

DOMINATION IN FUNCTIGRAPHS

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Abstract

Let G_1 and G_2 be disjoint copies of a graph G , and let $f : V(G_1) \rightarrow V(G_2)$ be a function. Then a *functigraph* $C(G, f) = (V, E)$ has the vertex set $V = V(G_1) \cup V(G_2)$ and the edge set $E = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2), v = f(u)\}$. A functigraph is a generalization of a *permutation graph* (also known as a *generalized prism*) in the sense of Chartrand and Harary. In this paper, we study domination in functigraphs. Let $\gamma(G)$ denote the domination number of G . It is readily seen that $\gamma(G) \leq \gamma(C(G, f)) \leq 2\gamma(G)$. We investigate for graphs generally, and for cycles in great detail, the functions which achieve the upper and lower bounds, as well as the realization of the intermediate values.

Keywords: domination, permutation graphs, generalized prisms, functigraphs.

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1. INTRODUCTION AND DEFINITIONS

Throughout this paper, $G = (V(G), E(G))$ stands for a finite, undirected, simple and connected graph with order $|V(G)|$ and size $|E(G)|$. A set $D \subseteq V(G)$ is a *dominating set* of G if for every vertex $v \in V(G) \setminus D$, there exists a vertex $u \in D$ such that v and u are adjacent. The *domination number* of a graph G , denoted by $\gamma(G)$, is the minimum of the cardinalities of all dominating sets of G . For earlier discussions on domination in graphs, see [3, 4, 10, 16]. For further reading on domination, refer to [13] and [14].

For any vertex $v \in V(G)$, the *open neighborhood* of v in G , denoted by $N_G(v)$, is the set of all vertices adjacent to v in G . The *closed neighborhood* of v , denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. Throughout the paper, we denote by $N(v)$ (resp., $N[v]$) the open (resp., closed) neighborhood of v in $C(G, f)$. The maximum degree of G is denoted by $\Delta(G)$. For a given graph G and $S \subseteq V(G)$, we denote by $\langle S \rangle$ the subgraph induced by S . Refer to [8] for additional graph theory terminology.

Chartrand and Harary studied planar permutation graphs in [7]. Hedetniemi introduced two graphs (not necessarily identical copies) with a function relation between them; he called the resulting object a “function graph” [15]. Independently, Dörfler introduced a “mapping graph”, which consists of two disjoint identical copies of a graph and additional edges between the two vertex sets specified by a function [11]. Later, an extension of permutation graphs, called *functigraph*, was rediscovered and studied in [9]. In the current paper, we study domination in functigraphs. We recall the definition of a functigraph in [9].

Definition. Let G_1 and G_2 be two disjoint copies of a graph G , and let f be a function from $V(G_1)$ to $V(G_2)$. Then a functigraph $C(G, f)$ has the vertex set $V(C(G, f)) = V(G_1) \cup V(G_2)$, and the edge set $E(C(G, f)) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2), v = f(u)\}$.

Throughout the paper, $V(G_1)$ denotes the *domain* of a function f ; $V(G_2)$ denotes the *codomain* of f ; $\text{Range}(f)$ denotes the *range* of f . For a set $S \subseteq V(G_2)$, we denote by $f^{-1}(S)$ the set of all pre-images of the elements of S ; i.e., $f^{-1}(S) = \{v \in V(G_1) \mid f(v) \in S\}$. Also, C_n denotes a cycle of length $n \geq 3$, and *id* denotes the identity function. Let $V(G_1) = \{u_1, u_2, \dots, u_n\}$ and $V(G_2) = \{v_1, v_2, \dots, v_n\}$. For simplicity, we sometimes refer to each vertex of the graph G_1 (resp., G_2) by the index i (resp., i') of its label u_i (resp., v_i) for $1 \leq i, i' \leq n$. When $G = C_n$, we assume that the vertices of G_1 and G_2 are labeled cyclically. It is readily seen that $\gamma(G) \leq \gamma(C(G, f)) \leq 2\gamma(G)$. We study the domination of $C(C_n, f)$ in great detail: for $n \equiv 0 \pmod{3}$, we characterize the domination number for an infinite class of functions and state conditions under which the upper bound is not achieved; for $n \equiv 1, 2 \pmod{3}$, we prove that, for any function f , the

domination number of $C(C_n, f)$ is strictly less than $2\gamma(C_n)$. These results extend and generalize a result by Burger, Mynhardt, and Weakley in [6].

Domination number on permutation graphs (generalized prisms) has been extensively investigated in a great many articles, among these are [1, 2, 5, 6, 12]; the present paper primarily deepens — and secondarily broadens — the current state of knowledge.

2. DOMINATION NUMBER OF FUNCTIGRAPHS

First we consider the lower and upper bounds of the domination number of $C(G, f)$.

Proposition 1. *For any graph G , $\gamma(G) \leq \gamma(C(G, f)) \leq 2\gamma(G)$.*

Proof. Let D be a dominating set of G . Since a copy of D in G_1 together with a copy of D in G_2 form a dominating set of $C(G, f)$ for any function f , the upper bound follows. For the lower bound, assume there is a dominating set D of $C(G, f)$ such that $|D| < \gamma(G)$. Let $D_1 = D \cap V(G_1) \neq \emptyset$ and $D_2 = D \cap V(G_2) \neq \emptyset$, with $D_1 \cup D_2 = D$. Now, for each $x \in D_1$, x dominates exactly one vertex in G_2 , namely $f(x)$. And so $D_2 \cup \{f(x) \mid x \in D_1\}$ is a dominating set of G_2 of cardinality less than or equal to $|D|$, but $|D| < \gamma(G_2)$ — a contradiction. ■

Next we consider realization results for an arbitrary graph G .

Theorem 2. *For any pair of integers a, b such that $1 \leq a \leq b \leq 2a$, there is a connected graph G for which $\gamma(G) = a$ and $\gamma(C(G, f)) = b$ for some function f .*

Proof. Let the star $S_i \cong K_{1,4}$ have center c_i for $1 \leq i \leq a$. Let G be a chain of a stars; i.e., the disjoint union of a stars such that the centers are connected to form a path of length a (and no other additional edges) — see Figure 1. Label the stars in the chain of the domain G_1 by S_1, S_2, \dots, S_a and label their centers by c_1, c_2, \dots, c_a , respectively. Likewise, label the stars in the chain of the codomain G_2 by S'_1, S'_2, \dots, S'_a and label their centers by c'_1, c'_2, \dots, c'_a , respectively. More generally, denote by v' the vertex in G_2 corresponding to an arbitrary v in G_1 .

We define $a + 1$ functions from G_1 to G_2 as follows. Let f_0 be the “identity function”; i.e., $f_0(v) = v'$. For each i from 1 to a , let f_i be the function which collapses S_1 through S_i to c'_1 through c'_i , respectively, and which acts as the “identity” on the remaining vertices: $f_i(S_j) = c'_j$ for $1 \leq j \leq i$ and $f_i(v) = v'$ for $v \notin \bigcup_{1 \leq j \leq i} V(S_j)$. (See Figure 1.) Notice $\gamma(G) = a$.

Claim. $\gamma(C(G, f_i)) = 2a - i$ for $0 \leq i \leq a$.

First, $\gamma(C(G, f_a)) = a$ because $D_a = \{c'_1, \dots, c'_a\}$ clearly dominates $C(G, f_a)$.

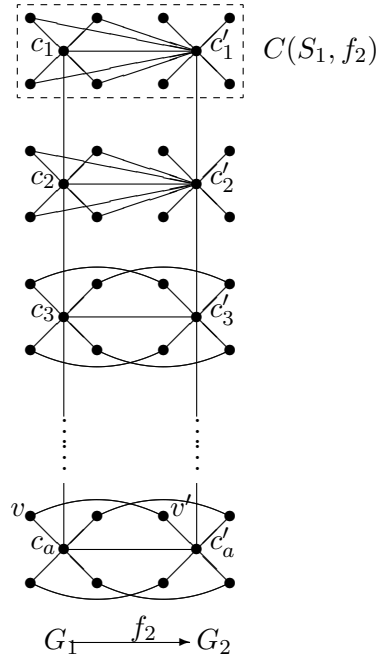


Figure 1. Realization graphs.

Second, consider $C(G, f_0)$. $D_0 = \{c_1, \dots, c_a, c'_1, \dots, c'_a\}$, the set of centers in G_1 or G_2 , is a dominating set; so $\gamma(C(G, f_0)) \leq 2a$ as noted earlier. It suffices to show that $\gamma(C(G, f_0)) \geq 2a$. It is clear that a dominating set D consisting only of the centers must have size $2a$ — for a pendant to be dominated, its neighboring center must be in D . We need to check that the replacement of centers by some (former) pendants (of G_1 or G_2) will only result in a dominating set D' such that $|D'| > |D_0|$. It suffices to check $C(S_i, f_0)$ at each i , a subgraph of $C(G, f_0)$ — since pendant domination is a local question: the closed neighborhood of each pendant of $C(S_i, f_0)$ is contained within $C(S_i, f_0)$. It is easy to see that the unique minimum dominating set of $C(S_i, f_0)$ consists of the two centers c_i and c'_i .

