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NOTE ON THE CORE MATRIX PARTIAL ORDERING

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Abstract

Complementing the work of Baksalary and Trenkler [2], we announce some results characterizing the core matrix partial ordering.

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1. Preliminaries

Let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ matrices with complex entries. We will denote the conjugate transpose, range (column space), and nullspace of $A \in \mathbb{C}^{m\times n}$ by A^*, $R(A)$, and $N(A)$, respectively. P_A will stand for the orthogonal projector on $R(A)$. We use I to denote an identity matrix with dimensions following from the context.

We start by stating several basic facts on generalized inverses. As references, one can consult [4, Sections 2.2–2.5] or [5, Sections 4.2–4.5].

We let A^- designate a generalized inverse of A, this being defined as a solution to the matrix equation $AXA = A$. A least squares generalized inverse of $A \in \mathbb{C}^{m\times n}$, written as A_ℓ^-, is defined to be a solution to the matrix equation $AX = P_A$ ([4, Theorem 2.5.14]). The collection of all A_ℓ^- is denoted by $\left\{A_\ell^-\right\}$. In light of Theorems 2.5.24 (ii) and 2.5.27 in [4], the

Moore-Penrose inverse of A is the unique element A^+ of $\{A_\ell^-\}$ with the property $R(A^+) = R(A^*)$. The general expression of A_ℓ^- can be written as $A_\ell^- = A^+ + (I - A^+A)U$, where $U \in \mathbb{C}^{n\times m}$ is arbitrary ([4, Theorem 2.5.17]). We will use the following simple fact ([4, Theorem 2.5.28 (iv)]): $A^+ = (A^*A)^+ A^*$.

We shall mostly be concerned with core matrices. Recall that a square matrix A is said to be core if $R(A)$ and $N(A)$ are complementary subspaces, which is equivalent to saying that $R(A) = R(A^2)$. Given a core matrix A, we let Q_A represent the projector which projects a vector on $R(A)$ along $N(A)$. A c-inverse A_c^- of a core matrix A is defined to be a solution to the matrix equation $XA = Q_A$ ([4, Definition 6.4.1]). We let $\{A_c^-\}$ denote the collection of all A_c^-. Among the c-inverses, those having $R(A_c^-) = R(A)$ are called χ-inverses ([4, Definition 2.4.1]). According to Theorem 2.4.3 and Remark 2.4.14 of [4], the group inverse $A^\#$ is the uniquely determined χ-inverse satisfying the following condition $N(A^\#) = N(A)$. It is evident that $A^\#$ is a reflexive generalized inverse of A such that $AA^\# = A^\#A$ ([4, Theorem 2.4.6]).

Following [2], we define the core inverse $A^{\oplus}$ by $A^{\oplus} = A^\# AA^+$. In fact, $A^{\oplus}$ is the unique generalized inverse of A, which is both a least squares inverse and a χ-inverse of A. In [2] there are presented some results on characterizations of $A^{\oplus}$. Finally, let us point out that the core inverse coincides with the hybrid inverse $A_{\rho^*\chi}^-$ defined by Rao and Mitra [5, Section 4.10.2].

2. Core matrix partial order

We will be concerned here with the core relation defined by Baksalary and Trenkler [2].

Definition 1. For a pair of core matrices $A, B \in \mathbb{C}^{n\times n}$ we define the core relation $<^{\oplus}$ by saying that $A <^{\oplus} B$ if the following condition is satisfied:

$$A^{\oplus}(B - A) = (B - A)A^{\oplus} = 0. \tag{1}$$

The lemma below gives two other conditions that are equivalent to (1).

Lemma 2. *Let A and B be core matrices of the same order. Then the following statements are equivalent:*

1. $A <^{\oplus} B$,
2. $A^{+}(B-A) = (B-A)A^{\#} = 0$,
3. $A^{*}A = A^{*}B$ *and* $BA = A^{2}$.

Proof. We first recall the well-known fact ([3, Fact 2.10.12]) that $rank(AB) = rank(A)$ if and only if $R(AB) = R(A)$. This result implies, and is in fact equivalent to, the statement that $rank(AB) = rank(B)$ if and only if $N(AB) = N(B)$.

To establish the claim, observe that $A^{\oplus}$, A^{+}, $A^{\#}$ and A have the same rank. Hence, $R\left(A^{\oplus}\right) = R\left(A^{\#}\right) = R\left(A\right)$ and $N\left(A^{\oplus}\right) = N\left(A^{+}\right) = N\left(A^{*}\right)$, from which the required result follows. ■

Let us mention here another equivalent formulation of condition (1). As observed in [2, (3.21)], $A <^{\oplus} B$ if and only if $A^{+}B = A^{+}A$ and $BA = A^{2}$.

