3-TRANSITIVE DIGRAPHS

César Hernández-Cruz

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Ciudad Universitaria, México, D.F., C.P. 04510, México

e-mail: cesar@matem.unam.mx

Abstract

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively.

A digraph D is 3-transitive if the existence of the directed path (u, v, w, x) of length 3 in D implies the existence of the arc $(u, x) \in A(D)$. In this article strong 3-transitive digraphs are characterized and the structure of non-strong 3-transitive digraphs is described. The results are used, e.g., to characterize 3-transitive digraphs that are transitive and to characterize 3-transitive digraphs with a kernel.

Keywords: digraph, kernel, transitive digraph, quasi-transitive digraph, 3-transitive digraph, 3-quasi-transitive digraph.

2010 Mathematics Subject Classification: 05C20.

1. Introduction

In this work, $D = (V(D), A(D))$ will denote a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. For general concepts and notation we refer the reader to [1, 4] and [7], particularly we will use the notation of [7] for walks, if $\mathscr{C} = (x_0, x_1, \ldots, x_n)$ is a walk and $i < j$ then $x_i \mathscr{C} x_j$ will denote the subwalk $(x_i, x_{i+1}, \ldots, x_{j-1}, x_j)$ of $\mathscr{C}$. Union of walks will be denoted by concatenation or with $\cup$. For a vertex $v \in V(D)$, we define the *out-neighborhood* of v in D as the set $N_D^+(v) = \{u \in V(D) \,|\, (v, u) \in A(D)\}$; when there is no possibility of confusion we will omit the subscript D. The elements of $N^+(v)$ are called the *out-neighbors* of v, and the *out-degree* of v, $d_D^+(v)$, is the number of out-neighbors of v. Definitions of in-neighborhood, in-neighbors and in-degree of v are analogously given. We say that a vertex u reaches a vertex v in

D if a directed uv-directed path (a path with initial vertex u and terminal vertex v) exists in D. An arc $(u, v) \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $(v, u) \notin A(D)$ (resp. $(v, u) \in A(D)$).

If D is a digraph and $X, Y \subseteq V(D)$, an XY-arc is an arc with initial vertex in X and terminal vertex in Y. If $X \cap Y = \emptyset$, $X \to Y$ will denote that $(x, y) \in A(D)$ for every $x \in X$ and $y \in Y$. Again, if X and Y are disjoint, $X \Rightarrow Y$ will denote that there are not YX-arcs in D. When $X \to Y$ and $X \Rightarrow Y$ we will simply write $X \mapsto Y$. If D_1, D_2 are subdigraphs of D, we will abuse notation to write $D_1 \to D_2$ or $D_1 D_2$-arc, instead of $V(D_1) \to V(D_2)$ or $V(D_1)V(D_2)$-arc, respectively. Also, if $X = \{v\}$, we will abuse notation to write $v \to Y$ or vY-arc instead of $\{v\} \to Y$ or $\{v\}Y$-arc, respectively. Analogously if $Y = \{v\}$.

A digraph is *strongly connected* (or strong) if for every $u, v \in V(D)$, there exists a uv-directed path, i.e., a directed path with initial vertex u and terminal vertex v. A *strong component* (or component) of D is a maximal strong subdigraph of D. The *condensation* of D is the digraph $D^\star$ with $V(D^\star)$ equal to the set of all strong components of D, and $(S, T) \in A(D^\star)$ if and only if there is an ST-arc in D. Clearly $D^\star$ is an acyclic digraph (a digraph without directed cycles), and thus, it has both vertices of out-degree equal to zero and vertices of in-degree equal to zero. A *terminal component* of D is a strong component T of D such that $d_{D^\star}^+(T) = 0$. An *initial component* of D is a strong component S of D such that $d_{D^\star}^-(S) = 0$.

A *biorientation* of the graph G is a digraph D obtained from G by replacing each edge $\{x, y\} \in E(G)$ by either the arc (x, y) or the arc (y, x) or the pair of arcs (x, y) and (y, x). A *semicomplete* digraph is a biorientation of a complete graph. An *orientation* of a graph G is an asymmetrical biorientation of G; thus, an *oriented* graph is an asymmetrical digraph. A tournament is an orientation of a complete graph. An orientation of a digraph D is a maximal asymmetrical subdigraph of D. A *complete* digraph is a biorientation of a complete graph obtained by replacing each edge $\{x, y\}$ by the arcs (x, y) and (y, x).

Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \dots, v_n\}$ and $H_1, H_2, \dots, H_n$ a family of vertex disjoint digraphs. The *composition* of digraphs $D[H_1, H_2, \dots, H_n]$ is the digraph having $\bigcup_{i=1}^n V(H_i)$ as its vertex set and arc set $\bigcup_{i=1}^n A(H_i) \cup \{(u, v) | u \in V(H_i), v \in V(H_j), (v_i, v_j) \in A(D)\}$. The dual (or converse) of D, $\overleftarrow{D}$ is the digraph with vertex set $V(\overleftarrow{D}) = V(D)$ and such that $(u, v) \in A(\overleftarrow{D})$ if and only if $(v, u) \in A(D)$. The directed cycle of length 3 will be denoted, as usual, by C_3.

A digraph is *transitive* if for every three distinct vertices $u, v, w \in V(D)$, $(u, v), (v, w) \in A(D)$ implies that $(u, w) \in A(D)$. Transitive digraphs have a lot of properties, many of which can be verified straightforward by using the following structural characterization, which can be found in [1] as an exercise.

