

DISTRIBUTIVE LATTICES OF t - k -ARCHIMEDEAN SEMIRINGS*

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Abstract

A semiring S in $\mathbb{S}\mathbb{L}^+$ is a t - k -Archimedean semiring if for all $a, b \in S$, $b \in \sqrt{Sa} \cap \sqrt{aS}$. Here we introduce the t - k -Archimedean semirings and characterize the semirings which are distributive lattice (chain) of t - k -Archimedean semirings. A semiring S is a distributive lattice of t - k -Archimedean semirings if and only if \sqrt{B} is a k -ideal, and S is a chain of t - k -Archimedean semirings if and only if \sqrt{B} is a completely prime k -ideal, for every k -bi-ideal B of S .

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1. INTRODUCTION

In 1942, A.H. Clifford [6] first defined the semilattice decompositions of semigroups. Thus the idea of studying a semigroup through its greatest semilattice decomposition was introduced. The idea consists of decomposing a given semigroup S into subsemigroups (components) which are possibly of considerably simpler structure, through a congruence η on S such that S/η is the greatest semilattice homomorphic image of S and each η -class is a component subsemigroup. Though the idea first appeared in [6] but much attention was given to the semigroups which are union of groups. In 1954, T. Tamura and N. Kimura [13] showed that every commutative semigroup is a semilattice of Archimedean semigroups. This well known result has since been generalized by M. Petrich, M.S.

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Putcha, T. Tamura, N. Kimura, S. Bogdanovic, M. Ciric, F. Kmet and many others [3, 4, 5, 7, 8, 9, 10, 11, 12]. Much attention in this area has been aimed to the semigroups which are decomposable into a semilattice of Archimedean semigroups.

In this article we introduce the t - k -Archimedean semirings and characterize the semirings which are distributive lattices(chain) of t - k -Archimedean semirings. The t - k -Archimedean semirings are the semirings analogue of t -Archimedean semigroups, in some sense. The k -bi-ideals play a crucial role in characterizing such semirings. A necessary and sufficient condition for a semiring S to be a distributive lattice of t - k -Archimedean semirings is that the k -radical of each k -bi-ideal B of S is a k -ideal of S .

The preliminaries and prerequisites we need have been discussed in Section 2. In Section 3, several equivalent characterizations have been made for the semirings which are distributive lattices of t - k -Archimedean semirings, which is the main theorem of this article. In Section 4, the semirings which are chains of t - k -Archimedean semirings has been characterized. A semiring S is a chain of t - k -Archimedean semirings if and only if k -radical of each k -bi-ideal of S is a completely prime k -ideal.

2. PRELIMINARIES

A *semiring* $(S, +, \cdot)$ is an algebra with two binary operations $+$ and \cdot such that both the *additive reduct* $(S, +)$ and the *multiplicative reduct* (S, \cdot) are semigroups and such that the following distributive laws hold:

$$x(y + z) = xy + xz \text{ and } (x + y)z = xz + yz.$$

Thus the semirings can be regarded as a common generalization of both rings and distributive lattices. By \mathbb{SL}^+ we denote the variety of all semirings $(S, +, \cdot)$ such that $(S, +)$ is a semilattice, i.e., a commutative and idempotent semigroup. Throughout this paper, unless otherwise stated, S is always a semiring in \mathbb{SL}^+ .

Let A be a nonempty subset of S . Then the *k -closure* \overline{A} of A in S is defined by

$$\overline{A} = \{x \in S \mid x + a_1 = a_2, \text{ for some } a_1, a_2 \in A\}.$$

Then we have, $A \subseteq \overline{A}$ and if $(A, +)$ is a subsemigroup of $(S, +)$ then $\overline{A} = \{x \in S \mid x + a \in A, \text{ for some } a \in A\}$, since $(S, +)$ is a semilattice. A is called a *k -set* if $\overline{A} \subseteq A$. An ideal K of S is called a *k -ideal* of S if it is a k -set. A subsemiring B of S is called a *k -bi-ideal* of S if $BSB \subseteq B$ and B is a k -set. For $a \in S$, the least k -bi-ideal $B_k(a)$ of S [1], which contains a is given by

$$B_k(a) = \{u \in S \mid u + a + a^2 + asa = a + a^2 + asa, \text{ for some } s \in S\}.$$

We note that $\overline{aSa} = \{x \in S \mid x + asa = asa, \text{ for some } s \in S\}$ is a k -bi-ideal of S but may not contain a . A nonempty subset A of S is called *completely prime* (resp. *semiprimary*) if for $x, y \in S$, $xy \in A$ implies $x \in A$ or $y \in A$ (resp. $x^n \in A$ or $y^n \in A$, for some $n \in \mathbb{N}$).

