{otherwise,} \end{cases}$$

is a minimal 2-dominating function of $G$ with $P_g = \{u_3, u_6\}$, $B_g = \{u_1, u_2, u_4, u_5\}$. However, the convex combination of two $MkDF$s of a graph $G$ need not be minimal, as shown in the following example.
is a minimal 2-dominating function of \( G \) with \( \mathcal{P}_g = \{u_3, u_6\}, \mathcal{B}_g = \{u_2, u_3, u_6, u_7\} \). Let \( h = \frac{1}{2}f + \frac{1}{2}g \). Then \( h(u_1) = h(u_3) = h(u_5) = h(u_6) = \frac{1}{2}, h(u_2) = h(u_4) = h(u_7) = 0, h(N_2[u_i]) = \frac{3}{2} \) for \( i \neq 2 \) and \( h(N_2[u_2]) = 1 \). Hence \( \mathcal{P}_h = \{u_1, u_3, u_5, u_6\} \) and \( \mathcal{B}_h = \{u_2\} \). Since \( u_5, u_6 \notin N_2[u_2] \) we have \( \mathcal{B}_h \) does not 2-dominate \( \mathcal{P}_h \) and hence the \( kDF \) \( h \) is not minimal.

**Observation 2.9.** If \( f \) and \( g \) are \( MkDFs \) of \( G \) and \( 0 < \lambda < 1 \), then \( h_\lambda = \lambda f + (1 - \lambda) g \) is an \( MkDF \) of \( G \) if and only if \( \mathcal{B}_f \cap \mathcal{B}_g \to_k \mathcal{P}_f \cup \mathcal{P}_g \).

**Observation 2.10.** For the cycle \( C_n \), the graph \( G = C_n^k \) is 2\( k \)-regular and hence it follows from Theorem 1.4 that \( \gamma_{kF}(C_n) = \frac{n}{2k+1} \).

We now proceed to determine the fractional \( k \)-domination number of several families of graphs.

**Proposition 2.11.** For the hypercube \( Q_n \), \( \gamma_{kF}(Q_n) = \frac{2^n}{\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}} \).

**Proof.** For any two vertices \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) in \( Q_n \), \( d(x, y) \leq k \) if and only if \( x \) and \( y \) differ in at most \( k \) coordinates and hence \( Q_n^k \) is \( r \)-regular where \( r = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} \). Hence by Theorem 1.4, we have \( \gamma_{kF}(Q_n) = \frac{2^n}{\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}} \).

**Proposition 2.12.** For the graph \( G = P_2 \Box C_n \), we have

\[
\gamma_{kF}(G) = \begin{cases} 
\frac{8}{7} & \text{if } n = 4 \text{ and } k = 2, \\
\frac{n}{2k} & \text{if } n \geq 5.
\end{cases}
\]

**Proof.** If \( n = 4 \) and \( k = 2 \), then \( G^2 \) is a 6-regular graph and hence \( \gamma_{2F}(G) = \frac{8}{7} \). If \( n \geq 5 \), \( G^k \) is a \((4k - 1)\)-regular graph and hence \( \gamma_{kF}(G) = \frac{2^n}{4k-1+1} = \frac{n}{2k} \).

**Theorem 2.13.** Let \( G = C_n \circ K_1 \). Then \( \gamma_{kF}(G) = \frac{n}{2k-1} \).

**Proof.** Let \( C_n = (v_1v_2 \ldots v_nv_1) \). Let \( u_i \) be the pendant vertex adjacent to \( v_i \). Clearly, \( |N_k[u_i] \cap V(C_n)| = 2k - 1 \) and \( N_k[u_i] \subset N_k[v_i], 1 \leq i \leq n \). Hence the function \( g : V(G) \to \{0, 1\} \) defined by

\[
g(x) = \begin{cases} 
0 & \text{if } x = u_i, \\
\frac{1}{2k-1} & \text{if } x = v_i
\end{cases}
\]

is a minimal \( k \)-dominating function of \( G \) with \( |g| = \frac{n}{2k-1} \). Also we have \( |N_k[v_i] \cap \{u_j : 1 \leq j \leq n\}| = 2k - 1, 1 \leq i \leq n \). Hence the function \( h : V(G) \to \{0, 1\} \) defined by

\[
h(x) = \begin{cases} 
\frac{1}{2k-1} & \text{if } x = u_i, \\
0 & \text{if } x = v_i
\end{cases}
\]