Theorem 1. *Let D be a digraph D with strong components $S_1, S_2, \ldots, S_n$. Then D is a transitive digraph if and only if $D = D^\star[S_1, S_2, \ldots, S_n]$, where S_i is a complete digraph for $1 \le i \le n$.*

But, the structure of transitive digraphs is so rich that, working on this family, many problems become trivial or have a very simple solution. In view of this situation, some generalizations of transitive digraphs have been studied. Without doubt, the most studied generalization of transitive digraphs is the family of quasi-transitive digraphs. A digraph is quasi-transitive if for every three distinct vertices $u, v, w \in V(D)$, $(u, v), (v, w) \in A(D)$ implies that $(u, w) \in A(D)$ or $(w, u) \in A(D)$. Clearly, every semicomplete digraph is a quasi-transitive digraph, so, quasi-transitive digraphs generalize both, transitive and semicomplete digraphs. Quasi-transitive have been characterized by Bang Jensen and Huang in [2], and their structure is very similar to the structure of transitive digraphs. Once again, this structural characterization has been very helpful to solve a large number of problems over this family, e.g., characterization of quasi-transitive digraphs with 3-kings, Hamiltonicity in quasi-transitive digraphs, or the Laborde-Payan-Xuong Conjecture for quasi-transitive digraphs.

Quasi-transitive digraphs were generalized with 3-quasi-transitive digraphs. A digraph D is 3-quasi-transitive if for every directed path, (v_0, v_1, v_2, v_3), either $(v_0, v_3) \in A(D)$ or $(v_3, v_0) \in A(D)$. Let us notice that in the definition of 3-quasi-transitive digraphs, the subdigraph (v_0, v_1, v_2, v_3) considered is a directed path, so it cannot happen that $v_0 = v_3$ and we can effectively work on digraphs without loops. The family of 3-quasi-transitive digraphs were introduced by Bang-Jensen in the context of arc-locally semicomplete digraphs, which generalize both, semicomplete digraphs and semicomplete bipartite digraphs. A digraph is arc-locally in-semicomplete if $(z, x), (x, y), (w, y) \in A(D)$ and $z \ne w$ implies that $(z, w) \in A(D)$ or $(w, z) \in A(D)$. A digraph is arc-locally out-semicomplete if $(x, z), (x, y), (y, w) \in A(D)$ and $z \ne w$ implies that $(x, w) \in A(D)$ or $(w, x) \in A(D)$. A digraph is arc-locally semicomplete if it is arc-locally in-semicomplete and arc-locally out-semicomplete. These families are defined to fulfill a property on some specific orientation of a path of length 3, in all of them, the existence of a (undirected) 4-cycle can be inferred from the existence of the specific orientation. There is one more orientation of a directed path of length 3 that induces the existence of a fourth family of digraphs. A digraph is often called of the type $\mathcal{H}_4$ if $(x, w), (x, y), (z, y) \in A(D)$ and $z \ne w$ implies that $(w, z) \in A(D)$ or $(z, w) \in A(D)$. The problem of finding structural characterizations of these four families of digraphs was proposed by Bang-Jensen. Besides transitive and quasi-transitive digraphs, also arc-locally semicomplete digraphs [8] and arc-locally in-semicomplete digraphs [13] have been characterized.

In [10], Galeana-Sánchez and the author introduce k-transitive and k-quasi-transitive digraphs. A digraph D is k-transitive if the existence of a directed
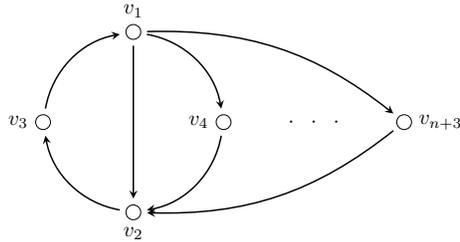
Figure 1. The family of digraphs F_n.

path $(v_0, v_1, \ldots, v_k)$ of length k in D implies that $(v_0, v_k) \in A(D)$. A digraph D is k-quasi-transitive if the existence of a directed path $(v_0, v_1, \ldots, v_k)$ of length k in D implies that $(v_0, v_k) \in A(D)$ or $(v_k, v_0) \in A(D)$. Also in [10], some basic properties on the structure of both k-transitive and k-quasi-transitive are proved. These properties are used to prove the existence of n-kernels in both families.

The aim of this article is to characterize strong 3-transitive digraphs and give a thorough description of the structure of non-strong 3-transitive digraphs. We will use the following characterization of strong 3-quasi-transitive digraphs given by Galeana-Sánchez, Goldfeder and Urrutia in [9].

Theorem 2 (Galeana-Sánchez, Goldfeder, Urrutia)**.** *Let D be a strong* 3*-quasi-transitive digraph of order n. Then D is either a semicomplete digraph, a semicomplete bipartite digraph or isomorphic to F_n (Figure* 1*).*

Thus, Section 2 will be devoted to prove some basic results about 3-transitive digraphs. In Section 3 the characterization of strong 3-transitive digraphs and the structural description of non-strong 3-transitive digraphs are given. In Section 4, one application of the results of Section 3 is given: A characterization of 3-transitive digraphs having a kernel. Also, an interesting problem concerning underlying graphs of 3-transitive and 3-quasi-transitive digraphs is proposed.

2.   Preliminary Results

We begin this section with a very simple remark that will be very useful through this work.

Remark 3. A digraph D is a 3-transitive digraph if and only if $\overleftarrow{D}$ is 3-transitive.