Finally, the set $D_i = \{c_{i+1}, \dots, c_a, c'_1, \dots, c'_a\}$ is a minimum dominating set of $C(G, f_i)$: in relation to $C(G, f_0)$, the subset $\{c_1, \dots, c_i\}$ of D_0 is not needed since the set $\{c'_1, \dots, c'_i\}$ dominates $\bigcup_{1 \leq j \leq i} V(S_j)$ in $C(G, f_i)$. The local nature of pendant domination and the fact that $f_i|_{S_j} = f_0|_{S_j}$ for $j > i$ ensure that D_i has minimum cardinality. ■

3. CHARACTERIZATION OF LOWER BOUND

We now present a characterization for $\gamma(C(G, f)) = \gamma(G)$, in analogy with what was done for permutation-fixers in [5].

Theorem 3. *Let G_1 and G_2 be two copies of a graph G in $C(G, f)$. Then $\gamma(G) = \gamma(C(G, f))$ if, and only if, there are sets $D_1 \subseteq V(G_1)$ and $D_2 \subseteq V(G_2)$ satisfying the following conditions:*

1. D_1 dominates $V(G_1) \setminus f^{-1}(D_2)$,
2. D_2 dominates $V(G_2) \setminus f(D_1)$,
3. $D_2 \cup f(D_1)$ is a minimum dominating set of G_2 ,
4. $|D_1| = |f(D_1)|$,
5. $D_2 \cap f(D_1) = \emptyset$, and
6. $D_1 \cap f^{-1}(D_2) = \emptyset$.

Proof. (\Leftarrow) Suppose there are sets $D_1 \subseteq V(G_1)$ and $D_2 \subseteq V(G_2)$ satisfying the specified conditions. Clearly $D_1 \cup D_2$ is a dominating set of $C(G, f)$. By assumption, $D_2 \cup f(D_1)$ is a minimum dominating set of G_2 . Since $|D_1| = |f(D_1)|$ and $D_2 \cap f(D_1) = \emptyset$, $\gamma(G) = \gamma(G_2) = |D_2| + |f(D_1)| = |D_2| + |D_1|$. Since $\gamma(G) \leq \gamma(C(G, f)) \leq |D_1| + |D_2| = \gamma(G)$, it follows that $\gamma(G) = \gamma(C(G, f))$.

(\Rightarrow) Let D be any minimum dominating set of $C(G, f)$. Suppose then that $\gamma(G) = \gamma(C(G, f))$ such that $D_1 = D \cap V(G_1)$ and $D_2 = D \cap V(G_2)$. So $\gamma(C(G, f)) = |D_1| + |D_2|$. Note that the only vertices in G_2 that are dominated by D_1 are the vertices in $f(D_1)$ and the only vertices in G_1 that are dominated by D_2 are the vertices in $f^{-1}(D_2)$. Since D is a dominating set of $C(G, f)$, D_2 must dominate every vertex in $V(G_2) \setminus f(D_1)$, and D_1 must dominate every vertex in $V(G_1) \setminus f^{-1}(D_2)$.

Clearly $D_2 \cup f(D_1)$ is a dominating set of G_2 . Note that $|D_1| \geq |f(D_1)|$. So $\gamma(G) = \gamma(C(G, f)) = |D_1| + |D_2| \geq |D_2| + |f(D_1)| \geq \gamma(G_2) = \gamma(G)$. But then these terms must all be equal. In particular, $|D_1| = |f(D_1)|$ and $D_2 \cup f(D_1)$ is a minimum dominating set of G_2 . Furthermore, $D_2 \cap f(D_1) = \emptyset$, else $D_2 \cup f(D_1)$ is a dominating set of G_2 with fewer than $\gamma(G_2)$ vertices. Finally, suppose there is a vertex $v \in D_1 \cap f^{-1}(D_2)$. So $v \in D_1$ and $v \in f^{-1}(D_2)$. But then $f(v) \in f(D_1)$ and $f(v) \in D_2$. But $f(D_1)$ and D_2 are disjoint. So, $D_1 \cap f^{-1}(D_2) = \emptyset$. ■

It is known that for cycles C_n ($n \geq 3$), $\gamma(C_n) = \lceil \frac{n}{3} \rceil$. We now apply Theorem 3 to characterize the lower bound of $\gamma(C(C_n, f))$.

Theorem 4. *For the cycle C_n ($n \geq 3$), let G_1 and G_2 be copies of C_n . Then $\gamma(C_n) = \gamma(C(C_n, f))$ if, and only if, there is a minimum dominating set $D = D_1 \cup D_2$ of $C(C_n, f)$ such that either:*

1. $D_1 = \emptyset$ and D_2 is a minimum dominating set of G_2 and $\text{Range}(f) \subseteq D_2$, or

2. $n \equiv 1 \pmod{3}$, D_2 is a minimum dominating set for $\langle V(G_2) \setminus \{v\} \rangle$,
 $D_1 = \{w\}$, $f(w) = v$, and $f(V(G_1) \setminus N[w]) \subseteq D_2$.

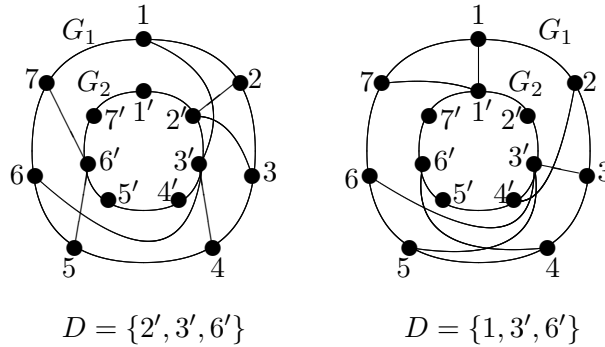


Figure 2. Examples of $\gamma(C(C_n, f)) = \gamma(C_n)$ for $n \equiv 1 \pmod{3}$.

Proof. (\Leftarrow) Suppose that there is a minimum dominating set D of $C(C_n, f)$ satisfying the specified conditions. So $\gamma(C(C_n, f)) = |D| = |D_1| + |D_2|$. If $D_2 \subseteq V(G_2)$ is a minimum dominating set of C_n and $\text{Range}(f) \subseteq D_2$, then $D_1 = \emptyset$. So $\gamma(C_n) = |D_2| = \lceil \frac{n}{3} \rceil$. Furthermore $\gamma(C(C_n, f)) = |D| = |D_1| + |D_2| = 0 + \gamma(G_2)$.

Suppose $n \equiv 1 \pmod{3}$, D_2 dominates all but one vertex v of G_2 , $D_1 = \{w\}$, $f(w) = v$, and $f(V(G_1) \setminus N[w]) \subseteq D_2$. Note that, since $n \equiv 1 \pmod{3}$, $n = 3k + 1$, for some positive integer k , and $\lceil \frac{n}{3} \rceil = k + 1$. By assumption, $\gamma(C(C_n, f)) = |D| = |D_1| + |D_2| = 1 + |D_2|$. Since $\gamma(C_n) = k + 1$, it remains to show that $\gamma(C(C_n, f)) = k + 1$, which is equivalent to showing that $|D_2| = k$. Since D_2 is a minimum dominating set for $\langle V(G_2) \setminus \{v\} \rangle$ and $\langle V(G_2) \setminus \{v\} \rangle$ has domination number k , $|D_2| = k$.

(\Rightarrow) Now suppose that $\gamma(C_n) = \gamma(C(C_n, f)) = \lceil \frac{n}{3} \rceil$. Let D be a minimum dominating set satisfying the conditions of Theorem 3. There are three cases to consider: $n \equiv 0 \pmod{3}$, $n \equiv 1 \pmod{3}$, and $n \equiv 2 \pmod{3}$. In each case, Theorem 3 implies that $D_2 \cup f(D_1)$ is a minimum dominating set of G_2 and $|D_1| = |f(D_1)|$. Since $f(D_1)$ must include all the vertices not dominated by D_2 , it follows that D must contain at least $|D_2| + (n - 3|D_2|) = n - 2|D_2|$ vertices. If $n \equiv 0 \pmod{3}$, then $n = 3k$ for some positive integer k and $\lceil \frac{n}{3} \rceil = k$. Note that D_2 dominates at most $3|D_2|$ vertices in G_2 . There are at least $n - 3|D_2|$ vertices in G_2 which are not dominated by D_2 . If $|D_2| < k$ then $\gamma(C(C_n, f)) = |D| \geq n - 2|D_2| > n - 2k = 3k - 2k = k$, contradicting the assumption that $\gamma(C(C_n, f)) = k$. So $|D_2| = k$. This implies $D_1 = \emptyset$. And this, in turn, implies that D_2 must dominate all the vertices in G_1 . So $\text{Range}(f) \subseteq D_2$.