is a maximal \( k \)-packing function of \( G \) with \( |h| = \frac{n}{2k-1} \). Hence by Observation 2.5, we have \( \gamma_{kF}(G) = \frac{n}{2k-1} \).
Theorem 2.14. For the grid graph \( G = P_2 \square P_n \), we have
\[
\gamma_{kf}(G) = \begin{cases} 
\frac{n(n+2k)}{2k(n+k)} & \text{if } n \equiv 0 \pmod{2k}, \\
\left\lceil \frac{n}{2k} \right\rceil & \text{otherwise.}
\end{cases}
\]

Proof. Let \( P_2 = (u_0, u_1) \) and \( P_n = (v_0, v_1, \ldots, v_{n-1}) \), so that \( V(G) = \{(u_i, v_j) : i = 0, 1, 0 \leq j \leq n-1\} \).

Case 1. \( n \equiv 0 \pmod{2k} \). Let \( n = 2kp, p > 1 \). Define \( f : V(G) \to [0, 1] \) by
\[
f((u_i, v_j)) = \begin{cases} 
\left(\frac{1}{2p+1}\right)(p - \left\lfloor \frac{j}{2k} \right\rfloor) & \text{if } j \equiv (k-1) \pmod{2k}, \\
\left(\frac{1}{2p+1}\right)(\left\lfloor \frac{j}{2k} \right\rfloor + 1) & \text{if } j \equiv k \pmod{2k}, \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( f \) is a \( k \)-dominating function of \( G \). Also, since \( f((u_0, v_j)) = f((u_1, v_j)) \) for all \( j \), we have \( |f| = 2(\sum_{j=0}^{n-1} f((u_0, v_j))) = \frac{2}{2p+1}(p+(p-1)+\cdots+3+2+1)+(1+2+3+\cdots+p) = \frac{2p(p+1)}{2p+1} = \frac{n(n+2k)}{2k(n+k)}. \) Now consider the function \( h : V(G) \to [0, 1] \) defined by
\[
h((u_i, v_j)) = \begin{cases} 
\left(\frac{1}{2p+1}\right)(p - \left\lfloor \frac{j}{2k} \right\rfloor) & \text{if } j \equiv 0 \pmod{2k}, \\
\left(\frac{1}{2p+1}\right)(\left\lfloor \frac{j}{2k} \right\rfloor + 1) & \text{if } j \equiv (2k-1) \pmod{2k}, \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( h \) is a \( k \)-packing function of \( G \) with \( |h| = \frac{2p(p+1)}{2p+1} = \frac{n(n+2k)}{2k(n+k)}. \) Hence \( \gamma_{k,f}(G) = \frac{n(n+2k)}{2k(n+k)}. \)

Case 2. \( n \not\equiv 0 \pmod{2k} \). Let \( n = 2kq+r, 1 \leq r \leq 2k-1 \). Let \( S = S_1 \cup S_2 \) and
\[
S_1 = \begin{cases} 
\{(u_0, v_j) : j \equiv 0 \pmod{4k}\} & \text{if } 1 \leq r \leq k, \\
\{(u_0, v_j) : j \equiv (k-1) \pmod{4k}\} & \text{if } k+1 \leq r \leq 2k-1.
\end{cases}
\]
\[
S_2 = \begin{cases} 
\{(u_1, v_j) : j \equiv 2k \pmod{4k}\} & \text{if } 1 \leq r \leq k, \\
\{(u_1, v_j) : j \equiv (3k-1) \pmod{4k}\} & \text{if } k+1 \leq r \leq 2k-1.
\end{cases}
\]

Let \( f \) be the characteristic function of \( S \). Since \( d(x,y) \geq 2k+1 \) for all \( x,y \in S \), it follows that \( f(N_k[u]) = 1 \) for all \( u \in V(G) \). Thus \( f \) is both a minimal \( k \)-dominating function and a maximal \( k \)-packing function of \( G \) and hence \( \gamma_{k,f}(G) = |f| = |S| = \left\lceil \frac{n}{2k} \right\rceil. \)

A special case of the above theorem gives the following result of Hare [7].