The following is another simple, yet useful, property of k-transitive digraphs.

Proposition 4. *If D is a k-transitive digraph with $k \geq 2$, then D is $k+n(k-1)$-transitive for every $n \in \mathbb{N}$.*

Proof. Let D be a k-transitive digraph. We will proceed by induction on n.

For $n = 1$, consider $(v_0, v_1, \ldots, v_{k+(k-1)})$, a directed path of length $k+(k-1)$. From the k-transitivity of D we have that $(v_0, v_k) \in A(D)$, so $(v_0, v_k, v_{k+1}, \ldots, v_{k+(k-1)})$ is a $v_0 v_k$-directed path of length k, and by the k-transitivity of D, we have that $(v_0, v_{k+(k-1)}) \in A(D)$.

Let us assume the result valid for $n - 1$ and let $(v_0, v_1, \ldots, v_{k+n(k-1)})$ be a directed path of length $k + n(k - 1)$ in D. By the induction hypothesis $(v_0, v_{k+(n-1)(k-1)}) \in A(D)$, and clearly $(v_0, v_{k+(n-1)(k-1)}, \ldots, v_{k+n(k-1)})$ is a directed path of length k in D.

It follows from the k-transitivity that $(v_0, v_{k+n(k-1)}) \in A(D)$. The result is now obtained by the Principle of Mathematical Induction. ∎

As a particular case of Proposition 4, we can observe that a 3-transitive digraph is n-transitive for every odd integer n. We can state this observation as the following corollary.

Corollary 5. *Let D be a 3-transitive digraph and $(v_0, v_1, \ldots, v_n)$ a directed path in D. Then $(v_0, v_i) \in A(D)$ for every odd integer $1 \leq i \leq n$.*

Proof. It is straightforward from Proposition 4. ∎

In [14], Wang and Wang proved some results describing the structure of non-strong 3-quasi-transitive digraphs. Since every 3-transitive digraph is also 3-quasi-transitive, the properties stated next hold also for 3-transitive digraphs.

Proposition 6 [14]**.** *Let D' be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph D and let $s \in V(D) \setminus V(D')$ with at least one arc from D' to s and $D' \Rightarrow s$. Then each of the following holds:*

1. *If D' is a bipartite digraph with bipartition (X, Y) and there exists a vertex of X which dominates s, then $X \mapsto s$.*

2. *If D' is a non-bipartite digraph, then $D' \mapsto s$.*

In the case of 3-transitive digraphs, the condition $D' \Rightarrow s$ in Proposition 6 not necessary. The following proposition is some kind of analogous of Proposition 6 for 3-transitive digraphs, emphasizing the behavior of certain strong subdigraphs.

Proposition 7. *Let D be a 3-transitive digraph and $v \in V(D)$. The following statements hold:*

1. *For every C_3 in D such that there is a $C_3 v$-arc in D, then $C_3 \to v$.*

2. *For every C_3 in D such that there is a $v C_3$-arc in D, then $v \to C_3$.*

3. *For every $\overleftrightarrow{K_n}$ in D, $n \geq 3$, such that there is a $\overleftrightarrow{K_n} v$-arc in D, then $\overleftrightarrow{K_n} \to v$.*

4. *For every $\overleftrightarrow{K_n}$ in D, $n \geq 3$, such that there is a $v \overleftrightarrow{K_n}$-arc in D, then $v \to \overleftrightarrow{K_n}$.*

5. *For every $\overleftrightarrow{K_{n,m}} = (X, Y)$ in D such that there is a Xv-arc in D, then $X \to v$.*

6. *For every $\overleftrightarrow{K_{n,m}} = (X, Y)$ in D such that there is a vX-arc in D, then $v \to X$.*

Proof. For 1. Let $C_3 = (x, y, z, x)$ be a cycle in D and $(x, v) \in A(D)$. The existence of the directed path (y, z, x, v) in D, implies that $(y, v) \in A(D)$. Finally, since (z, x, y, v) is a directed path of length 3 in D, $(z, v) \in A(D)$. Thus $C_3 \to v$.

For 2. It suffices to dualize 1 using Remark 3.

For 3. Let $D[S]$, with $S = \{1, 2, \ldots, n\}$, be a complete subdigraph of D and $(1, v) \in A(D)$. Let $i \in S \setminus \{1\}$ be an arbitrary vertex. Remember that $n \geq 3$, so there exists a vertex $j \in S \setminus \{1, i\}$. Now, since $D[S] = \overleftrightarrow{K_n}$, we have the existence of the directed path $(i, j, 1, v)$, which implies that $(i, v) \in A(D)$. But i is an arbitrary vertex of $D[S]$, and then we can conclude that $D[S] \to v$.

For 4. It suffices to dualize 3 using Remark 3.

For 5. Let $\overleftrightarrow{K_{n,m}} = (X, Y)$ be a complete subdigraph of D and $x \in X$. If $|X| = 1$, then we are done. If not, let $z \in X$ be a vertex such that $z \neq x$. Since $Y \neq \emptyset$, there is a vertex $y \in Y$. Also, $(z, y), (y, x) \in A(D)$, because $D[X \cup Y]$ is a complete bipartite digraph. So (z, y, x, v) is a directed path of length 3 in D and hence, $(z, v) \in A(D)$. Thus, $X \to v$.

For 6. It suffices to dualize 5 using Remark 3.   ∎

The following proposition is also due to Wang and Wang.