In the remaining two cases, where $n \equiv 1$ or $n \equiv 2 \pmod{3}$, then $n = 3k + 1$ or $n = 3k + 2$, respectively, for some positive integer k and $\gamma(C_n) = \lceil \frac{n}{3} \rceil = k + 1$.

From Theorem 3 it follows that $D_2 \cup f(D_1)$ is a minimum dominating set of G_2 . Since D_2 dominates at most $3|D_2|$ vertices in G_2 , D_1 must dominate at least $n - 3|D_2|$ vertices in G_2 . If $|D_2| < k$, then $\gamma(C(C_n, f)) = |D| \geq n - 2|D_2| > n - 2k = (3k+1) - 2k = k+1$, contradicting the assumption that $\gamma(C(C_n, f)) = k+1$. So $|D_2| \geq k$. Since $|D| = k + 1$, $|D_2| \leq k + 1$. If $|D_2| = k + 1$, then $D_1 = \emptyset$, $f(D_1) = \emptyset$ and $D_2 \cup f(D_1) = D_2$ is a minimum dominating set of G_2 . Since D is a dominating set of $C(C_n, f)$, it follows that D_2 must also dominate all the vertices in D_1 and, thus, $Range(f) \subseteq D_2$.

Let $n \equiv 1 \pmod{3}$. If $|D_2| = k$, then there is at least one vertex in G_2 not dominated by D_2 . If there are $c > 1$ vertices not dominated by D_2 then these vertices are a subset of $f(D_1)$ and Theorem 3 guarantees that $|D_1| = |f(D_1)| \geq c$ and, thus, $\gamma(C(C_n, f)) \geq k + c > k + 1$, contradicting our assumption. So $c = 1$. There is only one vertex $v \in V(G_2)$ which is not dominated by D_2 . D_1 can only contain a single vertex w (or $|D|$ will again be too large) and $f(w) = v$. Since w dominates $N[w]$ in G_1 , it follows that D_2 must dominate $V(G_1) \setminus N[w]$. So $f(V(G_1) \setminus N[w]) \subseteq D_2$.

Let $n \equiv 2 \pmod{3}$. If $|D_2| = k$, then there are at least two vertices in G_2 not dominated by D_2 . But then these vertices must be a subset of $f(D_1)$ and $|f(D_1)| \geq 2$. Since $|D_1| = |f(D_1)|$, $|D_1| \geq 2$. But then $k + 1 = \gamma(C(G, f)) = |D| = |D_1| + |D_2| \geq 2 + k$, which is a contradiction. So $|D_2| = k + 1$. ■

Next we consider the domination number of $C(C_3, f)$.

Lemma 5. *Let G_1 and G_2 be two copies of C_3 . Then $\gamma(C(C_3, f)) = 2\gamma(C_3)$ if and only if f is not a constant function.*

Proof. (\Leftarrow) Suppose that f is not a constant function. Then, for each vertex $v \in V(C(C_3, f))$, $\deg(v) \leq 4$ and hence $N[v] \subsetneq V(C(C_3, f))$. Thus $\gamma(C(C_3, f)) \geq 2$. Since there exists a dominating set consisting of one vertex from each of G_1 and G_2 , $\gamma(C(C_3, f)) = 2$.

(\Rightarrow) Suppose that f is a constant function, say $f(w) = a$ for some $a \in V(G_2)$ and for all $w \in V(G_1)$. Then $N[a] = V(C(C_3, f))$, and thus $\gamma(C(C_3, f)) = 1 = \gamma(C_3)$. ■

As an immediate consequence of Theorem 4 and Lemma 5, we have the following.

Corollary 6. *There is no permutation f such that $\gamma(C(C_n, f)) = \gamma(C_n)$ for $n = 3$ or $n \geq 5$.*

Now we consider $C(G, f)$ when $G = C_n$ ($n \geq 3$) and f is the identity function.

Theorem 7. *Let G_1 and G_2 be two copies of the cycle C_n for $n \geq 3$. Then*

$$\gamma(C(C_n, id)) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 2 \pmod{4}, \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proof. Since $C(C_n, id)$ is 3-regular, each vertex in $C(C_n, id)$ can dominate 4 vertices. We consider four cases.

Case 1. $n = 4k$. Since $|V(C(C_n, id))| = 8k$, we have $\gamma(C(C_n, id)) \geq \lceil \frac{8k}{4} \rceil = 2k$. Since $\cup_{j=0}^{k-1} \{4j+1, (4j+3)'\}$ is a dominating set of $C(C_n, id)$ with cardinality $2k$, we conclude that $\gamma(C(C_n, id)) = 2k = \lceil \frac{n}{2} \rceil$.

Case 2. $n = 4k + 1$. Since $|V(C(C_n, id))| = 2(4k + 1) = 8k + 2$, we have $\gamma(C(C_n, id)) \geq \lceil \frac{8k+2}{4} \rceil = 2k + 1$. Since $(\cup_{j=0}^k \{4j + 1\}) \cup (\cup_{i=0}^{k-1} \{(4i + 3)'\})$ is a dominating set of $C(C_n, id)$ with cardinality $2k + 1$, we have $\gamma(C(C_n, id)) = 2k + 1 = \lceil \frac{n}{2} \rceil$.

Case 3. $n = 4k + 2$. Notice that $(\cup_{j=0}^k \{4j + 1\}) \cup (\cup_{i=0}^{k-1} \{(4i + 3)'\}) \cup \{(4k + 2)'\}$ is a dominating set of $C(C_n, id)$ with cardinality $2k + 2 = \frac{n}{2} + 1$; thus $\gamma(C(C_n, id)) \leq 2k + 2$. Since $|V(C(C_n, id))| = 2(4k + 2) = 8k + 4$, $\gamma(C(C_n, id)) \geq \lceil \frac{8k+4}{4} \rceil = 2k+1$; indeed, $\gamma(C(C_n, id)) = 2k+1$ only if every vertex is dominated by exactly one vertex of a dominating set; i.e., no double domination is allowed. However, we show that there must exist a doubly-dominated vertex for any dominating set by the following *descent* argument: Let the graph A_0 be $P_{4k+3} \times K_2$ where the bottom row is labeled $1, 2, \dots, 4k + 2, 1$ and the top row is labeled $1', 2', \dots, (4k + 2)', 1'$; note that $C(C_n, id)$ is obtained by identifying the two end-edges each with end-vertices labeled 1 and $1'$. Without loss of generality, choose $1'$ to be in a dominating set D . For each vertex to be singly dominated, we delete vertices $1'(s), 1(s), 2'$, and $(4k + 2)'$, as well as their incident edges, to obtain a derived graph A_1 . In A_1 , vertices 2 and $4k + 2$ are end-vertices and neither may belong to D as each only dominates two vertices in A_1 . This forces support vertices 3 and $4k + 1$ in A_1 to be in D . Deleting vertices 2, 3, $3'$, 4, $4k + 2$, $4k + 1$, $(4k + 1)'$, and $4k$ and incident edges results in the second derived graph A_2 . After k iterations, A_k is the extension of $P_3 \times P_2$ by two leaves at both ends of either the top or the bottom row (see Figure 3); A_k , which has eight vertices, clearly requires three vertices to be dominated.

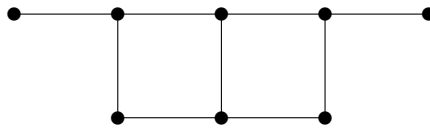


Figure 3. A_k in the $n = 4k + 2$ case.

Thus, we conclude that $\gamma(C(C_n, id)) = 2k + 2 = \frac{n}{2} + 1$.

Case 4. $n = 4k + 3$: Since $|V(C(C_n, id))| = 2(4k + 3) = 8k + 6$, we have $\gamma(C(C_n, id)) \geq \lceil \frac{8k+6}{4} \rceil = 2k + 2$. Since $\cup_{j=0}^k \{4j + 1, (4j + 3)'\}$ is a dominating set of $C(C_n, id)$ with cardinality $2k + 2$, we conclude that $\gamma(C(C_n, id)) = 2k + 2 = \lceil \frac{n}{2} \rceil$. ■

As a consequence of Theorem 7, we have the following result.

Corollary 8. (1) $\gamma(C(C_n, id)) = \gamma(C_n)$ if and only if $n = 4$.
 (2) $\gamma(C(C_n, id)) = 2\gamma(C_n)$ if and only if $n = 3$ or $n = 6$.

By Corollary 6 and Theorem 7, we have the following result.

Proposition 9. For a permutation f , $\gamma(C(C_n, f)) = \gamma(C_n)$ if and only if $C(C_n, f) \cong C(C_4, id)$.