Corollary 2.15. For the grid graph \( G = P_2 \square P_n \), we have
\[
\gamma_f(G) = \begin{cases} 
\frac{n(n+2)}{2(n+1)} & \text{if } n \text{ is even}, \\
\left\lfloor \frac{n}{2} \right\rfloor & \text{if } n \text{ is odd.}
\end{cases}
\]
3. Graphs with $\gamma_{kf}(G) = \gamma_k(G)$

In this section we obtain several families of graphs for which the fractional $k$-domination number and the $k$-domination number are equal.

Lemma 3.1. If a graph $G$ has an efficient $k$-dominating set, then $\gamma_{kf}(G) = \gamma_k(G)$.

Proof. Let $D$ be an efficient $k$-dominating set of $G$. Then $|N_k[u] \cap D| = 1$ for all $u \in V(G)$. Hence the characteristic function of $D$ is a minimal $k$-dominating function and a maximal $k$-packing function of $G$ and so $\gamma_{kf}(G) = \gamma_k(G)$.

Lemma 3.2. For any graph $G$, $\gamma_{kf}(G) = 1$ if and only if $\gamma_k(G) = 1$.

Proof. Suppose $\gamma_k(G) = 1$. Since $\gamma_{kf}(G) \leq \gamma_k(G)$, it follows that $\gamma_{kf}(G) = 1$. Conversely, let $\gamma_{kf}(G) = 1$. Then $\gamma_{f}(G^k) = 1$ and hence $\gamma(G^k) = 1$. Since $\gamma(G^k) = \gamma_k(G)$ the result follows.

Lemma 3.3. For any graph $G$, $p_{kf}(G) \leq \rho_{2k}(G) \leq P_{kf}(G)$.

Proof. Let $u \in V(G)$. Since $N_k[u] = N_{G^k}[u]$, we have $p_{kf}(G) = p_{f}(G^k)$, $P_{kf}(G) = P_{f}(G^k)$ and $\rho_{2k}(G) = P_{2}(G^k)$.

Hence the result follows from Theorem 1.1.

Corollary 3.4. For any graph $G$, $1 \leq p_{kf}(G) \leq \rho_{2k}(G) \leq P_{kf}(G) = \gamma_{kf}(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq \Gamma_{kf}(G)$.

Corollary 3.5. If $G$ is any graph with $\rho_{2k}(G) = \gamma_k(G)$, then $\gamma_{kf}(G) = \gamma_k(G)$.

Corollary 3.6. If $G$ is a block graph, then $\gamma_{kf}(G) = \gamma_k(G)$.

Proof. It follows from Theorem 1.5 that $\rho_{2k}(G) = \gamma_k(G)$ and hence the result follows.

Corollary 3.7. For any tree $T$, we have $\gamma_{kf}(T) = \gamma_k(T)$.

Theorem 3.8. For the graph $G = P_{k+1} \square P_n$ where $n \equiv 1 \pmod{(k+1)}$, $k \geq 1$, we have $\gamma_{kf}(G) = \gamma_k(G) = \lceil \frac{n}{k+1} \rceil$.

Proof. Let $n = (k+1)q + 1$, $q \geq 1$. Clearly $|V(G)| = n(k+1) = (k+1)^2q + (k+1)$. Let $P_{k+1} = (u_0, u_1, u_2, \ldots, u_k)$ and $P_n = (v_0, v_1, \ldots, v_{n-1})$ so that $V(G) = \{(u_i, v_j) : 0 \leq i \leq k, 0 \leq j \leq n-1\}$.

Now let $S_1 = \{(u_0, v_i) : i \equiv 0 \pmod{(k+1)}\}$, $S_2 = \{(u_k, v_i) : i \equiv (k+1) \pmod{(2k+1)}\}$ and $S = S_1 \cup S_2$. Clearly, $d(x, y) = (2k+1)r$, $r \geq 1$, for all $x, y \in S$ and $|S| = \lceil \frac{n}{k+1} \rceil = q + 1$. Also, $(u_0, v_0)$ and exactly one of
the vertices \((u_0, v_{n-1})\) or \((u_k, v_{n-1})\) are in \(S\) and each of these two vertices \(k\)-dominates \(\frac{(k+1)(k+2)}{2}\) vertices of \(G\). Also, if \(u \in N_k[x] \cap N_k[y]\), where \(x, y \in S\), then \(d(u, x) \leq k\), \(d(u, y) \leq k\) and so \(d(x, y) \leq d(x, u) + d(u, y) \leq 2k\), which is a contradiction. Thus \(N_k[x] \cap N_k[y] = \emptyset\) for all \(x, y \in S\). Each of the remaining vertices of \(S\) \(k\)-dominates \((k + 1)^2\) vertices of \(G\). Further, \(|V(G)| - (k + 1)(k + 2)\) is a multiple of \((k + 1)^2\) and hence it follows that \(S\) is an efficient \(k\)-dominating set of \(G\). Hence, by Lemma 3.1, we have \(\gamma_{kf}(G) = \gamma_k(G) = |S| = \left\lfloor \frac{n}{k+1} \right\rfloor\).