Proposition 8 [14]**.** *Let D' be a non-trivial strong subdigraph of a 3-quasi-transitive digraph D. For any $s \in V(D) \setminus V(D')$, if there exists a directed path between s and D', then s and D' are adjacent.*

In the case of 3-transitive digraphs we can be a little more specific. The proof of the following proposition will be omitted since it is almost the same as the one given by Wang and Wang in [14].

Proposition 9. *Let D' be a non-trivial strong subdigraph of a 3-transitive digraph D and $s \in V(D) \setminus V(D')$. Then each of the following holds:*

1. *If there exists an sD'-directed path in D, then an sD'-arc exists.*

2. *If there exists a $D's$-directed path in D, then a $D's$-arc exists.*

The following couple of propositions will be used later to characterize strong 3-transitive digraphs.

Proposition 10. *Let D be a strong 3-transitive digraph of order $n \geq 4$. If D is semicomplete, then D is complete.*

Proof. For any $(x, y) \in A(D)$, let $P = (y_0, y_1, \ldots, y_s)$ be a shortest path from y to x. If $s \geq 3$, then by Corollary 5 we can find a shorter path than P from y to

x. Suppose that $s = 2$, then (x, y, y_1, x) is a 3-cycle in D. Let $D' = D[\{x, y, y_1\}]$. Since the order of D is $n \geq 4$, there exists $v \in V(D) \setminus V(D')$. Also, D is strong, so a $D's$-directed path and an sD'-directed path exist in D. It follows from Propositions 7 (1 and 2) and 9 that $(y_1, v), (v, x) \in A(D)$. So (y, y_1, v, x) is a directed path of length 3 in D and hence, $(y, x) \in A(D)$. This contradicts that $s = 2$. Thus, $(y, x) \in A(D)$. ∎

Proposition 11. *Let D be a strong 3-transitive digraph. If D is semicomplete bipartite, then D is complete bipartite.*

Proof. Let (X, Y) be the bipartition of D. It suffices to prove that for any $(v, u) \in A(D)$, $(u, v) \in A(D)$. Since D is strong, there exists a path P from u to v of length n. Again, since D is bipartite and u and v belong to the different partite, n must be odd. By Corollary 5, $(u, v) \in A(D)$. ∎

3. THE STRUCTURE OF 3-TRANSITIVE DIGRAPHS

Let C_3^* and C_3^{**} be directed triangles with one and two symmetrical arcs, respectively. Digraphs C_3, C_3^* and C_3^{**} are shown in Figure 2.
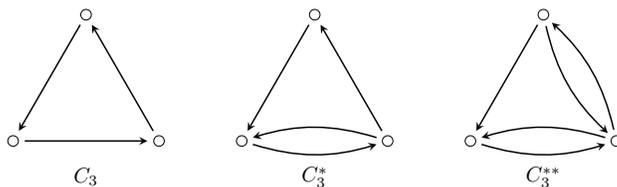

Figure 2. The digraphs C_3, C_3^* and C_3^{**}.

The characterization of strong 3-transitive digraphs is now proved.

Proposition 12. *A strong digraph D of order n is 3-transitive if and only if it is one of the following:*

1. *A complete digraph,*
2. *A complete bipartite digraph,*
3. *C_3, C_3^* or C_3^{**}.*

Proof. Since every 3-transitive digraph is 3-quasi-transitive, in virtue of Theorem 2, a strong 3-transitive digraph must be either semicomplete, semicomplete bipartite or isomorphic to F_n. But F_n is not 3-transitive, so a strong 3-transitive digraph must be either semicomplete or semicomplete bipartite. It is clear that every strong digraph of order less than or equal to 3 is either complete, complete

bipartite or one of the digraphs C_3, C_3^* or C_3^{**}. If D has order greater than or equal to 4, and it is a semicomplete digraph, it follows from Proposition 10 that D is complete. Finally, if D is semicomplete bipartite, it follows from Proposition 11 that D is complete bipartite. ∎

As immediate corollary from Proposition 12, we get the following result.

Corollary 13. *Let D be a 3-transitive digraph. Then D is Hamiltonian if and only if D is strong and it is not bipartite or it is regular.*

Let us recall that Proposition 7 describes the interaction of a single vertex with some subdigraphs of a 3-transitive digraph D. This covers the case when a strong component of D consists of a single vertex. In [14], the following proposition is proved.

Proposition 14. *Let D_1 and D_2 be two distinct non-trivial strong components of a 3-quasi-transitive digraph with at least one D_1D_2-arc. Then either $D_1 \mapsto D_2$ or the digraph induced by $D_1 \cup D_2$ is a semicomplete bipartite digraph.*

As it was noted before, every 3-transitive digraph is a 3-quasi-transitive digraph, so Proposition 14 is also valid for 3-transitive digraphs. In an attempt to be more explicit with the interaction between non-trivial strong components of a 3-transitive digraph, we state the following proposition. Nonetheless, we omit the proof, since it is very similar to the proof of Proposition 14.

Proposition 15. *Let D be a 3-transitive digraph and S_1, S_2 be distinct strong components of D such that there exists an S_1S_2-arc. The following statements hold:*

1. *If S_1 contains a subdigraph isomorphic to C_3, then $S_1 \to S_2$.*
2. *If S_2 contains a subdigraph isomorphic to C_3, then $S_1 \to S_2$.*
3. *If S_i is a complete bipartite digraph with bipartition (X_i, Y_i) for $i \in \{1, 2\}$ and if the S_1S_2-arc is an X_1X_2-arc, then $X_1 \to X_2$.*
4. *If S_i is a complete bipartite digraph with bipartition (X_i, Y_i) for $i \in \{1, 2\}$ and there exist an X_1X_2-arc and a Y_1X_2-arc, then $S_1 \to S_2$.*
5. *If S_i is a complete bipartite digraph with bipartition (X_i, Y_i) for $i \in \{1, 2\}$ and there exist an X_1X_2-arc and an X_1Y_2-arc, then $S_1 \to S_2$.*

As a direct consequence of Propositions 9 and 15, we have the following corollary.