Proof. (\Leftarrow) If $C(C_4, f) \cong C(C_4, id)$, then $\gamma(C_4) = 2 = \gamma(C(C_4, id))$ by Theorem 7.

(\Rightarrow) Let $\gamma(C(C_n, f)) = \gamma(C_n)$ for $n \geq 3$. By Corollary 6, $n = 4$. If f is a permutation, then $C(C_4, f)$ is isomorphic to the graph (A) or (B) in Figure 4 (refer to [7, 9] for details).

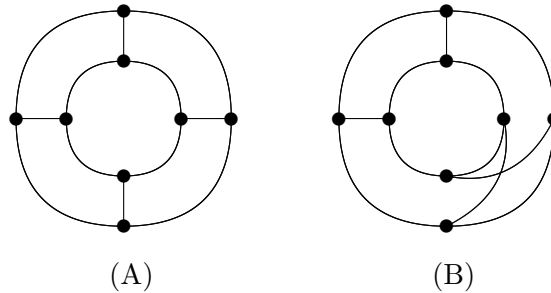


Figure 4. Two non-isomorphic graphs of $C(C_4, f)$ for a permutation f .

If $C(C_4, f) \cong C(C_4, id)$, then we are done. If $C(C_4, f)$ is as in (B) of Figure 4, we claim that $\gamma(C(C_4, f)) \geq 3$.

Since $|V(C(C_4, f))| = 8$ and $C(C_4, f)$ is 3-regular, $D = \{w_1, w_2\}$ dominates $C(C_4, f)$ only if no vertex in $C(C_4, f)$ is dominated by both w_1 and w_2 . It suffices to consider two cases, using the fact that $C(C_4, f) \cong C(C_4, f^{-1})$.

- (i) $D = \{w_1, w_2\} \subseteq V(G_1)$,
- (ii) $w_1 \in V(G_1)$ and $w_2 \in V(G_2)$.

Also, we only need to consider w_1 and w_2 such that $w_1w_2 \notin E(C(C_4, f))$. By symmetry, there is only one specific case to check in case (i). In case (ii), by fixing a vertex in $V(G_1)$, we see that there are three cases to check. In each case, for any $D = \{w_1, w_2\}$, $N[w_1] \cap N[w_2] \neq \emptyset$. Thus $\gamma(C(C_4, f)) > 2$. ■

4. UPPER BOUND OF $\gamma(C(C_n, f))$

In this section we investigate domination number of functigraphs for cycles: We show that $\gamma(C(C_n, f)) < 2\gamma(C_n)$ for $n \equiv 1, 2 \pmod{3}$. For $n \equiv 0 \pmod{3}$, we characterize the domination number for an infinite class of functions and state conditions under which the upper bound is not achieved. Our result in this section generalizes a result of Burger, Mynhardt, and Weakley in [6] which states that no cycle other than C_3 and C_6 is a *universal doubler* (i.e., only for $n = 3, 6$, $\gamma(C(C_n, f)) = 2\gamma(C_n)$ for any permutation f).

4.1. A characterization of $\gamma(C(C_{3k+1}, f))$

Proposition 10. *For any function f , $\gamma(C(C_{3k+1}, f)) < 2\gamma(C_{3k+1})$ for $k \in \mathbb{Z}^+$.*

Proof. Without loss of generality, we may assume that $u_1v_1 \in E(C(C_n, f))$. Since $D = \{v_1\} \cup \{u_{3j}, v_{3j} \mid 1 \leq j \leq k\}$ is a dominating set of $C(C_{3k+1}, f)$ with $|D| = 2k + 1$ for any function f , $\gamma(C(C_{3k+1}, f)) < 2\gamma(C_{3k+1})$ for $k \in \mathbb{Z}^+$. ■

4.2. A characterization of $\gamma(C(C_{3k+2}, f))$

We begin with the following example showing $\gamma(C(C_5, f)) < 2\gamma(C_5)$ for any function f .

Example 11. For any function f , $\gamma(C(C_5, f)) < 2\gamma(C_5)$.

Proof. Let $G = C_5$, $V(G_1) = \{1, 2, 3, 4, 5\}$, and $V(G_2) = \{1', 2', 3', 4', 5'\}$. If $|Range(f)| \leq 2$, we can choose a dominating set consisting of all vertices in the range and, if necessary, an additional vertex. If $|Range(f)| = 3$, then we can choose the range as a dominating set.

So, let $|Range(f)| \geq 4$. Then f is bijective on at least three vertices in the domain and their image. By the pigeonhole principle, there exist two adjacent vertices, say 1 and 2, on which f is bijective. Let $f(1) = 1'$. Then, by relabeling if necessary, $f(2) = 2'$ or $f(2) = 3'$. Suppose $f(2) = 3'$. Then $D = \{1', 3', 4\}$ forms a dominating set, and we are done. Suppose then $f(2) = 2'$. We consider two cases.

Case 1. $|Range(f)| = 4$. By symmetry, $5' \notin Range(f)$ is the same as $3' \notin Range(f)$. So, consider two distinct cases, $5' \notin Range(f)$ and $4' \notin Range(f)$. If $5' \notin Range(f)$, then $D = \{1, 3', 4'\}$ forms a dominating set. If $4' \notin Range(f)$, then $D = \{1, 3', 5'\}$ forms a dominating set. In either case, we have $\gamma(C(C_5, f)) < 2\gamma(C_5)$.

Case 2. f is a bijection (permutation). Recall $f(1) = 1'$ and $f(2) = 2'$; there are thus $3! = 6$ permutations to consider. Using the standard cycle notation,

the permutations are (3, 4), (3, 5), (4, 5), (3, 4, 5), (3, 5, 4), and identity. However, they induce only four non-isomorphic graphs, since (3, 4) and (4, 5) induce isomorphic graphs and (3, 4, 5) and (3, 5, 4) induce isomorphic graphs. If f is either (3, 4) or (3, 4, 5), then $D = \{2, 3', 5'\}$ is a dominating set. If f is (3, 5), then $D = \{1', 3, 3'\}$ is a dominating set. When f is the identify function, $D = \{1', 3, 5'\}$ is a dominating set. It is thus verified that $\gamma(C(C_5, f)) < 2\gamma(C_5)$. ■

Remark 12. Example 11 has the following implication. Given $C(C_{3k+2}, f)$ for $k \in \mathbb{Z}^+$, suppose there exist five consecutive vertices being mapped by f into five consecutive vertices. Then $\gamma(C(C_{3k+2}, f)) < 2\gamma(C_{3k+2}) = 2k + 2$, and here is a proof. Relabeling if necessary, we may assume that $\{u_1, u_2, u_3, u_4, u_5\}$ are mapped into $\{v_1, v_2, v_3, v_4, v_5\}$; let $S = \{u_i, v_i \mid 1 \leq i \leq 5\}$. Then $\langle S \rangle$ in $C(C_{3k+2}, f)$ and the additional edge set $\{u_1u_5, v_1v_5\}$ form a graph isomorphic to a $C(C_5, f)$, which has a dominating set S_0 with $|S_0| \leq 3$. In $C(C_{3k+2}, f)$, if S is dominated by S_0 , then $D = S_0 \cup \{u_{3j+1} \mid 2 \leq j \leq k\} \cup \{v_{3j+1} \mid 2 \leq j \leq k\}$ forms a dominating set for $C(C_{3k+2}, f)$ with at most $2k + 1$ vertices. If u_1 is not dominated by S_0 in $C(C_{3k+2}, f)$, then it is dominated solely by u_5 of S_0 in $C(C_5, f)$. But then u_6 is dominated by u_5 in $C(C_{3k+2}, f)$ and we can replace $\{u_{3j+1} \mid 2 \leq j \leq k\}$ with $\{u_{3j+2} \mid 2 \leq j \leq k\}$ to form D . Similarly, if u_5 is not dominated by S_0 in $C(C_{3k+2}, f)$, then it is dominated solely by u_1 of S_0 in $C(C_5, f)$. Then u_{3k+2} is dominated by u_1 in $C(C_{3k+2}, f)$ and we can replace $\{u_{3j+1} \mid 2 \leq j \leq k\}$ with $\{u_{3j} \mid 2 \leq j \leq k\}$ to form D . The cases where v_1 or v_5 is not dominated by S_0 in $C(C_{3k+2}, f)$ can be likewise handled. Thus, if five consecutive vertices are mapped by f into five consecutive vertices, then $\gamma(C(C_{3k+2}, f)) \leq 2k + 1 < 2k + 2 = 2\gamma(C_{3k+2})$.

Remark 13. Unlike $C(C_5, f)$, it is easily checked that $\gamma(C(P_5, f)) = 2\gamma(P_5)$ for the function f given in Figure 5, where P_5 is the path on five vertices.