**Theorem 3.9.** For the graph \(G = P_3 \square P_n\), we have \(\gamma_{2f}(G) = \gamma_2(G) = \left\lceil \frac{n}{3} \right\rceil\).

**Proof.** If \(n \equiv 1 \pmod{3}\), then the result follows from Theorem 3.8. Suppose \(n \equiv 0 \pmod{3}\) or \(2 \pmod{3}\). Let \(n = 3q, q \geq 1\) or \(n = 3q + 2, q \geq 0\). Let \(P_3 = (u_0, u_1, u_2)\) and \(P_n = (v_0, v_1, \ldots, v_{n-1})\) so that \(V(G) = \{(u_i, v_j) : 0 \leq i \leq 2, 0 \leq j \leq n - 1\}\). Now \(D = \{(u_i, v_j) : j \equiv 1 \pmod{3}\}\) is a \(\gamma_2\)-set of \(G\) with \(|D| = \left\lfloor \frac{n}{3} \right\rfloor\) and hence \(\gamma_2(G) = \left\lceil \frac{n}{3} \right\rceil\). Further \(f = \chi_D\) is a 2-dominating function of \(G\) with \(|f| = \left\lceil \frac{n}{3} \right\rceil\). Also let \(S_1 = \{(u_0, v_j) : j \equiv 0 \pmod{6}\}\), \(S_2 = \{(u_2, v_j) : j \equiv 3 \pmod{6}\}\) and \(S = S_1 \cup S_2\). Then \(g = \chi_S\) is a 2-packing function of \(G\) with \(|g| = \left\lceil \frac{n}{3} \right\rceil\). Hence \(\gamma_{2f}(G) = \left\lceil \frac{n}{3} \right\rceil\).

**Observation 3.10.** The graph \(G = P_3 \square P_3\) does not have an efficient 2-dominating set. In fact the set \(S = \{(u_0, v_0), (u_2, v_3)\}\) efficiently 2-dominates 14 vertices of \(G\) and the vertex \((u_0, v_4)\) is not \(2\)-dominated by \(S\). Further if \(S\) is any \(2\)-dominating set of \(G\) with \(|S| = \gamma_2(G) = 2\), then at least one vertex of \(G\) is \(2\)-dominated by both vertices of \(S\). This shows that the converse of Lemma 3.1 is not true.

**Theorem 3.11.** For the linear benzenoid chain \(G = B(h)\), we have

\[
\gamma_{kf}(G) = \gamma_k(G) = \begin{cases} \frac{h}{2} + 1 & \text{if } k = 2 \text{ and } h \equiv 0 \pmod{2}, \\ \left\lceil \frac{h}{2} \right\rceil & \text{if } k \geq 3 \text{ and } h \equiv \left\lfloor \frac{h}{2} \right\rfloor \pmod{k}. \end{cases}
\]

**Proof.** Since \(G = B(h)\) is a subgraph of \(P_2 \square P_{2h+1}\), we take \(V(G) = \{(u_i, v_j) : i = 0, 1, 0 \leq j \leq 2h\}\), where \(P_2 = (u_0, u_1)\) and \(P_{2h+1} = (v_0, v_1, \ldots, v_{2h})\). Clearly, \(|V(G)| = 4h + 2\). Any vertex \(u \in V(G)\) \(k\)-dominates at most \(4k\) vertices of \(G\) and hence \(\gamma_k(G) \geq \left\lceil \frac{4h+2}{4k} \right\rceil\).