Corollary 16. *Let D be a 3-transitive digraph and S_1 a strong component of D which contains a subdigraph isomorphic to C_3. If $S_1 \to v$ for some vertex $v \in V$, then $S_1 \to u$ for every vertex $u \in V$ that can be reached from v. Dually, if $v \to S_1$ for some vertex $v \in V$, then $u \to S_1$ for every vertex $u \in V$ that reaches v.*

We have already proved that the structure of 3-transitive digraphs is very similar to the structure of transitive digraphs. The following results are devoted to a deeper exploration of the similarities between these families of digraphs. A structural characterization of 3-transitive digraphs that are transitive is given.

Theorem 17. *Let D be a non-strong 3-transitive digraph with strong components $S_1, S_2, \ldots, S_p$. Then $D = D^\star[S_1, S_2, \ldots, S_p]$ if and only if, for every pair of strong components S_i, S_j of D, such that an S_iS_j-arc exists in D, then:*

1. *If S_i, S_j are complete bipartite digraphs, then $D[S_i \cup S_j]$ is not bipartite.*

2. *If one of S_i and S_j is a complete bipartite digraph and the other consists of a single vertex, then $D[S_i \cup S_j]$ is not bipartite.*

Proof. The necessity is trivial. In order to prove the sufficient, let S_i and S_j be two distinct strong components of D such that there is an S_iS_j-arc. If both S_i and S_j are both non-trivial digraphs, then by 1 of the theorem and Proposition 14, we have that $S_i \to S_j$. Since the converse of a 3-transitive digraph is still a 3-transitive digraph, we assume, without loss of generality, that S_i is a non-trivial complete bipartite digraph with bipartition (X_i, Y_i) and $S_j = \{v\}$. Since $D[S_i \cup S_j]$ is not a bipartite digraph, then there is a vertex $x \in X_i$ such that $x \to v$ and there is a vertex $y \in Y_i$ such that $y \to v$. By Proposition 6.1, we have that $S_i \to v$. ∎

Theorem 18. *Let D be a 3-transitive digraph. Then $D^\star$ is a transitive digraph if and only if for every triplet of strong components S_1, S_2, S_3 of D, such that: S_i consists of a single vertex v_i, $i \in \{1,3\}$; S_2 is either a single vertex v_2 or a complete bipartite digraph with bipartition (X, Y) and $v_1 \to v_2 \to v_3$ or $v_1 \to X \to v_3$ but there are neither v_1Y-arcs nor Yv_3-arcs in D, respectively, then $(v_1, v_3) \in A(D)$.*

Proof. Let D be a 3-transitive digraph. If $D^\star$ is a transitive digraph, then for every triplet of strong components S_1, S_2 and S_3 of D, such that there is an S_1S_2-arc in D and an S_2S_3-arc in D, then there is an S_1S_3-arc in D. In particular, if S_1 and S_3 consist of single vertices v_1 and v_3 respectively, then $(v_1, v_3) \in A(D)$.

For the converse, let D be a 3-transitive digraph and S_1, S_2 and S_3 strong components of D, such that there is an S_1S_2-arc in D and an S_2S_3-arc in D. We will prove that there is an S_1S_3-arc in D. If S_1 contains an isomorphic copy of C_3, then, by Corollary 16, we have that $S_1 \to S_3$ in D. If S_3 contains an isomorphic copy of C_3, again, by Corollary 16, we have that $S_1 \to S_3$. So, let us assume that neither S_1 nor S_3 contains an isomorphic copy of C_3.

It follows from Proposition 12 that S_1 and S_3 are either a single vertex or complete bipartite digraphs. If S_1 is not a single vertex, then it is a complete bipartite digraph with bipartition (X_1, Y_1). Let us assume without loss of generality that the S_1S_2-arc is an X_1S_2-arc. Let (x_1, u) be the S_1S_2-arc in D. Since

S_2 is a strong component of D, we have, by Propositions 12 and 15, two cases. The first case is that a vertex $s_3 \in V(S_3)$ exists, such that $(u, s_3) \in A(D)$. In this case is clear that, for any vertex $y_1 \in Y_1$ (recall that $Y_1 \neq \emptyset$), (y_1, x_1, u, s_3) is a directed path of length 3 in D. By the 3-transitivity of D, we have that $(y_1, s_3) \in A(D)$, the desired S_1S_3-arc. The second case is that vertices $v \in V(S_2)$ and $s_3 \in V(S_3)$ exist, such that $(u, v), (v, s_3) \in A(D)$. Again, it is clear that (x_1, u, v, s_3) is a directed path of length 3 and thus, $(x_1, s_3) \in A(D)$, the desired S_1S_3-arc. The case when S_3 is a complete bipartite digraph can be obtained dualizing the previous argument using Remark 3.

