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WEAK COMPACTNESS IN THE SPACE OF OPERATOR VALUED MEASURES $M_{ba}(\Sigma, \mathcal{L}(X,Y))$ AND ITS APPLICATIONS

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Abstract

In this note we present necessary and sufficient conditions characterizing conditionally weakly compact sets in the space of (bounded linear) operator valued measures $M_{ba}(\Sigma, \mathcal{L}(X,Y))$. This generalizes a recent result of the author characterizing conditionally weakly compact subsets of the space of nuclear operator valued measures $M_{ba}(\Sigma, \mathcal{L}_1(X,Y))$. This result has interesting applications in optimization and control theory as illustrated by several examples.

Keywords: space of operator valued measures, weak compactness, semigroups of bounded linear operators, optimal structural control.

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1. Introduction

Necessary and sufficient conditions for weak compactness in the space of vector measures has been a subject of great interest over half a century. One of the seminal results in this topic is the well known Bartle-Dunford-Schwartz theorem [1, Theorem 5, p. 105] for countably additive bounded vector measures with values in Banach spaces satisfying, along with their duals, the Radon-Nikodym property. This result was extended to finitely additive vector measures by Brooks [3] and Brooks and Dinculeanu [1, Corollary 6, 106]. Weak sequential compactness for regular vector measures have been studied by Kuo [4, Theorem 1.6, Theorem 3.3] where he gives several results on weak sequential compactness based on set-wise weak convergence.

In physical sciences and engineering involving control theory and optimization one has the freedom to choose from a given class of operator valued measures or functions the best one that minimizes certain functionals. This is where compactness is useful. These problems arise naturally in the area of system identification, Kalman Filtering, Structural control and optimization etc [8, 9, 10].

The rest of the paper is organized as follows: In Section 2, basic notation are introduced. In Section 3, the main result of this paper is presented with complete proof. In Section 4, the result is applied to a class of structurally controlled semilinear evolution equations on Banach spaces.

2. Some notation

Let D be a compact Hausdorff space and $\Sigma \equiv \sigma(D)$ the sigma algebra of subsets of the set D. For any Banach space Z and any set function $\mu : \Sigma \longrightarrow Z$, we let $|\mu|(K)$ denote the variation of μ over the set $K \in \Sigma$ and $v(\mu)(\cdot) \equiv |\mu|(\cdot)$ the positive measure induced by the variation. Let $M_{ba}(\Sigma, Z)$ denote the space of finitely additive Z valued vector measures having bounded total variation. This is a Banach space with respect to the norm topology induced by the variation $\| \mu \|_v \equiv \| \mu \| \equiv |\mu|(D)$. The semivariation of μ is given by

$$\| \mu \|_{sv} \equiv \sup \{ |z^* \mu|(D), \, z^* \in B_1(Z^*) \}$$

where $B_1(Z^*)$ denotes the closed unit ball in Z^* centered at the origin. Clearly, it follows from the definition that $\| \mu \|_{sv} \leq \| \mu \|_v$. The class of countably additive Z valued vector measures denoted by $M_{ca}(\Sigma, Z)$ and equipped with the total variation norm is a closed subspace of $M_{ba}(\Sigma, Z)$ and hence it is also a Banach space.

Consider any pair of real Banach spaces $\{X, Y\}$ and let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators from X to Y. Furnished with the uniform operator topology, this is a Banach space. Let $\mathcal{L}_s(X, Y) \equiv (\mathcal{L}(X, Y), \tau_{so})$ denote the space $\mathcal{L}(X, Y)$ endowed with the strong operator topology and $\mathcal{L}_w(X, Y) \equiv (\mathcal{L}(X, Y), \tau_{wo})$ the same furnished with weak operator topology. It is well known that these are all locally convex sequentially complete linear topological vector spaces. By $\mathcal{L}_1(X, Y)$ we denote the space of nuclear operators from X to Y. This class of operators are used in

Section 4 to illustrate some applications of the compactness results proved in this paper. Further notation will be introduced as and when necessary.

Let $M_{ba}(\Sigma, \mathcal{L}(X, Y))$ denote the space of all finitely additive operator valued measures defined on Σ and taking values from the space of bounded linear operators from X to Y. Additivity and variation of elements of $M_{ba}(\Sigma, \mathcal{L}(X, Y))$ depend on the particular topology used for the space $\mathcal{L}(X, Y)$. Let $\{\sigma_i\}$ be a pairwise disjoint family of Σ-measurable subsets of the set D. An element $B \in M_{ba}(\Sigma, \mathcal{L}(X, Y))$ is said to be countably additive in the uniform operator topology if for every such family

$$(1): \qquad \lim_{n\to\infty} \left\| B\left(\bigcup \sigma_i\right) - \sum_{i=1}^{n} B(\sigma_i) \right\|_{\mathcal{L}(X,Y)} = 0,$$

and it is said to be countably additive in the strong operator topology if for every $x \in X$,

$$(2): \qquad \lim_{n\to\infty} \left| B\left(\bigcup \sigma_i\right)x - \sum_{i=1}^{n} B(\sigma_i)x \right|_Y = 0.$$

Similarly one can define countable additivity in the weak operator topology and verify that countable additivity in the strong and weak operator topologies are equivalent.