**Case 1.** \(k = 2\) and \(h \equiv 0 \pmod{2}\). In this case we have \(\gamma_2(G) \geq \left\lceil \frac{4h+2}{8} \right\rceil = \frac{h}{2} + 1\). Now let \(S_1 = \{(u_0, v_j) : j \equiv 0 \pmod{8}\}\), \(S_2 = \{(u_1, v_j) : j \equiv 4 \pmod{8}\}\) and \(S = S_1 \cup S_2\). Clearly, for any \(x, y \in S\), \(d(x, y) \geq 5\) and hence \(N_2[x] \cap N_2[y] = \emptyset\). Also \(|S| = \left\lceil \frac{2h+1}{4} \right\rceil = \frac{h}{2} + 1\). Now \((u_0, v_0)\) and exactly one of the vertices \((u_0, v_{2h})\) or \((u_1, v_{2h})\) is in \(S\) and each of these two vertices \(2\)-dominates exactly \(5\) vertices of \(G\). Each of the remaining vertices of \(S\) \(2\)-dominates \(8\) vertices of \(G\). Further \(|V(G)| - 10 = 4h - 8 = \frac{8h}{2} - 1\), which is a multiple of \(8\) and hence it follows that \(S\) is an efficient \(2\)-dominating set of \(G\). Hence \(\gamma_{2f}(G) = \gamma_2(G) = |S| = \frac{h}{2} + 1\).
Case 2. \( k \geq 3 \) and \( h \equiv \left\lfloor \frac{k^2}{2} \right\rfloor \) (mod \( k \)). Let \( h = kq + \left\lfloor \frac{k}{2} \right\rfloor, q \geq 1 \). In this case we have \( \gamma_k(G) \geq \left\lceil \frac{1}{k} \left( \frac{4h+2}{k^2} \right) \right\rceil = \left\lceil \frac{h}{k} \right\rceil \). Now let \( S_1 = \{ (u_0, v_j) : j \equiv (k-1) \) (mod \( 4k \}) \}, \( S_2 = \{ (u_1, v_j) : j \equiv (3k-1) \) (mod \( 4k \}) \} \) and \( S = S_1 \cup S_2 \). Clearly, \( d(x, y) = (2k+1)r, r \geq 1 \) for all \( x, y \in S \), hence \( N_k[x] \cap N_k[y] = \emptyset \). Also \( |S| = \left\lceil \frac{2h-(k-1)}{2k} \right\rceil = \left\lceil \frac{h}{k} \right\rceil \).

Now, when \( k \) is odd, exactly one of the vertices \((u_0, v_{2h})\) or \((u_1, v_{2h})\) is in \( S \) and it \( k \)-dominates \( 2k+1 \) vertices. When \( k \) is even, exactly one of the vertices \((u_0, v_{2h-1})\) or \((u_1, v_{2h-1})\) are in \( S \) and it \( k \)-dominates \( 2k+3 \) vertices. The vertex \((u_0, v_{k-1})\) \( k \)-dominates \( 4k-1 \) vertices. In both cases the number of vertices of \( G \) which are not \( k \)-dominated by these two vertices is a multiple of \( 4k \) and each of the remaining vertices of \( S \) \( k \)-dominates \( 4k \) vertices of \( G \). Hence it follows that \( S \) is an efficient \( k \)-dominating set of \( G \) so that \( \gamma_kf(G) = \gamma_k(G) = |S| = \left\lceil \frac{h}{k} \right\rceil \).

Conclusion. In this paper we have determined the fractional \( k \)-domination number of several families of graphs. We have also obtained several families of graphs for which \( \gamma_kf(G) = \gamma_k(G) \). The study of the fractional version of distance \( k \)-irredundance and distance \( k \)-independence remains open. Slater has mentioned several efficiency parameters such as redundance and influence in Chapter 1 of [10]. One can investigate these parameters for fractional distance domination. The following are some interesting problems for further investigation.

1. Characterize the class of graphs \( G \) for which \( \gamma_kf(G) = \frac{n}{k+1} \).
2. Characterize the class of graphs \( G \) with \( \gamma_kf(G) = \gamma_k(G) \).
3. Determine \( \gamma_kf(P_r \square P_s) \) for \( r, s \geq 4 \).

Acknowledgement

We are thankful to the National Board for Higher Mathematics, Mumbai, for its support through the project 48/5/2008/R&D-II/561, awarded to the first author. The second author is thankful to the UGC, New Delhi for the award of FIP teacher fellowship during the XIth plan period. We are also thankful to the referees for their helpful suggestions.

References


Received 22 December 2010
Revised 12 August 2011
Accepted 16 August 2011