Let F be a subsemiring of S . F is called a left (right) filter of S if:

- (i) for any $a, b \in S$, $ab \in F \Rightarrow b \in F$ ($a \in F$); and
- (ii) for any $a \in F$, $b \in S$, $a + b = b \Rightarrow b \in F$. F is a filter of S if it is both a left and a right filter of S . The least filter of S containing a is denoted by $N(a)$. Let \mathcal{N} be the equivalence relation on S defined by

$$\mathcal{N} = \{(x, y) \in S \times S \mid N(x) = N(y)\}.$$

Lemma 1. *Let S be a semiring in \mathbb{SL}^+ .*

- (a) *For $a, b \in S$ the following statements are equivalent:*
 - (i) *There are $s_1, s_2, t_1, t_2 \in S$ such that $b + s_1at_1 = s_2at_2$;*
 - (ii) *There is $s \in S$ such that $b + sas = sas$.*
- (b) *If $a, b, c, d \in S$ such that $c + xa = xa$ and $d + yb = yb$ for some $x, y \in S$, then there is some $z \in S$ such that $c + za = za$ and $d + zb = zb$.*
- (c) *If $a, b \in S$ such that $a + bxb = byb$ for some $x, y \in S$, then there is some $z \in S$ such that $a + bzb = bzb$.*

Proof. (a) Since (ii) \Rightarrow (i) is clear, we assume (i). For $x = s_1 + s_2 + t_1 + t_2$ one gets $s_1at_1 + xax = s_2at_2 + xax = xax$, since $(S, +)$ is a semilattice, and then $b + s_1at_1 + xax = s_2at_2 + xax$ implies that $b + xax = xax$.

Hence (i) implies (ii).

(b) Clearly, $z = x + y$ is such an element.

(c) Again, $z = x + y$ is such an element. ■

Let A be a non-empty subset of a semiring S . Then the k -radical of A in S is given by $\sqrt{A} = \{x \in S \mid (\exists n \in \mathbb{N}) x^n \in \overline{A}\}$. Thus $\sqrt{A} = \{x \in S \mid (\exists n \in \mathbb{N}) x^n + a_1 = a_2, \text{ for some } a_1, a_2 \in \overline{A}\}$. It is interesting to note that for every $a \in S$, $\sqrt{aSa} = \sqrt{B_k(a)}$, though $aSa \subseteq B_k(a)$ and the inclusion is likely to be proper.

Let S be a semiring in \mathbb{SL}^+ . Define a binary relation σ on S by: for $a, b \in S$,

$$a\sigma b \Leftrightarrow b \in \sqrt{SaS} \Leftrightarrow b^n \in \overline{SaS} \text{ for some } n \in \mathbb{N}.$$

Then $a^3 \in SaS \subseteq \overline{SaS}$ shows that σ is reflexive. So the transitive closure $\rho = \sigma^*$ is reflexive and transitive, and therefore the symmetric relation $\eta = \rho \cap \rho^{-1}$ is an equivalence relation. This equivalence relation η is the least distributive lattice congruence on S .

Lemma 2 [2]. *For any S in \mathbb{SL}^+ , η is the least distributive lattice congruence on S .*

Definition. A semiring S in \mathbb{SL}^+ is called left (right) k -Archimedean if for all $a \in S$, $S = \sqrt{Sa}(\sqrt{aS})$ and t - k -Archimedean semiring if it is both a left k -Archimedean semiring and a right k -Archimedean semiring.

Then by Lemma 1, a semiring S is t - k -Archimedean if and only if for $a, b \in S$ there exist $n \in \mathbb{N}$ and $x \in S$ such that $b^n + xa = xa$ and $b^n + ax = ax$. A semiring S is called a *distributive lattice(chain) of t - k -Archimedean semirings* if there exists a congruence ρ on S such that S/ρ is a distributive lattice(chain) and each ρ -class is a t - k -Archimedean semiring.