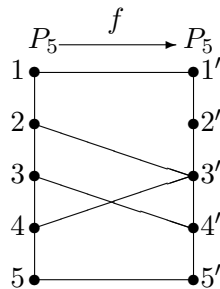


Figure 5. An example where $\gamma(C(P_5, f)) = 2\gamma(P_5)$.

Now we consider the domination number of $C(C_{3k+2}, f)$ for a non-permutation function f , where $k \in \mathbb{Z}^+$.

Theorem 14. *Let $f : V(C_{3k+2}) \rightarrow V(C_{3k+2})$ be a function which is not a permutation. Then $\gamma(C(C_{3k+2}, f)) < 2\gamma(C_{3k+2}) = 2k + 2$.*

Proof. Suppose f is a function from C_{3k+2} to C_{3k+2} and f is not a permutation. There must be a vertex v_1 in G_2 such that $\deg(v_1) \geq 4$ in $C(C_{3k+2}, f)$. Define the sets $V_1 = \{v_{3i+1} \mid 0 \leq i \leq k\}$, $V_2 = \{v_{3i+2} \mid 0 \leq i \leq k\}$, and $V_3 = \{v_{3i} \mid 1 \leq i \leq k\} \cup \{v_1\}$. Notice that each of these three sets is a minimum dominating set of G_2 of cardinality $k + 1$. Also, notice that $|f^{-1}(V_1)| + |f^{-1}(V_2)| + |f^{-1}(V_3)|$ counts every vertex in the pre-image of $V(G_2) \setminus \{v_1\}$ once and every vertex in the pre-image of $\{v_1\}$ twice, so $|f^{-1}(V_1)| + |f^{-1}(V_2)| + |f^{-1}(V_3)| \geq 3k + 4$. By the Pigeonhole Principle, $|f^{-1}(V_i)| \geq \lceil \frac{3k+4}{3} \rceil = k + 2$ for some i . Set $D_2 = V_i$ for this i and notice that D_2 is a dominating set of G_2 with cardinality $k + 1$ and $|f^{-1}(D_2)| \geq k + 2$.

Without loss of generality, we may assume that u_1 is in $f^{-1}(D_2)$. If there exists $0 \leq i \leq k$ such that u_{3i+2} is also in the pre-image of D_2 , then $D_1 = \{u_{3j} \mid 1 \leq j \leq i\} \cup \{u_{3j+1} \mid i+1 \leq j \leq k\}$ dominates the remaining vertices of G_1 . Otherwise, there are at least $k + 1$ vertices in $f^{-1}(D_2) \cap \{u_{3j}, u_{3j+1} \mid 1 \leq j \leq k\}$. By the Pigeonhole Principle, there exist two vertices u_{3j_0} and u_{3j_0+1} in $f^{-1}(D_2)$ which are adjacent in G_1 . Then $D_1 = \{u_1\} \cup \{u_{3j+1} \mid 1 \leq j \leq j_0 - 1\} \cup \{u_{3j'} \mid j_0 + 1 \leq j' \leq k\}$ dominates the remaining vertices of G_1 . In either case, $D_1 \cup D_2$ is a dominating set of $C(C_{3k+2}, f)$ with $2k + 1$ vertices. ■

For $G_i \subseteq C(G, f)$ ($i = 1, 2$), the distance between x and y in $\langle V(G_i) \rangle$ is denoted by $d_{G_i}(x, y)$.

Theorem 15. *Let $f : V(C_{3k+2}) \rightarrow V(C_{3k+2})$ be a function, where $k \in \mathbb{Z}^+$. For the cycle C_{3k+2} , if there exist two vertices x and y in G_1 such that $d_{G_1}(x, y) \equiv 1 \pmod{3}$ and $d_{G_2}(f(x), f(y)) \not\equiv 1 \pmod{3}$, then $\gamma(C(C_{3k+2}, f)) < 2\gamma(C_{3k+2})$.*

Proof. Let $x = 1$ and $y = 3a + 2$ for a nonnegative integer a . By relabeling, if necessary, we may assume that $f(x) = 1'$. Note that $D_1 = (\cup_{i=1}^a \{3i\}) \cup (\cup_{i=a+1}^k \{3i + 1\})$ dominates vertices in $V(G_1) \setminus \{x, y\}$. If $f(x) = 1' = f(y)$, let D_2 be any minimum dominating set of G_2 containing $1'$. Then $D = D_1 \cup D_2$ is a dominating set of $C(C_{3k+2}, f)$ with $|D| \leq 2k + 1$. Thus, we assume that $f(x) \neq f(y)$. Since $d_{G_2}(f(x), f(y)) \not\equiv 1 \pmod{3}$, $f(y) = (3\ell)'$ or $f(y) = (3\ell + 1)'$ for some ℓ ($1 \leq \ell \leq k$). First, consider when $\ell > 1$. If $f(y) = (3\ell)'$, let $D_2 = (\cup_{i=1}^{\ell-1} \{(3i + 1)'\}) \cup (\cup_{i=\ell+1}^k \{(3i)'\}) \cup \{1', (3\ell)'\}$; and if $f(y) = (3\ell + 1)'$, let $D_2 = (\cup_{i=1}^{\ell-1} \{(3i + 1)'\}) \cup (\cup_{i=\ell+1}^k \{(3i + 1)'\}) \cup \{1', (3\ell + 1)'\}$. Second, consider when $\ell = 1$. If $f(y) = (3\ell)'$, let $D_2 = (\cup_{i=1}^k \{(3i)'\}) \cup \{1'\}$; if $f(y) = (3\ell + 1)'$, let $D_2 = (\cup_{i=1}^k \{(3i + 1)'\}) \cup \{1'\}$. Notice that D_2 dominates $V(G_2) \cup \{x, y\}$ in each case. Thus $D = D_1 \cup D_2$ is a dominating set of $C(C_{3k+2}, f)$ with $|D| = |D_1| + |D_2| = k + k + 1 = 2k + 1 < 2\gamma(C_{3k+2}) = 2k + 2$. ■

Next we consider $C(C_{3k+2}, f)$ for a permutation f .

Lemma 16. *Let f be a monotone increasing function from $S = \{1, 2, \dots, n\}$ to \mathbb{Z} such that $f(1) = 1$. If $|j - i| \equiv 1 \pmod{3}$ implies $|f(j) - f(i)| \equiv 1 \pmod{3}$ for any $i, j \in S$, then $f(i) \equiv i \pmod{3}$.*

Proof. The monotonicity of f — and the rest of the hypotheses — provides that $f(i + 1) - f(i) \equiv 1 \pmod{3}$, for each $1 \leq i < n$; apply it inductively to reach the conclusion. ■

Theorem 17. *Let $G = C_{3k+2}$ for a positive integer k , and let $f : V(G_1) \rightarrow V(G_2)$ be a permutation, where the vertices in both the domain and codomain are labeled 1 through $3k + 2$. Assume*

$$(1) \quad d_{G_2}(f(x), f(y)) \equiv 1 \pmod{3} \text{ whenever } d_{G_1}(x, y) \equiv 1 \pmod{3}.$$

If $f(1) = 1$, then $C(C_{3k+2}, f) \cong C_{3k+2} \times K_2$.

Proof. Denote by $F(n)$ the sequence of inequalities $f(1) < f(2) < \dots < f(n - 1) < f(n)$. By cyclically relabeling (equivalent to going to an isomorphic graph) if necessary, we may assume $F(3)$; now the graph $C(C_{3k+2}, f)$, along with the labeling of all its vertices, is fixed. Without loss of generality, let $f(1) = 1$, $f(2) = 3y_0 + 2$, and $f(3) = 3z_0 + 3$ for $0 \leq y_0 \leq z_0 < k$. Notice $|x - y| \equiv 1 \pmod{3}$ if and only if $d_G(x, y) \equiv 1 \pmod{3}$ for $G = C_{3k+2}$; we will use $|\cdot|$ in distance considerations. We will prove that f is monotone increasing on vertices in G_1 (and hence f is the identity function) in two steps: Step I is the extension to $F(5)$ from $F(3)$. Step II is the extension to $F(3(m + 1) + 2)$ from $F(3m + 2)$ if $1 \leq m \leq k - 1$.

Step I. Suppose for the sake of contradiction that $F(5)$ is false. We first prove $F(4)$ and then $F(5)$.

Suppose $f(4) < f(3)$. This means, by condition (1), that $f(4) \equiv 2 \pmod{3}$. If $f(5) < f(4)$, then condition (1) implies $f(5) \equiv 1 \pmod{3}$. If $f(5) > f(4)$, then condition (1) implies $f(5) \equiv 0 \pmod{3}$. Now notice $|1 - 5| \equiv 1 \pmod{3}$. If $f(5) < f(4)$, then $|f(1) - f(5)| = f(5) - f(1) \equiv 0 \pmod{3}$; if $f(5) > f(4)$, then $|f(1) - f(5)| = f(5) - f(1) \equiv 2 \pmod{3}$. In either case, condition (1) is violated. Thus $f(3) < f(4)$, and $f(4) \equiv 1 \pmod{3}$.