So, the remaining cases are when S_1 and S_3 consist of single vertices. We have again two cases. First, when S_2 contains a subdigraph isomorphic to C_3, then $S_2 \to S_3$. So, there exist vertices $s_1 \in V(S_1), u, v \in V(S_2), s_3 \in V(S_3)$ such that $(s_1, u), (u, v), (v, s_3) \in A(D)$. Thus, (s_1, u, v, s_3) is a directed path of length 3 in D. By the 3-transitivity of D, $(s_1, s_3) \in A(D)$ is the desired S_1S_3-arc. If S_2 does not contain a subdigraph isomorphic to C_3, then S_2 is a single vertex or complete bipartite. If S_2 is a single vertex v_2 or a complete bipartite digraph with bipartition (X, Y) such that $v_1 \to v_2 \to v_3$ or $v_1 \to X \to v_3$ but there are neither v_1Y-arcs nor Yv_3-arcs in D, respectively, then, by hypothesis $(v_1, v_3) \in A(D)$. Hence, we have the existence of an S_1S_3-arc. The remaining case is that S_2 is a complete bipartite digraph with bipartition (X, Y) such that $v_1 \to X \to v_3$, and either a v_1Y-arc or a Yv_3-arc exists. In the first case we have by Proposition 15 that $v_1 \to S_2$, and thus, vertices $u \in X$, $v \in Y$ exist such that $(v_1, v), (u, v_3) \in A(D)$. So, (v_1, v, u, v_3) is a directed path of length 3 in D. For the second case, again by Proposition 15, it follows that $S_2 \to v_3$. Then, vertices $u \in X$ and $v \in Y$ exist such that $(v_1, u), (v, v_3) \in A(D)$. Therefore, (v_1, u, v, v_3) is a directed path of length 3 in D. In either case, it follows by the 3-transitivity of D that $(v_1, v_3) \in A(D)$. So an S_1S_3-arc exists.

Since the cases are exhaustive, we have that $D^\star$ is transitive. ∎

Corollary 19. *Let D be a 3-transitive digraph. Then D is a transitive digraph if and only if every strong component of D is a complete digraph and, for every triplet of strong components S_1, S_2, S_3 of D, such that: S_i consists of a single vertex v_i, $i \in \{1, 3\}$; S_2 is either a single vertex v_2 or a symmetrical arc $(v_2, v_2') \in A(D)$ and $v_1 \to v_2 \to v_3$ but $(v_1, v_2'), (v_2', v_3) \notin A(D)$, then $(v_1, v_3) \in A(D)$.*

Proof. It is clear from Theorems 1, 17 and 18. ∎

Corollary 20. *Let D be a 3-transitive digraph. If every strong component of D is a complete digraph of order greater than or equal to 3, then D is transitive.*

Proof. Let D be a 3-transitive digraph such that every strong component of D is a complete digraph of order greater than or equal to 3. Then, by Theorem 18, it is clear that $D^\star$ is transitive. Also, in virtue of Theorem 15, we can observe
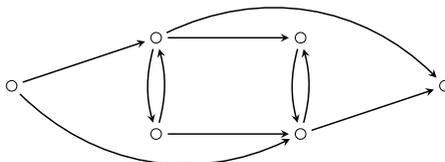
Figure 3. A 3-transitive digraph without 3-transitive condensation.

that $S_i \to S_j$ for every pair of strong components S_i, S_j of D such that there exists an $S_i S_j$-arc in D. Thus, $D = D^\star[S_1, S_2, \ldots, S_n]$, where $\{S_1, S_2, \ldots, S_n\}$ is the set of strong components of D and $D^\star$ is transitive. So, by Theorem 1, D is transitive. ∎

As we have already shown, the structure of 3-transitive digraphs is very similar to the structure of transitive digraphs. We know that the condensation of a transitive digraph is again transitive. A characterization of 3-transitive digraphs with a transitive condensation has been already given, but a natural question arises. Is the condensation of a 3-transitive digraph 3-transitive again? Sadly, the answer is no, Figure 3 shows a counterexample to this fact.

Following similar ideas to those used to characterize the 3-transitive digraphs with a transitive condensation in Theorem 18, we can characterize 3-transitive digraphs with a 3-transitive condensation. The 'bad' configurations, preventing the condensation of a 3-transitive digraph to be 3-transitive, are pointed out in the following theorem.

Theorem 21. *Let D be a 3-transitive digraph. Then $D^\star$ is a 3-transitive digraph if and only if for every 4-set, $\{S_1, S_2, S_3, S_4\}$, of strong components of D such that: S_i consists of a single vertex v_i, $i \in \{1, 4\}$ and one of the following conditions is fulfilled:*

1. *S_2 consists of single vertex v_2 and S_3 is a complete bipartite digraph with bipartition (X, Y), such that $v_1 \to v_2 \to X$ and $Y \to v_4$, but there are neither $v_2 Y$-arcs nor $X v_4$-arcs in D;*

2. *S_2 is a complete bipartite digraph with bipartition (X, Y) and S_3 consists of single vertex v_3, such that $v_1 \to X$ and $Y \to v_3 \to v_4$, but there are neither $v_1 Y$-arcs nor $X v_3$-arcs in D;*

3. *S_j is a complete bipartite digraph with bipartition (X_j, Y_j), $j \in \{2, 3\}$, such that $v_1 \to X_2 \to X_3$ and $Y_3 \to v_4$, but there are neither $v_1 Y_2$-arcs, $v_1 X_3$-arcs, $Y_2 v_4$-arcs, nor $X_3 v_4$-arcs, and $D[V(S_2) \cup V(S_3)]$ is a semicomplete bipartite digraph,*

then $(v_1, v_4) \in A(D)$.