3. DISTRIBUTIVE LATTICES OF t - k -ARCHIMEDEAN SEMIRINGS

In this section we describe the semirings S by radical of k -bi-ideal of S . In the following proofs we will use that from $b + c = c$ for $b, c \in S$ in any semiring S in \mathbb{SL}^+ it follows that $b^n + c^n = c^n$ for every $n \in \mathbb{N}$. This can be proved by induction. Since the case $n = 1$ is given, we may assume $b^n + c^n = c^n$ for some $n \in \mathbb{N}$. Then $b^{n+1} + c^n b = c^n b$ and by adding c^{n+1} on both sides we get $b^{n+1} + c^n(b + c) = c^n(b + c)$, and hence $b^{n+1} + c^{n+1} = c^{n+1}$.

Lemma 3. *Let S be a semiring in \mathbb{SL}^+ such that for all $a, b \in S$, $ab \in \sqrt{Sa} \cap \sqrt{bS}$. Then*

1. *for all $a, b \in S$, $a \in \overline{bSb} \Rightarrow a \in \sqrt{b^{2^r} S b^{2^r}}$ for all $r \in \mathbb{N}$.*

2. for all $a, b \in S$, $a \in \sqrt{bSb}$ implies that $\sqrt{aS a} \subseteq \sqrt{bSb}$.
3. the least distributive lattice congruence η on S is given by: for $a, b \in S$,

$$a\eta b \Leftrightarrow b \in \sqrt{aS a} \text{ and } a \in \sqrt{bSb}.$$

Proof. (1) Let $a, b \in S$ such that $a \in \sqrt{bSb}$. Then there exists $s \in S$ such that $a + bsb = bsb$. By hypothesis, there exist $m \in \mathbb{N}$ and $u \in S$ such that $(b^2s)^m + ub = ub$. Then $a^{m+1} + (bsb)^{m+1} = (bsb)^{m+1}$ gives $a^{m+1} + bsub^2 = bsub^2$. Again, there are $n \in \mathbb{N}$ and $v \in S$ such that $(bsub^2)^n + b^2v = b^2v$. Then we have $a^{(m+1)(n+1)} + b^2vbsub^2 = b^2vbsub^2$ which yields $a \in \sqrt{b^2Sb^2}$. Thus the result is true for $r = 1$. Let $a \in \sqrt{b^{2^k}Sb^{2^k}}$ for some $k \in \mathbb{N}$. Then there are $p \in \mathbb{N}$ and $w \in S$ such that

$$a^p + b^{2^k}wb^{2^k} = b^{2^k}wb^{2^k}$$

Now iterating this implication as above, we get $a \in \sqrt{b^{2^{k+1}}Sb^{2^{k+1}}}$. Therefore, by the method of principle of mathematical induction, we have: for every $r \in \mathbb{N}$, $a \in \sqrt{b^{2^r}Sb^{2^r}}$.

(2) For $a \in \sqrt{bSb}$ there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n + bsb = bsb$. Let $x \in \sqrt{aS a}$. Then there exists $m \in \mathbb{N}$ such that $x^m \in \overline{aS a}$. Suppose $r \in \mathbb{N}$ be such that $2^r > n$. Then by (1), we find $p \in \mathbb{N}$ and $u \in S$ such that $x^p + a^{2^r}ua^{2^r} = a^{2^r}ua^{2^r}$ which implies $x^p + bsba^{2^r-n}ua^{2^r-n}bsb = bsba^{2^r-n}ua^{2^r-n}bsb$, and so $x \in \sqrt{bSb}$. Thus the result.

(3) From Theorem 3.4 [2], we have the least distributive lattice congruence η on S as follows:

$$\eta = \rho \cap \rho^{-1}, \text{ where } \rho = \sigma^* \text{ and } a\sigma b \Leftrightarrow b \in \sqrt{SaS}.$$