With respect to the uniform operator topology, the variation of $M \in M_{ba}(\Sigma, \mathcal{L}(X, Y))$ on any set $J \in \Sigma$ is given by

$$|M|_u(J) \equiv \sup_{\pi} \sum_{\sigma\in\pi} \| M(\sigma) \|_{\mathcal{L}(X,Y)} \tag{1}$$

where π denotes any finite, disjoint, Σ measurable partition of the set J and the supremum is taken over all such partitions. It is well known [1, 2] that, furnished with the total variation norm, $M_{ba}(\Sigma, \mathcal{L}(X, Y))$ is also a Banach space. The class of bounded linear operator valued measures, countably additive in the uniform operator topology, is denoted by $M_{ca}(\Sigma, \mathcal{L}(X, Y))$. It is a closed subspace of $M_{ba}(\Sigma, \mathcal{L}(X, Y))$ and hence a Banach space. For details on vector measures, see the well known books of Diestel and Uhl [1] and Dunford and Schwartz [2].

In general, for an operator valued measure $M : \Sigma \longrightarrow \mathcal{L}(X, Y)$, the total variation norm based on the uniform operator topology is rather strong for some applications. Using other types of variations derived from weaker topologies, such as strong or weak operator topologies, one has a broader class of operator valued measures. Let $B_1(Z)$ denote the closed unit ball in any Banach space Z. With respect to the strong operator topology, we define the variations of M on the set $J \in \Sigma$ by

$$(2) \quad |M|_s(J) \equiv \sup\Big\{ \Big\| \sum_{i=1}^{n} M(J \cap \sigma_i) x_i \Big\|_Y, x_i \in B_1(X), 1 \leq i \leq n, n \in N \Big\},$$

while for the weak operator topology it is given by

$$(3) \quad \begin{aligned} |M|_w(J) \equiv \sup\Big\{ \Big| \sum_{i=1}^{n} (y^*, M(J \cap \sigma_i) x_i) \Big|,\ x_i \in B_1(X), \\ y^* \in B_1(Y^*),\ 1 \leq i \leq n,\ n \in N \Big\} \end{aligned}$$

where $\{\sigma_i, 1 \leq i \leq n\}$ is a finite family of Σ measurable disjoint partitions of D and the supremum in equation (2), as well as (3), is taken over all such finite partitions and $\{x_i\} \in B_1(X)$ and $y^* \in B_1(Y^*)$. The strong and weak variations of the operator valued measure M on D are then given by

$$|M|_s \equiv \sup\{|M|_s(\sigma), \sigma \in \Sigma\} \quad \text{and} \quad |M|_w \equiv \sup\{|M|_w(\sigma), \sigma \in \Sigma\}$$

respectively. Clearly, $|M|_w \leq |M|_s \leq |M|_u$ and one can easily construct examples of operator valued measures having finite weak or strong variation while the uniform variation is infinity. As seen later, the variation with respect to strong operator topology is equivalent to semivariation.

3. MAIN RESULT

Our main concern is to find necessary and sufficient conditions under which a set $\Gamma \subset M_{ba}(\Sigma, \mathcal{L}(X, Y))$ is weakly conditionally compact. Let $M_{casbsv}(\Sigma, \mathcal{L}(X, Y)) \subset M_{ba}(\Sigma, \mathcal{L}(X, Y))$ denote the class of operator valued measures which are countably additive in the strong operator topology having bounded semivariation. Let $B_\infty(D, X)$ denote the vector space of bounded Σ-measurable functions with values in X which are uniform limits of Σ-measurable simple functions. Furnished with the sup norm topology, this is a Banach space. For any $f \in B_\infty(D, X)$ and $T \in M_{casbsv}(\Sigma, \mathcal{L}(X, Y))$, define the set function

$$(Tf)(\sigma) \equiv \int_\sigma T(ds) f(s), \sigma \in \Sigma,$$

where the integral is understood in the sense of Dobrakov [5, 6]. Thus, for each $f \in B_\infty(D, X)$, the set

$$\Gamma(f) \equiv \{\mu : \mu(\cdot) = (Tf)(\cdot),\ T \in \Gamma\}$$

is a well defined family of vector measures contained in $M_{ca}(\Sigma, Y)$.