Proof. Let D be a 3-transitive digraph. If $D^\star$ is a 3-transitive digraph, then for every 4-set of strong components $\{S_1, S_2, S_3, S_4\}$ of D, such that there is an $S_i S_{i+1}$-arc in D, $i \in \{1, 2, 3\}$, then there is an $S_1 S_3$-arc in D. In particular, if S_1 and S_4 consist of single vertices v_1 and v_4 respectively, then $(v_1, v_4) \in A(D)$.

Conversely, let $\{S_1, S_2, S_3, S_4\}$ be a 4-set of strong components of D such that there is an $S_i S_{i+1}$-arc in D, $i \in \{1, 2, 3\}$. If S_1 or S_4 are non-trivial, then by Proposition 9, there exists an $S_1 S_4$-arc in D. So let us assume without loss of generality that S_i consists of a single vertex S_i, $i \in \{1, 4\}$. Suppose that S_2 or S_3 contains C_3 as a subdigraph. It can be easily derived from Corollary 16 the existence of an $S_1 S_4$-arc in D. So, we have 3 cases.

Before the analysis of the cases, let us recall that, by Proposition 7, if $S = (X, Y)$ is a bipartite strong component of D and $v \in V(D) \setminus V(S)$ such that a vX-arc exists, then $v \to X$; and if an Xv-arc exists, then $X \to v$.

The first case is when S_2 consists of single vertex v_2 and S_3 is a complete bipartite digraph with bipartition (X, Y). Clearly, if a $v_2 X$-arc, and an Xv_4-arc exist, then $v_2 \to X \to v_4$. Thus, a $v_1 v_4$-directed path of length 3 exists and $(v_1, v_4) \in A(D)$ by the 3-transitivity of D. Analogously, if a $v_2 Y$-arc and a $Y v_4$-arc exist in D, clearly $(v_1, v_4) \in A(D)$. So, we can assume without loss of generality that $v_2 \to X, Y \to v_4$ and there are neither $v_2 Y$-arcs nor Xv_4-arcs in D. Then, by hypothesis, $(v_1, v_4) \in A(D)$.

The second case is when S_2 is a complete bipartite digraph with bipartition (X, Y) and S_3 consists of single vertex v_3. But this case is just the dual of the first case, so, using Remark 3, it can be easily shown that $(v_1, v_4) \in A(D)$.

The third case is when S_j is a complete bipartite digraph with bipartition (X_j, Y_j), $j \in \{2, 3\}$. Let us assume without loss of generality that $v_1 \to X_2$ and $Y_3 \to v_4$. If $X_2 \to Y_3$, then $v_1 \to X_2 \to Y_3 \to v_4$ and clearly $(v_1, v_4) \in A(D)$. If $Y_2 \to X_3$, it is easy to observe that $X_2 \to Y_3$. So, we can suppose that $X_2 \to X_3$ (thus $Y_2 \to Y_3$) and that there are neither $X_2 Y_3$-arcs nor $Y_2 X_3$-arcs. Thus, $D[V(S_2) \cup V(S_3)]$ is semicomplete bipartite. If $v_1 \to Y_2$, then $v_1 \to Y_2 \to Y_3 \to v_4$ and we are done. If $v_1 \to X_3$, then $v_1 \to X_3 \to Y_3 \to v_4$ and $(v_1, v_4) \in A(D)$. Symmetrically, if $Y_2 \to v_4$ or $X_3 \to v_4$ we can conclude that $(v_1, v_4) \in A(D)$. Hence, we can suppose that there are neither $v_1 Y_2$-arcs, $v_1 X_3$-arcs, $Y_2 v_4$-arcs, nor $X_3 v_4$-arcs in D. By hypothesis $(v_1, v_4) \in A(D)$.

Since the cases are exhaustive, we have that $D^\star$ is 3-transitive. ∎

4.   Consequences

4.1.   Existence of kernels

Let D be a digraph and $N \subseteq V(D)$. We say that N is l-absorbent if for every vertex $u \in V(D) \setminus N$, there is a vertex $v \in N$ such that $d(u, v) \leq l$ in D. The set

N is k-independent if for every $u, v \in N$, we have that $d(u, v), d(v, u) \geq N$. We call N a (k, l)-kernel of D if D is k-independent and l-absorbent. A $(k, k-1)$-kernel is a k-kernel and a 2-kernel is simply a kernel. In [12], von Neumann and Morgenstern introduce the concept of kernel of a digraph in the context of Game Theory. Since then, kernels have been largely studied for their applications within many branches of Mathematics, we can find in [5] a very good survey on the subject. Also, in [6] is proved that the problem of determining if a given digraph has a kernel is NP-complete, so, finding sufficient conditions for a digraph to have a kernel or finding large families of digraphs with a kernel is a very valuable progress.

Theorem 22. *Let D be a 3-transitive digraph. Then D has a kernel if and only if it has no terminal strong component isomorphic to C_3.*

Proof. The 'only if' part will be proved by contrapositive. Let D be a 3-transitive digraph such that a terminal strong component S is isomorphic to C_3. Let $V(S) = \{v_0, v_1, v_2\}$ and $A(S) = \{(v_i, v_{i+1})\}_{i=0}^{2}$ (mod 3). Since S is terminal, we have that $d^+(v) = 1$ for every $v \in V(S)$. Thus, the only out- neighbor of v_i is v_{i+1} (mod 3). It is clear that S has no kernel and vertices in S cannot be absorbed by any other vertex in D, thus, D has no kernel.