Let us define a binary relation ξ on S by: for $a, b \in S$,

$$a\xi b \Leftrightarrow b \in \sqrt{aS a} \text{ and } a \in \sqrt{bSb}.$$

We will show $\xi = \eta$. Clearly $\sqrt{aS a} \subseteq \sqrt{SaS}$. Now let $x \in \sqrt{SaS}$. Then there are $n \in \mathbb{N}$ and $s \in S$ such that $x^n + sas = sas$. Again, $sas \in \sqrt{Ssa} \subseteq \sqrt{Sa}$ implies that $(sas)^m + ta = ta$ for some $m \in \mathbb{N}$ and $t \in S$. Also, there are $p \in \mathbb{N}$ and $u \in S$ such that $(ta)^p + au = au$, which gives $x^{nm(p+1)} + aut = aut$, and so $x \in \sqrt{aS a}$. Thus $\sqrt{SaS} = \sqrt{aS a}$. Now $a\eta b$ implies that there are $c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_m \in S$ such that $a\sigma c_1, c_1\sigma c_2, \dots, c_{n-1}\sigma c_n, c_n\sigma b$ and $b\sigma d_1, d_1\sigma d_2, \dots, d_{m-1}\sigma d_m, d_m\sigma a$. These give $c_1 \in \sqrt{aS a}, c_2 \in \sqrt{c_1S c_1}, \dots, b \in \sqrt{c_nS c_n}$ and $d_1 \in \sqrt{bSb}, d_2 \in \sqrt{d_1S d_1}, \dots, a \in \sqrt{d_mS d_m}$ so that $b \in \sqrt{aS a}$ and $a \in \sqrt{bSb}$, by (2). Thus $a\xi b$. Again $a\xi b$ implies $b \in \sqrt{aS a}$ and $a \in \sqrt{bSb}$ which yields $a\sigma b$ and $b\sigma a$ to get $a\eta b$. Thus $\xi = \eta$. ■

Remark 4. Let S be a semiring in \mathbb{SL}^+ and $a \in S$. Then $\sqrt{aSa} = \sqrt{B_k(a)}$. Thus it follows that if for all $a, b \in S$, $ab \in \sqrt{Sa} \cap \sqrt{bS}$ then the least distributive lattice congruence η on S is given by: for $a, b \in S$,

$$a\eta b \Leftrightarrow a \in \sqrt{B_k(b)} \text{ and } b \in \sqrt{B_k(a)}.$$

Now we prove the main theorem of this article.

Theorem 5. *The following conditions on a semiring S in \mathbb{SL}^+ are equivalent:*

1. S is a distributive lattice of t - k -Archimedean semirings;
2. For all $a, b \in S$, $b \in \overline{SaS}$ implies $b \in \sqrt{Sa} \cap \sqrt{aS}$;
3. For all $a, b \in S$, $ab \in \sqrt{Sa} \cap \sqrt{bS}$;
4. \sqrt{B} is a k -ideal of S , for every k -bi-ideal B of S ;
5. $\sqrt{B_k(a)} = \sqrt{aSa}$ is a k -ideal of S , for all $a \in S$;
6. $N(a) = \{x \in S \mid a \in \sqrt{Sx} \cap \sqrt{xS}\}$ for all $a \in S$.

Proof. Scheme of the proof: (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1),(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (3), (3) \Leftrightarrow (6).

(1) \Rightarrow (2) : Let ν be a distributive lattice congruence on S such that the ν -classes $T_\alpha; \alpha \in S/\nu$ are t - k -Archimedean semirings. Let $a, b \in S$ be such that $b \in \overline{SaS}$. Then there is $s \in S$ such that $b+sas = sas$, which gives $b^3+uau = uau$, where $u = bs + sb$. Now $uau\nu au^2\nu au\nu va$ implies that $uau, au, ua \in T_\alpha$, for some $\alpha \in S/\nu$. Then there exist $m \in \mathbb{N}$ and $v \in T_\alpha$ such that $(uau)^m + auv = auv$ and $(uau)^m + vua = vua$. From these we get $b^{3m} + auv = auv$ and $b^{3m} + vua = vua$, and so $b \in \sqrt{aS} \cap \sqrt{Sa}$.

(2) \Rightarrow (3) : Let $a, b \in S$. Now $(ab)^2 \in \overline{SaS} \cap \overline{SbS}$ implies that there exist $m, n \in \mathbb{N}$ such that $(ab)^{2m} \in \overline{Sa}$ and $(ab)^{2n} \in \overline{bS}$. Thus $ab \in \sqrt{Sa} \cap \sqrt{bS}$.

(3) \Rightarrow (1) : By the Lemma 3, the least distributive lattice congruence η on S is given by: for $a, b \in S$

$$a\eta b \Leftrightarrow a \in \sqrt{bSb} \text{ and } b \in \sqrt{aSa}.$$

Let T be an η -class. Then T is a subsemiring of S , since η is a distributive lattice congruence. Let $a, b \in T$. Then there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n + bsb = bsb$ and $b^n + asa = asa$. Then we have $a^{n+1} + (a+abs)b = (a+abs)b$ and $a^{n+1} + b(sba+a) = b(sba+a)$. Now $a\eta(a+abs)\eta(sba+a)$ implies $a+abs, sba+a \in T$, and so $a \in \sqrt{Tb} \cap \sqrt{bT}$. Hence T is a t - k -Archimedean semiring.