Let $\mathcal{S}(D, X)$ denote the class of Σ measurable simple functions with values in X. The semivariation of the operator valued measure T on $\sigma \in \Sigma$ is defined by the extended real valued subadditive set function $\hat{T}(\cdot)$ given by

$$\hat{T}(\sigma) \equiv \sup\left\{\Big\| \int_\sigma T(ds)f(s)\Big\|_Y : f \in \mathcal{S}(D, X), \| f \|_{B_\infty(D,X)} \leq 1\right\}, \tag{4}$$

and the semivariation of T is given by $\hat{T}(D) \equiv \sup\{\hat{T}(\sigma), \sigma \in \Sigma\}$. Note that this semivariation is precisely the variation in the strong operator topology as defined by the expression (2). Thus $\hat{T}(D) = |T|_s$, and it follows from (4) that

$$\| Tf \|_Y \equiv \Big\| \int_D T(ds)f(s)\Big\|_Y \leq |T|_s \| f \|_{B_\infty(D,X)} .$$

The set function induced by the semivariation of an element $T \in M_{ba}(\Sigma, \mathcal{L}(X, Y))$ is generally finitely sub additive unlike the variation which is additive. For $T \in M_{casbsv}(\Sigma, \mathcal{L}(X, Y))$, its semivariation is countably sub additive unlike the variation which is countably additive.

We are interested in the characterization of weakly compact sets in $M_{ba}(\Sigma, \mathcal{L}(X, Y))$. We are able to do so only for weakly compact sets in the class $M_{casbsv}(\Sigma, \mathcal{L}(X, Y))$.

Let us recall few well known facts. A compact Hausdorff space is said to be an $\mathcal{F}$-space, if every pair of disjoint open F_σ sets has disjoint closures. A Banach space Ξ is said to be a Grothendieck space if weak star convergence in its dual Ξ^* is equivalent to weak convergence. Clearly, Grothendieck spaces contain reflexive Banach spaces. For other characterizations of Grothendieck spaces see [1, p. 179].

Our main result is stated as follows.

Theorem 1. *Let D be a compact Hausdorff $\mathcal{F}$-space and $\{X, Y\}$ a pair of Banach spaces with Y being reflexive and consider the space of operator valued measures $M_{ba}(\Sigma, \mathcal{L}(X, Y))$. A set $\Gamma \subset M_{casbsv}(\Sigma, \mathcal{L}(X, Y)) \subset$*

$M_{ba}(\Sigma, \mathcal{L}(X,Y))$ *is conditionally weakly compact if, and only if, the following conditions hold:*

(i) Γ *is bounded, that is,* $\sup\{\hat{T}(D) \equiv |T|_s, T \in \Gamma\} < \infty$.

(ii) *for each* $f \in B_\infty(D,X)$, *the set of measures* $\{v(\mu)(\cdot) \equiv |\mu|(\cdot) : \mu \in \Gamma(f)\}$ *is uniformly countably additive.*

Proof. Given that Γ is bounded, it is clear that for each $f \in B_\infty(D,X)$ (ia): the set $K^f \equiv \Gamma(f)$ is a bounded subset of $M_{ca}(\Sigma, Y)$. Further, assumption (ii) is the same as (iia): for each $f \in B_\infty(D,X)$, K^f is uniformly countably additive. By our assumption, Y is reflexive. Thus, it follows from a well known result of Brooks [3, Main Theorem, Corollary 1, p. 284] that the conditions (ia) and (iia) are necessary and sufficient for K^f to be conditionally weakly compact. Hence, it suffices to verify that conditional weak compactness of K^f for each $f \in B_\infty(D,X)$ implies conditional weak compactness of Γ and conversely. By conditional weak compactness we mean $\bar{\Gamma} \equiv wc\ell(\Gamma)$ is weakly compact in $M_{casbsv}(\Sigma, \mathcal{L}(X,Y))$. First we prove that conditional weak compactness of $K^f \equiv \Gamma(f)$, for each $f \in B_\infty(D,X)$, implies conditional weak compactness of Γ. Since D is an $\mathcal{F}$ space, and reflexivity of Y implies that Y^* is also reflexive, we conclude that $C(D, Y^*)$ is a Grothendieck space [4, Theorem 2.2]. In other words, w^* convergence in its dual $M_{ca}(\Sigma, Y)$ is equivalent to weak convergence. Thus every sequence $\mu_n^f \in K^f$, for any $f \in B_\infty(D,X)$, has a subsequence, relabeled as the original sequence, which is weakly convergent to a unique element $\mu^f \in M_{ca}(\Sigma, Y)$. Therefore, for every $g \in B_\infty(D, Y^*)$, we have