Suppose $f(5) < f(4)$. This means, by condition (1), that $f(5) \equiv 0 \pmod{3}$. Then $|f(1) - f(5)| = f(5) - f(1) \equiv 2 \pmod{3}$, which contradicts condition (1) since, again, $|1 - 5| \equiv 1 \pmod{3}$. Thus we have $f(4) < f(5)$, and $f(5) \equiv 2 \pmod{3}$.

Step II. Suppose $F(3m + 2)$ for $1 \leq m \leq k - 1$; we will show $F(3(m + 1) + 2)$. Observe that

$$(2) \quad f(3m + 5) - f(1) \equiv 1 \pmod{3} \text{ implies } f(3m + 5) \equiv 2 \pmod{3}.$$

First, assume $f(3m+3) < f(3m+2)$. This means, by condition (1) and Lemma 16, that $f(3m+3) \equiv 1 \pmod{3}$. Assuming $f(3m+4) > f(3m+3)$, then $f(3m+4) \equiv 2 \pmod{3}$; which in turn implies that $f(3m+5) \equiv 0$ or $1 \pmod{3}$, either way a contradiction to (2). Assuming $f(3m+4) < f(3m+3)$, then $f(3m+4) \equiv 0 \pmod{3}$; however, comparing with $f(3)$, $f(3m+4) \equiv 1$ or $2 \pmod{3}$, either way a contradiction again. We have thus shown that $f(3m+3) > f(3m+2)$, which means $f(3m+3) \equiv 0 \pmod{3}$.

Second, assume $f(3m+4) < f(3m+3)$. This means, by condition (1) and Lemma 16, that $f(3m+4) \equiv 2 \pmod{3}$. Assuming $f(3m+5) > f(3m+4)$, we have $f(3m+5) \equiv 0 \pmod{3}$. Assuming $f(3m+5) < f(3m+4)$, we have $f(3m+5) \equiv 1 \pmod{3}$. Either way we reach a contradiction to (2). We have thus shown that $f(3m+4) > f(3m+3)$, which means $f(3m+4) \equiv 1 \pmod{3}$.

Finally, assume $f(3m+5) < f(3m+4)$. This means, by condition (1) and Lemma 16, that $f(3m+5) \equiv 0 \pmod{3}$, which is a contradiction to (2). Thus, $f(3m+5) > f(3m+4)$ and $f(3m+5) \equiv 2 \pmod{3}$. ■

Theorem 18. *For any function f , $\gamma(C(C_{3k+2}, f)) < 2\gamma(C_{3k+2})$, where $k \in \mathbb{Z}^+$.*

Proof. Combine Theorem 7, Theorem 14, Theorem 15, and Theorem 17. ■

4.3. Towards a characterization of $\gamma(C(C_{3k}, f))$

Definition. Let f be a function from $S = \{1, 2, \dots, 3k\}$ to itself. We say f is a *three-translate* if $f(x+3i) = f(x) + 3i$ for $x \in \{1, 2, 3\}$ and $i \in \{0, 1, \dots, k-1\}$. Let $\tilde{f} = f|_{\{1,2,3\}}$.

Notation. Denote by $\tilde{f} = (a_1, a_2, a_3)$ the function such that $\tilde{f}(1) = a_1$, $\tilde{f}(2) = a_2$, and $\tilde{f}(3) = a_3$. We use $C(C_{3k}, f)$ and $C(C_{3k}, \tilde{f})$ interchangeably when f is a three-translate.

First consider $C(C_{3k}, f)$ for a three-translate permutation f .

Theorem 19. *Let f be a three-translate permutation and let $k \geq 4$. Then $\gamma(C(C_{3k}, f)) = 2k = 2\gamma(C_{3k})$ if and only if \tilde{f} is $(2, 1, 3)$ or $(1, 3, 2)$.*

Proof. Notice that \tilde{f} is one of the six permutations: identity, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, and $(3, 2, 1)$. First, the identity does not attain the upper bound for $k \geq 3$ by Corollary 8. Second, the permutations $(2, 3, 1)$ and $(3, 1, 2)$ are inverses of each other and induce isomorphic graphs in $C(C_{3k}, f)$; they do not attain the upper bound for $k \geq 4$: $D = \{1, 4, 8, 4', 7', 11', 12'\}$ is a dominating set of $C(C_{12}, f)$ where $\tilde{f} = (2, 3, 1)$ (see (B) of Figure 6). Third, the transposition $(3, 2, 1)$ fails to attain the upper bound for $k \geq 3$: $D = \{1, 6, 8, 1', 6'\}$ is a dominating set of $C(C_9, f)$ (see (C) of Figure 6). When \tilde{f} is $(2, 3, 1)$ or $(3, 1, 2)$ or $(3, 2, 1)$, one can readily see how to extend a dominating set from k to $k+1$. Lastly, the transpositions $(1, 3, 2)$ and $(2, 1, 3)$ induce isomorphic graphs in $C(C_{3k}, f)$.

Claim. If \tilde{f} is $(1, 3, 2)$ or $(2, 1, 3)$, then $\gamma(C(C_{3k}, f)) = 2k = 2\gamma(C_{3k})$ for each $k \geq 3$.

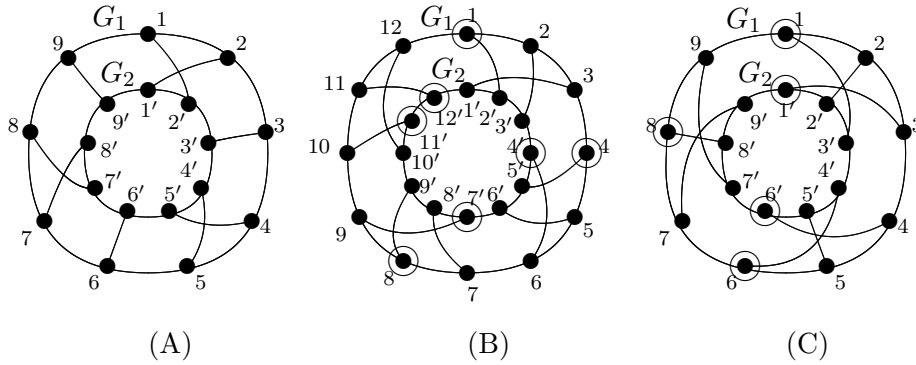


Figure 6. Examples of $C(C_{3k}, f)$ for three-rotate permutations f when $k \geq 3$.

For definiteness, let $\tilde{f} = (2, 1, 3)$ (see (A) of Figure 6). For the sake of contradiction, assume $\gamma(C(C_{3k}, f)) < 2\gamma(C_{3k}) = 2k$ and consider a minimum dominating set D for $C(C_{3k}, f)$. We can partition the vertices into k sets $S_i = \{u_{3i-2}, u_{3i-1}, u_{3i}, v_{3i-2}, v_{3i-1}, v_{3i}\}$ for $1 \leq i \leq k$. By the Pigeonhole Principle, $|D \cap S_i| \leq 1$ for some i . Without loss of generality, we assume that $|D \cap S_1| \leq 1$. Since neither u_2 nor v_2 has a neighbor that is not in S_1 , $D \cap S_1$ must be either $\{u_1\}$ or $\{v_1\}$ — in order for both u_2 and v_2 to be dominated by only one vertex.

Notice that u_3 and v_3 are dominated neither by u_1 nor by v_1 , so $D \cap S_2$ must contain both u_4 and v_4 . But then either $|D \cap S_2| \geq 3$ or u_6 and v_6 are not dominated by any vertex in $D \cap S_2$: if $|D \cap S_2| \geq 3$, we start the argument anew at S_3 ; thus we may, without loss of generality, assume u_6 and v_6 are not dominated by any vertex in $D \cap S_2$ and $|D \cap S_2| = 2$. This forces u_7 and v_7 to be in D , but this still leaves u_9 and v_9 un-dominated by any vertex in $\cup_{i=1}^3 (D \cap S_i)$. Again, if $|D \cap S_3| \geq 3$, we start the argument anew at S_4 . Thus, we may assume u_9 and v_9 are not dominated by any vertex in $\cup_{i=1}^3 (D \cap S_i)$.

This pattern (allowing restarts) is forced to persist if $\gamma(C(C_{3k}, f)) < 2k$. Now, one of two situations prevails for U_k . First, the argument begins anew at U_k . In this case, even if u_{3k-2} and v_{3k-2} are dominated by vertices outside S_k , one still has $|D \cap S_k| \geq 2$, and hence $|D| \geq 2k$. Second, the vertices u_{3k-2} and v_{3k-2} are already in D . And if $|D \cap S_k| = 2$, then either u_{3k} or v_{3k} is left un-dominated. Therefore, $|D \cap S_k| \geq 3$; this means $|D| \geq 2k$, contradicting the original hypothesis. ■

Remark 20. For $k \in \mathbb{Z}^+$, one can readily check that $\gamma(C(C_{12k}, (2, 3, 1))) = \gamma(C(C_{12k}, (3, 1, 2))) \leq 7k$ and $\gamma(C(C_{9k}, (3, 2, 1))) \leq 5k$.