(3) \Rightarrow (4) : Let B be a k -bi-ideal of S and let $u, v \in \sqrt{B}, c \in S$. Then there exist $n \in \mathbb{N}$ and $b \in B$ such that $u^n + b = b$ and $v^n + b = b$. By (3), there exist $x, y \in S$ and $m_1, t_1 \in \mathbb{N}$ such that $(uc)^{m_1} + xu = xu$, $(ux)^{t_1} + yu = yu$. Then $(uc)^{m_1(t_1+1)} + x(ux)^{t_1}u = x(ux)^{t_1}u$ implies $(uc)^{m_2} + x_1u^2 = x_1u^2$, where $m_2 = m_1(t_1 + 1)$ and $x_1 = xy$. Also, there exist $s \in S$ and $t_2 \in \mathbb{N}$ such

that $(u^2x_1)^{t_2} + su^2 = su^2$. Iterating similarly we find that for every $r \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that $(uc)^p + x_ru^{2^r} = x_ru^{2^r}$. Let $r \in \mathbb{N}$ be such that $2^r > n$. Then there exists $q \in \mathbb{N}$ such that $(uc)^q + x_ru^{2^r} = x_ru^{2^r}$. By (3), there exist $t \in \mathbb{N}$ and $z \in S$ such that $(x_ru^{2^r})^t + u^{2^r}z = u^{2^r}z$. Then we have $(uc)^{q(t+1)} + bu^{2^r-n}zx_ru^{2^r-n}b = bu^{2^r-n}zx_ru^{2^r-n}b$. Hence $uc \in \sqrt{B}$. Similarly, $cu \in \sqrt{B}$. Again we have $(u+b)^n + sbs + sb + bs = u^n + sbs + sb + bs$, for some $s \in S$, i.e., $(u+b)^n + b + sbs + sb + bs = b + sbs + sb + bs$. Then for $w = (u+b)s + s(u+b) + u + b$, we have $(u+b)^{n+2} + bw = bw$. Also, there are $m \in \mathbb{N}$ and $y \in S$ such that $(bw)^m + ywb = ywb$. Then we have $(u+b)^{(n+2)(m+1)} + bwbywb = bwbywb$. Again there exist $p \in \mathbb{N}$ and $z \in S$ such that $(bwbywb)^p + bz = bz$. Then we get $(u+b)^{(n+2)(m+1)(p+1)} + bzwbwywb = bzwbwywb$. This implies $(u+b) \in \sqrt{B_k(b)} \subseteq \sqrt{B}$. Again for some $t \in S$ we have $(u+v)^n + tut + tu + ut = v^n + tut + tu + ut$ which implies that $(u+v)^n + t(u+b)t + t(u+b) + (u+b)t + (u+b) = t(u+b)t + t(u+b) + (u+b)t + (u+b)$ which again implies that $(u+v)^{n+2} + s(u+b)s = s(u+b)s$, where $s = (u+v)t + t(u+v) + u + v$. Now $s(u+b)s \in \sqrt{B}$ shows that there are $m \in \mathbb{N}$ and $b \in B$ such that $(s(u+b)s)^m + b = b$ and hence $(u+v)^{(n+2)m} + b = b$. Thus $(u+v) \in \sqrt{B}$, and so \sqrt{B} is an ideal of S . Let $s \in S$, $a \in \sqrt{B}$ such that $s + a = a$. Then there is $b \in B$ and $n \in \mathbb{N}$ such that $a^n + b = b$. Now $s^n + a^n = a^n$ gives $s^n + b = b$ which yields $s \in \sqrt{B}$. Thus \sqrt{B} is a k -ideal of S .

(4) \Rightarrow (5) : Obvious.

(5) \Rightarrow (3) : Let $a, b \in S$. Then \sqrt{aSa} and \sqrt{bSb} are k -ideals of S . Then $ab \in \sqrt{aSa} \cap \sqrt{bSb}$ and hence $ab \in \sqrt{Sa} \cap \sqrt{bS}$.