Another concept referred to is the minus partial ordering (see, for example, [4, Chapter 3]). We say that $A \in \mathbb{C}^{m\times n}$ is below $B \in \mathbb{C}^{m\times n}$ under the minus partial order, and write $A <^{-} B$, if $(A-B)A^{-} = 0$ and $A^{-}(A-B) = 0$ for some generalized inverse A^{-}.

It is worth making the following Proposition, which includes Theorem 8 in [2].

Proposition 3. *If* $A <^{\oplus} B$ *then* $A <^{-} B$, $R(A) \subset R(B)$, $R\left(A^{*}\right) \subset R\left(B^{*}\right)$. *The relation* $<^{\oplus}$ *is reflexive and antisymmetric.*

The following Theorem describes a new property of the core relation $<^{\oplus}$.

Theorem 4. $A <^{\oplus} B$ *if and only if* $\left\{B_{\ell}^{-}\right\} \subset \left\{A_{\ell}^{-}\right\}$ *and* $\left\{B_{c}^{-}\right\} \subset \left\{A_{c}^{-}\right\}$.

Proof. For proof of necessity, assume that $G \in \left\{B_{\ell}^{-}\right\}$. Since $A <^{\oplus} B$, we have $A^{*}A = A^{*}B$ and $R(A) \subset R(B)$. Therefore $A^{*}AG = A^{*}BB^{+} = A^{*}$. Premultiplying this relationship by $A\left(A^{*}A\right)^{+}$ yields $AG = AA^{+}$, which justifies $\left\{B_{\ell}^{-}\right\} \subset \left\{A_{\ell}^{-}\right\}$. Suppose next that $G \in \left\{B_{c}^{-}\right\}$. Since $BA = A^{2}$, we get $GA = GA^{2}A^{\#} = GBAA^{\#} = Q_{B}AA^{\#} = AA^{\#}$. This proves that $\left\{B_{c}^{-}\right\} \subset \left\{A_{c}^{-}\right\}$.

To show sufficiency, note that our assumption $\left\{B_{c}^{-}\right\} \subset \left\{A_{c}^{-}\right\}$ forces $A = B^{\#}A^{2}$. Then, clearly, $R(A) \subset R(B)$, and consequently, $BA = BB^{\#}A^{2} = A^{2}$, as needed. Next, to establish $A^{*}A = A^{*}B$, we consider the general expression $B_{\ell}^{-} = B^{+} + (I - B^{+}B)U$. If $\left\{B_{\ell}^{-}\right\} \subset \left\{A_{\ell}^{-}\right\}$, then $AB_{\ell}^{-} = AB^{+}$, and consequently, $A(I - B^{+}B)U = 0$ for every $U \in \mathbb{C}^{n\times n}$, which implies that $A = AB^{+}B$. Hence $R(A^{*}) \subset R(B^{*})$. Moreover, $\{B_{\ell}^{-}\} \subset \{A_{\ell}^{-}\}$ guarantees that $A^{*} = A^{*}AB^{+}$. Therefore $A^{*}B = A^{*}AB^{+}B = A^{*}A$, as required. ■

Theorem 4 guarantees that the core relation is transitive. On account of Proposition 3, we obtain that the relation $<^{\oplus}$ defines a matrix partial ordering ([2, Theorem 6]).

In the following we shall link different types of partial orders together. The following terminology will be required ([4, Definitions 6.3.1, 6.5.2]).

For $A, B \in \mathbb{C}^{m\times n}$, we define the left star relation $*<$ by saying that $A*< B$ if $R(A) \subset R(B)$ and $A^{*}A = A^{*}B$.

For core matrices $A, B \in \mathbb{C}^{n\times n}$ we define the right sharp relation $<\#$ by setting $A <\# B$ if $R(A^{*}) \subset R(B^{*})$ and $A^{2} = BA$.

The star relation is due to Baksalary and Mitra [1]. As is well known, the left star and the right sharp relation are partial orders ([1], [4, Corollary 6.3.10])

Proposition 3 permits us to conclude with the following

Proposition 5. *$A <^{\oplus} B$ if and only if $A*< B$ and $A <\# B$.*

As a matter of fact, Proposition 5 states that the core relation is an intersection partial ordering ([4, Definition A.8.1]).

Some remarks are due. It was our intention here to present a fairly simple and selfcontained proof of Theorem 4. However, once Proposition 5 is established, Theorem 4 may be achieved by appealing to characterizations of one-sided orders as given by Theorems 6.4.8 and 6.5.17 in [4].

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