Next we consider $C(C_{3k}, f)$ for a non-permutation three-translate f . Note that constant three-translates (i.e., $\tilde{f} = \text{constant}$) never achieve the upper bound.

Remark 21. It is easy to check that there are five non-isomorphic and non-constant three-translates which are not permutations for $k \geq 3$. That is,

- (i) $C(C_{3k}, (1, 1, 2)) \cong C(C_{3k}, (1, 1, 3)) \cong C(C_{3k}, (1, 2, 2)) \cong C(C_{3k}, (2, 2, 3)) \cong C(C_{3k}, (1, 3, 3)) \cong C(C_{3k}, (2, 3, 3));$
- (ii) $C(C_{3k}, (1, 2, 1)) \cong C(C_{3k}, (2, 1, 2)) \cong C(C_{3k}, (2, 3, 2)) \cong C(C_{3k}, (3, 2, 3));$
- (iii) $C(C_{3k}, (2, 1, 1)) \cong C(C_{3k}, (2, 2, 1)) \cong C(C_{3k}, (3, 2, 2)) \cong C(C_{3k}, (3, 3, 2));$
- (iv) $C(C_{3k}, (1, 3, 1)) \cong C(C_{3k}, (3, 1, 3));$
- (v) $C(C_{3k}, (3, 1, 1)) \cong C(C_{3k}, (3, 3, 1)).$

Theorem 22. *Let f be a three-translate which is not a permutation and let $k \geq 3$. Then $\gamma(C(C_{3k}, \tilde{f})) = 2k = 2\gamma(C_{3k})$ if and only if $C(C_{3k}, \tilde{f}) \cong C(C_{3k}, (1, 1, 2))$ or $C(C_{3k}, \tilde{f}) \cong C(C_{3k}, (1, 2, 1))$ or $C(C_{3k}, \tilde{f}) \cong C(C_{3k}, (1, 3, 1))$.*

Proof. There are 21 functions which are not permutations from $S = \{1, 2, 3\}$ to itself. The three constant functions obviously fail to achieve the upper bound (if $\tilde{f} \equiv \text{constant}$, then $\gamma(C(C_{3k}, \tilde{f})) = \gamma(C_{3k}) = k$); so there are 18 non-permutation functions to consider. By Remark 21, we need to consider five non-isomorphic classes.

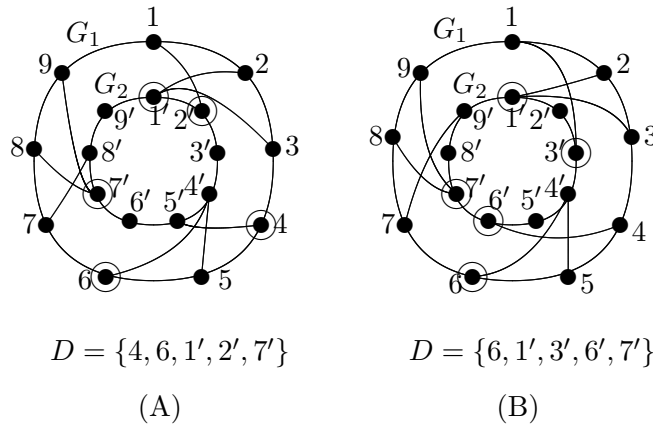


Figure 7. Examples of $\gamma(C(C_{3k}, f))$ such that $\gamma(C(C_{3k}, f)) < 2\gamma(C_{3k})$ for non-permutation three-translates f and for $k \geq 3$.

First, we consider when the domination number of $C(C_{3k}, f)$ is less than $2\gamma(C_{3k}) = 2k$. If $C(C_{3k}, \tilde{f}) \cong C(C_{3k}, (2, 1, 1))$, then $D = \{4, 6, 1', 2', 7'\}$ is a dominating set of $C(C_{3k}, (2, 1, 1))$ (see (A) of Figure 7). If $C(C_{3k}, \tilde{f}) \cong C(C_{3k}, (3, 1, 1))$, then $D = \{6, 1', 3', 6', 7'\}$ is a dominating set

of $C(C_9, (3, 1, 1))$ (see (B) of Figure 7). In each case, $|D| = 5 < 2\gamma(C_9)$, and one can readily see how to extend a dominating set from k to $k + 1$ such that $\gamma(C(C_{3k}, \tilde{f})) < 2\gamma(C_{3k}) = 2k$.

Second, we consider $C(C_{3k}, \tilde{f}) \cong C(C_{3k}, (1, 1, 2))$ or $C(C_{3k}, \tilde{f}) \cong C(C_{3k}, (1, 2, 1))$ or $C(C_{3k}, \tilde{f}) \cong C(C_{3k}, (1, 3, 1))$ (see Figure 8). In all three cases, $\gamma(C(C_{3k}, \tilde{f})) = 2\gamma(C_{3k})$ and our proofs for the three cases agree in the main idea but differ in details.

Here is the main idea. Since one can explicitly check the few cases when $k < 3$, assume $k \geq 3$. In all three cases, we view $C(C_{3k}, \tilde{f})$ as the union of k subgraphs $\langle U_i \rangle$ for $1 \leq i \leq k$, where $U_i = \{u_{3i-2}, u_{3i-1}, u_{3i}, v_{3i-2}, v_{3i-1}, v_{3i}\}$, together with two additional edges between U_i and U_j exactly when $i - j \equiv -1$ or $1 \pmod k$. For each i , the presence of internal vertices in U_i (vertices which can not be dominated from outside of U_i) imply the inequality $|D \cap U_i| \geq 1$. Assuming, for the sake of contradiction, that there exists a minimum dominating set D with $|D| < 2k$, we conclude, by the pigeonhole principle, the existence of a “deficient U_p ” (i.e., $|D \cap U_p| = 1 < 2$). Starting at this U_p and sequentially going through each U_i , we can argue that this deficient U_p is necessarily compensated (or “paired off”) by an “excessive U_q ” (i.e., $|D \cap U_q| > 2$). Going through all indices in $\{1, 2, \dots, k\}$, we are forced to conclude that $|D| \geq 2k$, contradicting our hypothesis. To avoid undue repetitiveness, we provide a detailed proof only in one of the three cases, the case of $C(C_{3k}, (1, 3, 1))$, which is isomorphic to $C(C_{3k}, (3, 1, 3))$.

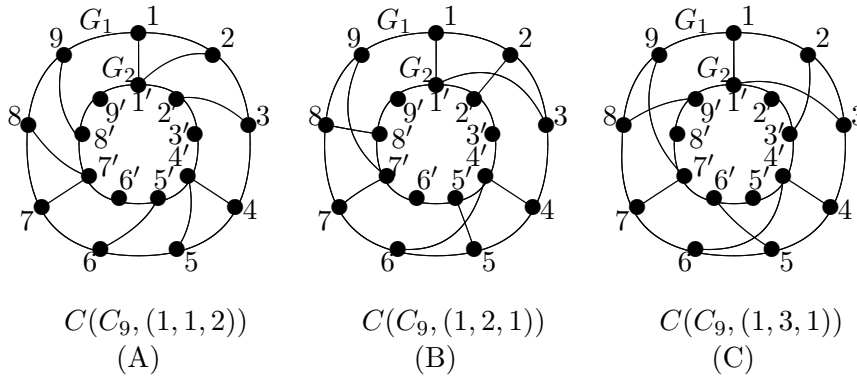


Figure 8. Examples of $C(C_{3k}, f)$ such that $\gamma(C(C_{3k}, f)) = 2\gamma(C_{3k})$ for non-permutation three-translates f and for $k \geq 3$.

Claim. If $C(C_{3k}, \tilde{f}) \cong C(C_{3k}, (3, 1, 3))$, then $\gamma(C(C_{3k}, f)) = 2k = 2\gamma(C_{3k})$.

Proof of Claim. The assertion may be explicitly verified for $k < 4$; so let $k \geq 4$. For the sake of contradiction, assume $\gamma(C(C_{3k}, f)) < 2k$ and consider a minimum dominating set D for $C(C_{3k}, f)$. We can partition the vertices into

k sets $U_i = \{u_{3i-2}, u_{3i-1}, u_{3i}, v_{3i-2}, v_{3i-1}, v_{3i}\}$ for $1 \leq i \leq k$. By the Pigeonhole Principle, $|D \cap U_i| \leq 1$ for some i . Without loss of generality, we assume that $|D \cap U_1| \leq 1$. Since neither u_2 nor v_2 has a neighbor that is not in U_1 , $D \cap U_1$ must be $\{v_1\}$ — the only vertex to dominate both u_2 and v_2 .