(3) \Rightarrow (6) : Let $a \in S$ and $F = \{x \in S \mid a \in \sqrt{Sx} \cap \sqrt{xS}\}$ and $y, z \in F$. Then there exist $n \in \mathbb{N}$ and $u \in S$ such that $a^n + uy = uy$, $a^n + uz = uz$ and $a^n + yu = yu$, $a^n + zu = zu$ so that $a^n + u(y+z) = u(y+z)$, and $a^n + (y+z)u = (y+z)u$, which imply $y+z \in F$. By (3), there exist $m \in \mathbb{N}$ and $v \in S$ such that $(yu)^m + vy = vy$ and $(uz)^m + zv = zv$. Now we can write $a^{n(m+1)} + u(yu)^mz = u(yu)^mz$ and $a^{n(m+1)} + y(uz)^mu = y(uz)^mu$, which give $a^{n(m+1)} + uv(yz) = uv(yz)$ and $a^{n(m+1)} + (yz)vu = (yz)vu$ so that $yz \in F$. Thus F is a subsemiring of S . Let $y \in F$ and $c \in S$ such that $y + c = c$. Then there exist $n \in \mathbb{N}$ and $u \in S$ such that $a^n + uy = uy$ and $a^n + yu = yu$, which imply $a^n + uc = uc$ and $a^n + cu = cu$ so that $c \in F$.

Now let $y, z \in S$ such that $yz \in F$. Then there exist $n \in \mathbb{N}$ and $u \in S$ such that $a^n + (uy)z = (uy)z$ and $a^n + y(zu) = y(zu)$. By (3), $(yz)^m + sy = sy$ and $(yz)^m + zs = zs$, for some $m \in \mathbb{N}$ and $s \in S$. Since F is a subsemiring, $(yz)^m \in F$. Then there exist $r \in \mathbb{N}$ and $v \in S$ such that $a^r + v(yz)^m = v(yz)^m$, and $a^r + (yz)^mv = (yz)^mv$. This implies $a^r + (vs)y = (vs)y$, and $a^r + z(sv) = z(sv)$. Hence $y, z \in F$. Thus F is a filter of S containing a . Let T be a filter of S containing a and let $y \in F$. Then $a^n + sy = sy$, for some $n \in \mathbb{N}$ and $s \in S$. Now since T is a filter, $a \in T$ implies $a^n \in T$ and so $a^n + sy = sy$ implies that $y \in T$.

Thus $F = N(a)$. It also follows directly that $\{x \in S \mid a \in \sqrt{Sx} \cap \sqrt{xS}\} = \{x \in S \mid a \in \sqrt{xSx}\}$.

(6) \Rightarrow (3) : Let $a, b \in S$. Then $a, b \in N(ab)$, since $N(ab)$ is a filter of S . So by (6), $ab \in \sqrt{Sa} \cap \sqrt{bS}$. \blacksquare

4. CHAINS OF t - k -ARCHIMEDEAN SEMIRINGS

In this section we characterize the semirings which are chains of t - k -Archimedean semirings. Let $(T, +, \cdot)$ be a distributive lattice with the partial order defined by $a \leq b \Leftrightarrow a + b = b$ for all $a, b \in S$. It is well known that (T, \leq) is a chain if and only if $ab = b$ or $ab = a$ for all $a, b \in T$.

Theorem 6. *The following conditions on a semiring S in \mathbb{SL}^+ are equivalent:*

1. S is a chain of t - k -Archimedean semirings;
2. S is a distributive lattice of t - k -Archimedean semirings and for all $a, b \in S$,

$$b \in \sqrt{aSa} \text{ or } a \in \sqrt{bSb};$$

3. For all $a, b \in S$, $N(a) = \{x \in S \mid a \in \sqrt{xSx}\}$ and $N(ab) = N(a) \cup N(b)$;
4. $\eta = \mathcal{N}$ is the least chain congruence on S such that each of its congruence classes is t - k -Archimedean.

Proof. (1) \Rightarrow (2) : Let S be a chain \mathcal{C} of t - k -Archimedean semirings $S_\alpha (\alpha \in \mathcal{C})$. Then S is a distributive lattice of t - k -Archimedean semirings too. Let $a, b \in S$. Then there exist $\alpha, \beta \in \mathcal{C}$ such that $a \in S_\alpha, b \in S_\beta$. Since \mathcal{C} is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$ then $a, ab \in S_\alpha$. Since S_α is t - k -Archimedean, there exist $n \in \mathbb{N}$ and $x \in S_\alpha$ such that $a^n + abxab = abxab$. Since S is a distributive lattice of t - k -Archimedean semirings, there exist $m \in \mathbb{N}$ and $y \in S$ such that $(abxab)^m + bxaby = bxaby$, by Theorem 5. Then we have $a^{n(m+1)} + bxabyabxab = bxabyabxab$, i.e., $a \in \sqrt{bSb}$. If $\alpha\beta = \beta$, then $b, ab \in S_\beta$. Similarly, proceeding as above we have $b \in \sqrt{aSa}$.