The 'if' implication will be proved by induction on the number of strong components of D. Let us assume that D is strong. It can be directly verified that the digraphs mentioned in Proposition 12, except for C_3 have a kernel. So, let us assume that every 3-transitive digraph such that no terminal strong component isomorphic to C_3 and with n strong components has a kernel. Let D be a 3-transitive digraph such that no terminal strong component isomorphic to C_3 and with $n+1$ strong components. Let us recall that $D^\star$ is an acyclic digraph, so, we can consider an initial strong component S of D. By induction hypothesis, $D - S$ has a kernel N. If S is not a complete bipartite digraph, then, either S consists of a single vertex or contains a subdigraph isomorphic to C_3. If S consists of a single vertex v, and v is absorbed by N, we are done. If v is not absorbed by N, since S is initial, $N \cup \{v\}$ is independent and thus a kernel of D. If D contains a subdigraph isomorphic to C_3, we can use Corollary 16 to prove that $S \mapsto S_t$ for some terminal strong component S_t of D. But since S_t is terminal, at least one vertex of S_t must belong to N, and thus S is absorbed by N. So, N is a kernel of D. If S is a complete bipartite digraph, we must consider three cases. Let (X, Y) be the bipartition of S. If neither X nor Y is absorbed by N, then we consider $N \cup X$. Since S is an initial component, every arc between X and N must be an XN-arc. But if such arc exists, we would have by Proposition 7.5 that $X \to N$, contradicting our assumption. So $N \cup X$ is an independent set, and $Y \to X$ because S is a complete bipartite digraph. Thus, $N \cup X$ is a kernel for D. If some vertex of X is absorbed by N, then by Proposition 7.5 X is absorbed

by N. So let us assume that Y is not absorbed by N. Once again, since S is an initial component, every arc between N and Y must be a YN-arc, but no such arc can exist. So, $N \cup Y$ is an independent absorbent set of D, and hence a kernel of D. The case when Y is absorbed but X is not is analogous. Finally, if S is absorbed by N, we have that N is the desired kernel of D.

Since in every case D has a kernel, the result follows from the Principle of Mathematical Induction. ∎

In [10], Galeana-Sánchez and the author proved that a k-transitive digraph D has a n-kernel for every $n \geq k$. Thus, Theorem 22 completes the study of existence of k-kernels in 3-transitive digraphs.

4.2. One further problem

A graph G is a comparability graph if it can be oriented as an asymmetrical transitive digraph. In [11], Ghouila-Houri proved that the underlying graphs of asymmetrical quasi-transitive digraphs are comparability graphs. That is to say, a graph G can receive a transitive orientation if and only if G can receive a quasi-transitive orientation. In view of this result, and considering the great similarity between the structure of transitive and 3-transitive digraphs, we propose the following conjecture.

Conjecture 23. *Let D be an asymmetrical 3-quasi-transitive digraph, then the underlying graph of D, $UG(D)$, admit a 3-transitive asymmetrical orientation.*

Acknowledgements

References

[1] J. Bang-Jensen and G. Gutin, Digraphs. Theory, Algorithms and Applications (Springer-Verlag Berlin Heidelberg New York, 2002).

[2] J. Bang-Jensen and J. Huang, *Quasi-transitive digraphs*, J. Graph Theory **20** (1995) 141–161.
doi:10.1002/jgt.3190200205

[3] J. Bang-Jensen, J. Huang and E. Prisner, *In-tournament digraphs*, J. Combin. Theory (B) **59** (1993) 267–287.
doi:10.1006/jctb.1993.1069

[4] C. Berge, Graphs (North-Holland, Amsterdam New York, 1985).

[5] E. Boros and V. Gurvich, *Perfect graphs, kernels and cores of cooperative games*, Discrete Math. **306** (2006) 2336–2354.
doi:10.1016/j.disc.2005.12.031

[6] V. Chvátal, On the computational complexity of finding a kernel, Report No. CRM-300, 1973, Centre de recherches mathématiques, Université de Montréal.

[7] R. Diestel, Graph Theory 3rd Edition (Springer-Verlag Berlin Heidelberg New York, 2005).

[8] H. Galeana-Sánchez and I.A. Goldfeder, *A classification of arc-locally semicomplete digraphs*, Publicaciones Preliminares del Instituto de Matemáticas, UNAM **859** (2010).

[9] H. Galeana-Sánchez, I.A. Goldfeder and I. Urrutia, *On the structure of 3-quasi-transitive digraphs*, Discrete Math. **310** (2010) 2495–2498.
doi:10.1016/j.disc.2010.06.008

[10] H. Galeana-Sánchez and C. Hernández-Cruz, *k-kernels in k-transitive and k-quasi-transitive digraphs*, Submitted (2010).

[11] A. Ghouila-Houri, *Caractérization des graphes non orientés dont on peut orienter les arrêtes de manière à obtenir le graphe d'une relation d'rdre*, Comptes Rendus de l'Académie des Sciences Paris **254** (1962) 1370–1371.

[12] J. Von Neumann and O. Morgenstern, Theory of Games and Economic Behavior (Princeton University Press, Princeton, 1953).

[13] S. Wang and R. Wang, *The structure of arc-locally in-semicomplete digraphs*, Discrete Math. **309** (2009) 6555–6562.
doi:10.1016/j.disc.2009.06.033

[14] S. Wang and R. Wang, *Independent sets and non-augmentable paths in arc-locally in-semicomplete digraphs and quasi-arc-transitive digraphs*, Discrete Math. **311** (2010) 282–288.
doi:10.1016/j.disc.2010.11.009