Notice that u_3 and v_3 are not dominated by v_1 , the only vertex in $D \cap U_1$, so $D \cap U_2$ must contain both u_4 and v_4 . But then either $|D \cap U_2| \geq 3$ or u_6 is not dominated by any vertex in $D \cap U_2$, if $|D \cap U_2| \geq 3$, we start the argument anew at U_3 ; thus we may, without loss of generality, assume u_6 is not dominated by any vertex in $D \cap U_2$. This forces u_7 , which dominates u_6 , u_8 , and v_9 , to be in D . Now, for v_7 and v_8 to be dominated, one of them must be in D . But this still leaves u_9 un-dominated by any vertex in $\cup_{i=1}^3 U_i$. Again, if $|D \cap U_3| \geq 3$, we start the argument anew at U_4 . Thus, we may, without loss of generality, assume u_9 is not dominated by any vertex in $\cup_{i=1}^3 U_i$.

This pattern (allowing restarts) is forced to persist if $\gamma(C(C_{3k}, f)) < 2k$. Now, one of two situations prevails for U_k : first, the argument begins anew at U_k . In this case, even if u_{3k-2} and v_{3k-2} are dominated by vertices outside of U_k , one still has $|D \cap U_k| \geq 2$, and hence $|D| \geq 2k$. Second, the vertices u_{3k-2} and either v_{3k-2} or v_{3k-1} are already in D . And if $|D \cap U_k| = 2$, then u_{3k} (and, for that matter, u_1) is left un-dominated. Therefore, $|D \cap U_k| \geq 3$ and $|D| \geq 2k$, contradicting the original hypothesis. ■

Now, we consider sufficient conditions for $\gamma(C(C_{3k}, f)) < 2\gamma(C_{3k})$ in terms of the maximum and the average degree of $C(C_{3k}, f)$, respectively.

Proposition 23. *If $\Delta(C(C_{3k}, f)) \geq k + 5$, then $\gamma(C(C_{3k}, f)) < 2\gamma(C_{3k})$.*

Proof. Suppose $C(C_{3k}, f)$ is a functigraph with maximum degree at least $k + 5$. Without loss of generality, we assume that the degree of v_1 is at least $k + 5$. Partition the vertices of G_1 into k sets $U_i = \{u_{3i-2}, u_{3i-1}, u_{3i}\}$, where $1 \leq i \leq k$. If $N[v_1]$ contains any set U_i , say $U_1 \subseteq N[v_1]$, then $\{u_i \mid i \geq 5 \text{ and } i \equiv 2 \pmod{3}\} \cup \{v_i \mid i \equiv 1 \pmod{3}\}$ is a dominating set of $C(C_{3k}, f)$ with $2k - 1$ vertices. Thus, we may assume that $|N[v_1] \cap U_i| \leq 2$ for each i . It follows that $|N[v_1] \cap U_i| = 2$ for at least 3 different values of i , say $i = p, q$, and r . Let x, y , and z be the vertices in G_1 that are in U_p, U_q, U_r (respectively) and not in $N[v_1]$.

Suppose one of x, y , and z , say x , maps to a vertex v_{3j+1} for some j . Then $\{u_\ell \mid \ell \equiv 2 \pmod{3} \text{ and } \ell \neq 3p - 1\} \cup \{v_\ell \mid \ell \equiv 1 \pmod{3}\}$ is a dominating set of $C(C_{3k}, f)$ with $2k - 1$ vertices. Otherwise, two of x, y , and z , say x and y , map to vertices v_s and v_t such that $s \equiv t \pmod{3}$, say $s \equiv t \equiv 0 \pmod{3}$, without loss of generality. But then the set $\{u_\ell \mid \ell \equiv 2 \pmod{3}, \ell \neq 3p - 1, \text{ and } \ell \neq 3q - 1\} \cup \{v_1\} \cup \{v_\ell \mid \ell \equiv 0 \pmod{3}\}$ is a dominating set of $C(C_{3k}, f)$ with $2k - 1$ vertices. ■

The following example shows that the bound provided in Proposition 23 is nearly sharp. Namely, there exists a function $f : V(C_{3k}) \rightarrow V(C_{3k})$ such that the resulting functigraph has $\Delta(C(C_{3k}, f)) = k+3$ and $\gamma(C(C_{3k}, f)) = 2\gamma(C_{3k}) = 2k$.

Example 24. For $k \in \mathbb{Z}^+$, let $f : V(C_{3k}) \rightarrow V(C_{3k})$ be a function defined by

$$f(u_i) = \begin{cases} v_i & \text{if } i \equiv 1 \pmod{3}, \\ v_{i+1} & \text{if } i \equiv 2 \pmod{3}, \\ v_{3k} & \text{if } i \equiv 0 \pmod{3}. \end{cases}$$

Then $\gamma(C(C_{3k}, f)) = 2k = 2\gamma(C_{3k})$.

Proof. Notice that $\Delta(C(C_{3k}, f)) = \deg(v_{3k}) = k + 3$. For $1 \leq i \leq k$, define $S_i = \{u_{3i}, u_{3i-1}, u_{3i-2}, v_{3i}, v_{3i-1}, v_{3i-2}\}$, and notice that $\cup_{i=1}^k S_i$ is a partition of $V(C(C_{3k}, f))$. Let D be any dominating set of $C(C_{3k}, f)$; we need to show that $|D| \geq 2k$. Observe that $|D \cap S_i| \geq 1$ since neither u_{3i-1} nor v_{3i-1} can be dominated from outside of S_i for $1 \leq i \leq k$. We will argue in an inductive fashion starting at k and descending to 1.

Suppose $|D| < 2k$; choose the biggest $j \leq k$ such that $|D \cap S_j| = 1$. Of necessity $v_{3j} \in D$, as it is the only vertex in S_j dominating both u_{3j-1} and v_{3j-1} . Then $|D \cap S_{j-1}| \geq 2$, since to dominate u_{3j-2} and v_{3j-2} in S_j , D must contain both u_{3j-3} and v_{3j-3} in S_{j-1} .

Now, if $|D \cap S_{j-1}| \geq 3$, then it is “paired off” with S_j . We will choose the biggest $\ell < j$ such that $|D \cap S_\ell| = 1$ and restart at S_ℓ our inductive argument. Of course, S_j may be paired off with S_q where $j > q \geq 1$ and $|D \cap S_q| \geq 3$; in this case, of necessity, $|D \cap S_p| = 2$ for $j > p > q$, and we restart the argument after S_q when $q > 1$. Therefore, one of the following cases must hold for S_1 .

- (i) $|D \cap S_1| \geq 3$, then S_1 may be paired off with the least j such that $|D \cap S_j| = 1$, if necessary.
- (ii) $|D \cap S_1| = 2$ and every S_j with $|D \cap S_j| = 1$ is paired off with S_q such that $q < j$ and $|D \cap S_q| \geq 3$.
- (iii) $|D \cap S_1| = 2$ and there exists $j > 1$ with $|D \cap S_j| = 1$ which is not paired off with some S_q such that $q < j$ and $|D \cap S_q| \geq 3$. If $j = k$, then by examining S_k, S_{k-1} , and S_1 , we will readily see that the assumption is impossible (u_1 is not dominated). If $j < k$, then there must exist $q > j$ such that $|D \cap S_q| \geq 3$ (in order to dominate $u_{3(j+1)-2}$).
- (iv) $|D \cap S_1| = 1$, then there must exist $q > 1$ such that $|D \cap S_q| \geq 3$ (in order to dominate u_4).

In each case, we conclude $|D| \geq 2k$, contradicting our original supposition. ■

Proposition 25. *Suppose $C(C_{3k}, f)$ is a functigraph with domain G_1 and codomain G_2 . Partition G_2 into three sets V_1, V_2 , and V_3 such that $V_i = \{v_j \mid j \equiv i \pmod{3}\}$. If there is some i such that the average degree over all vertices in V_i is strictly greater than 4, then $\gamma(C(C_{3k}, f)) < 2\gamma(C_{3k})$.*

Proof. Suppose $C(C_{3k}, f)$ is a functigraph with codomain G_2 and that there is some i , say $i = 1$, such that the average degree over all vertices in V_1 is strictly greater than 4. Then $|N[V_1] \cap V(G_1)| \geq 2k + 1$. Let U_1 be the vertices in $V(G_1)$ that are not in $N[V_1]$ and notice that $|U_1| \leq k - 1$. Then $U_1 \cup V_1$ is a dominating set of $C(C_{3k}, f)$. ■

Remark 26. The result obtained in Proposition 25 is sharp as shown in Example 24. In the example, the average degree of the vertices in V_3 is exactly 4.

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