(2) \Rightarrow (3) : Since S is a distributive lattice of t - k -Archimedean semirings, by Theorem 5, $N(a) = \{x \in S \mid a \in \sqrt{xSx}\}$. Let $a, b \in S$. Then $ab \in N(ab) \Rightarrow a \in N(ab)$ and $b \in N(ab)$. Then $N(a) \subseteq N(ab)$ and $N(b) \subseteq N(ab) \Rightarrow N(a) \cup N(b) \subseteq N(ab)$. By hypothesis, either $a \in \sqrt{bSb}$ or $b \in \sqrt{aSa}$. If $a \in \sqrt{bSb}$, then there exist $m \in \mathbb{N}$ and $x \in S$ such that $a^m + bxb = bxb$. Now $a^{m+1} + abxb = abxb$. Since S is a distributive lattice of t - k -Archimedean semirings, by Theorem 5, there exist $n \in \mathbb{N}$ and $y \in S$ such that $(babx)^n + yba = yba$. Then $a^{m+1} + abxb = abxb$ implies $a^{(m+1)(n+1)} + abxybab = abxybab$ so that $a \in \sqrt{abSab}$, i.e. $ab \in N(a)$, and so $N(ab) \subseteq N(a)$. If $b \in \sqrt{aSa}$, then similarly we have $N(ab) \subseteq N(b)$. Thus $N(ab) \subseteq N(a) \cup N(b)$ and hence $N(ab) = N(a) \cup N(b)$.

(3) \Rightarrow (4) : By Theorem 5, S satisfies $ab \in \sqrt{Sa} \cap \sqrt{bS}$. Then by Lemma 3, the least distributive lattice congruence η on S is given by: for all $a, b \in S$, $a\eta b \Leftrightarrow a \in \sqrt{bSb}$ and $b \in \sqrt{aSa}$. Let $a, b \in S$ be such that $a\eta b$. Then $a \in \sqrt{bSb}$ and $b \in \sqrt{aSa}$ which, by Lemma 3, implies that $\sqrt{aSa} = \sqrt{bSb}$. Then

$$\begin{aligned}
 x \in N(a) &\Leftrightarrow a \in \sqrt{xSx} \\
 &\Leftrightarrow \sqrt{aSa} \subseteq \sqrt{xSx}, \text{ by Lemma 3 and Theorem 5} \\
 &\Leftrightarrow \sqrt{bSb} \subseteq \sqrt{xSx} \\
 &\Leftrightarrow b \in \sqrt{xSx} \\
 &\Leftrightarrow x \in N(b)
 \end{aligned}$$

shows that $N(a) = N(b)$ and so $a\mathcal{N}b$. Again for $a, b \in S$, $a\mathcal{N}b$ implies that $N(a) = N(b)$. Then $x \in \sqrt{aSa} \Leftrightarrow a \in N(x) \Leftrightarrow N(a) \subseteq N(x) \Leftrightarrow N(b) \subseteq N(x) \Leftrightarrow b \in N(x) \Leftrightarrow x \in \sqrt{bSb}$ shows that $\sqrt{aSa} = \sqrt{bSb}$. Thus $a\eta b$. Hence $\eta = \mathcal{N}$. Let $a, b \in S$. Then $ab \in N(a)$ or $ab \in N(b)$ which implies $N(ab) \subseteq N(a) \subseteq N(ab)$ or $N(ab) \subseteq N(b) \subseteq N(ab)$, i.e. $ab\mathcal{N}a$ or $ab\mathcal{N}b$, and thus \mathcal{N} is a chain congruence. Also by Theorem 5, each η -class is a t - k -Archimedean semiring.

(4) \Rightarrow (1) : The proof is obvious. \blacksquare

Theorem 7. *The following conditions on a semiring S in \mathbb{SL}^+ are equivalent:*

1. S is a chain of t - k -Archimedean semirings;
2. \sqrt{B} is a completely prime k -ideal of S for every k -bi-ideal B of S ;
3. $\sqrt{B_k(a)} = \sqrt{aSa}$ is a completely prime k -ideal of S for every $a \in S$;
4. $\sqrt{B_k(ab)} = \sqrt{B_k(a)} \cap \sqrt{B_k(b)}$ for all $a, b \in S$ and every k -bi-ideal of S is semiprimary.

Proof. (1) \Rightarrow (2) : Let S be a chain \mathcal{C} of t - k -Archimedean semirings $\{S_\alpha \mid \alpha \in \mathcal{C}\}$. Consider a k -bi-ideal B of S . Then \sqrt{B} is a k -ideal of S , by Theorem 5. Let $x, y \in S$ such that $xy \in \sqrt{B}$. Then there exists $m \in \mathbb{N}$ such that $u = (xy)^m \in \overline{B} = B$. Suppose $\alpha, \beta \in \mathcal{C}$ be such that $x \in S_\alpha$ and $y \in S_\beta$. Since \mathcal{C} is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $x, u \in S_\alpha$. Then $x \in \sqrt{uSu} \subseteq \sqrt{B}$, and so $x \in \sqrt{B}$. If $\alpha\beta = \beta$, then similarly we have $y \in \sqrt{B}$.

(2) \Rightarrow (3) : Obvious.

(3) \Rightarrow (4) : Let $a, b \in S$. Then $\sqrt{B_k(a)}$, $\sqrt{B_k(b)}$ and $\sqrt{B_k(ab)}$ are completely prime k -ideals of S . Let $x \in \sqrt{B_k(ab)}$. Then there exist $m \in \mathbb{N}$ and $u \in S$ such that $x^m + abuab = abuab$. Again there exist $n \in \mathbb{N}$ and $v \in S$ such that $(abuab)^n + va = va$, by Theorem 5. Then we have $x^{m(n+1)} + abuabva = abuabva$, whence $x \in \sqrt{B_k(a)}$. Therefore $\sqrt{B_k(ab)} \subseteq \sqrt{B_k(a)}$. Similarly, $\sqrt{B_k(ab)} \subseteq \sqrt{B_k(b)}$. Thus $\sqrt{B_k(ab)} \subseteq \sqrt{B_k(a)} \cap \sqrt{B_k(b)}$. Let $y \in \sqrt{B_k(a)} \cap \sqrt{B_k(b)}$. Then there exist $n \in \mathbb{N}$ and $s \in S$ such that $y^n + asa = asa$ and $y^n + bsb = bsb$. Again there exist $m \in \mathbb{N}$ and $u \in S$ such that $(asabsb)^m + uasab = uasab$, and we get $y^{2nm} + uasab = uasab$. Again there exist $p \in \mathbb{N}$ and $v \in S$ such that $(uasab)^p + av = av$. Then we have $y^{2nm(p+1)} + avuasab = avuasab$, and so $y \in \sqrt{B_k(ab)}$. Thus $\sqrt{B_k(ab)} = \sqrt{B_k(a)} \cap \sqrt{B_k(b)}$. Let B be a k -bi-ideal of S and $a, b \in S$ be such that $ab \in B$. Then $ab \in \sqrt{B_k(ab)}$ implies that $a \in \sqrt{B_k(ab)}$ or $b \in \sqrt{B_k(ab)}$. Thus $a^n \in B_k(ab) \subseteq B$ or $b^n \in B_k(ab) \subseteq B$, for some $n \in \mathbb{N}$, i.e., $a^n \in B$ or $b^n \in B$ and hence B is semiprimary.

(4) \Rightarrow (1) : Let $a, b \in S$. Then $ab \in \sqrt{B_k(a)} \cap \sqrt{B_k(b)}$. Then there are $m, n \in \mathbb{N}$ and $s \in S$ such that $(ab)^n + asa = asa$ and $(ab)^m + bsb = bsb$, i.e., $ab \in \sqrt{Sa} \cap \sqrt{bS}$. Then by Lemma 3 and Theorem 5, the least distributive lattice congruence η on S is given by : for $a, b \in S$, $a\eta b \Leftrightarrow \sqrt{aSa} = \sqrt{bSb} \Leftrightarrow \sqrt{B_k(a)} =$

$\sqrt{B_k(b)}$, and each η -class is a t - k -Archimedean semiring. Now there exists $n \in \mathbb{N}$ such that $a^n \in B_k(ab)$ or $b^n \in B_k(ab)$. Then $\sqrt{B_k(ab)} \subseteq \sqrt{B_k(a)} \subseteq \sqrt{B_k(ab)}$ or $\sqrt{B_k(ab)} \subseteq \sqrt{B_k(b)} \subseteq \sqrt{B_k(ab)}$, i.e., $\sqrt{B_k(ab)} = \sqrt{B_k(a)}$ or $\sqrt{B_k(ab)} = \sqrt{B_k(b)}$. Hence $ab\eta a$ or $ab\eta b$. Thus η is a chain congruence and so S is a chain of t - k -Archimedean semirings. ■

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