$$\int_D < g(s), \mu_n^f(ds) > \longrightarrow \int_D < g(s), \mu^f(ds) >, \tag{5}$$

where $< \cdot >$ denotes the (Y^*, Y) pairing. Since $\mu_n^f \in K^f$, by definition there exists a sequence $T_n \in \Gamma$ so that

$$\mu_n^f(\sigma) = \int_\sigma T_n(ds) f(s)$$

for every $\sigma \in \Sigma$. The limit expressed by (5) holds for every choice of $f \in B_\infty(D,X)$ and $g \in B_\infty(D, Y^*)$. Thus for every $\sigma \in \Sigma$, and for $f(s) = \chi_\sigma(s)x, x \in X$, and $g(s) = \chi_\sigma(s)y^*, y^* \in Y^*$, with χ_σ denoting the characteristic function of the set $\sigma \in \Sigma$, (with slight abuse of notation) it follows from (5) and the above expression that

$$< y^*, T_n(\sigma)x > \equiv < y^*, \mu_n^x(\sigma) > \longrightarrow < y^*, \mu^x(\sigma) > . \tag{6}$$

Since weak convergence implies norm boundedness, $\sup_{n\in N} \| \mu_n^x(\sigma) \|_Y < \infty$ for each $\sigma \in \Sigma$ and $x \in X$, and hence, by Nikodym uniform boundedness theorem [1, Theorem 1, p. 14], $\{\mu_n^x, n \in N\}$ is uniformly bounded, that is,

$$\sup_{n\in N} \| \mu_n^x \|_{sv} (D) < \infty.$$

Hence, it follows from [1, Proposition 11, p. 4] that the range of these measures $\{\mu_n^x\}$ is bounded. Thus, for each $x \in X$, there exists a finite positive number C_x such that

$$\sup\{\| \mu_n^x(\sigma) \|_Y = \| T_n(\sigma)x \|_Y, \sigma \in \Sigma, n \in N\} \leq C_x < \infty.$$

Now, by virtue of the well known uniform boundedness principle for linear operators, we conclude from this that there exists a finite positive number b such that

$$\sup_n \sup_{\sigma\in\Sigma} \| T_n(\sigma) \|_{\mathcal{L}(X,Y)} \leq b.$$

This holds for any sequence $T_n \in \Gamma$ and hence for any $T \in \Gamma$. This means that, for sufficiently large $b > 0$, we have $\{T(\sigma), \sigma \in \Sigma, T \in \Gamma\} \subset B_b(\mathcal{L}(X,Y))$, the closed ball of radius b in $\mathcal{L}(X,Y)$ centered at the origin. Since Y is a reflexive Banach space, a closed bounded convex subset of $\mathcal{L}(X,Y)$ is compact in the weak operator topology. Thus, for every $\sigma \in \Sigma$, there exists a $T_o(\sigma) \in B_b(\mathcal{L}(X,Y))$, such that, along a generalized subsequence (sub net) if necessary,

$$T_n(\sigma) \xrightarrow{\tau_{wo}} T_o(\sigma). \tag{7}$$

In other words, for arbitrary $x \in X$ and $y^* \in Y^*$, and $\sigma \in \Sigma$, we have

$$< y^*, T_n(\sigma)x > \longrightarrow < y^*, T_o(\sigma)x > . \tag{8}$$

Thus it follows from (6) and (8) that $\mu^x(\sigma) = T_o(\sigma)x$. Since the choice of $f \in B_\infty(D,X)$ and $g \in B_\infty(D,Y^*)$ is arbitrary, this is true for every $x \in X$, $y^* \in Y^*$ and every $\sigma \in \Sigma$. This way we obtain a set function, again denoted by $T_o : \Sigma \longrightarrow \mathcal{L}(X,Y)$, which satisfies the range condition, $T_o(\varpi) \in B_b(\mathcal{L}(X,Y))$ for all $\varpi \in \Sigma$. Then, in view of (5) and (8) we may conclude that, along a generalized subsequence if necessary, $T_n \xrightarrow{w} T_o$ in $M_{ba}(\Sigma, \mathcal{L}(X,Y))$. We must show that $T_o \in \bar{\Gamma}$. In order to show this, it is only necessary to prove that $T_o \in M_{casbsv}(\Sigma, \mathcal{L}(X,Y))$. That is, T_o has bounded semivariation and that it is countably additive in the strong operator topology. First we verify that T_o has bounded semivariation. For any given $f \in \mathcal{S}(D,X) \cap B_\infty(D,X)$, it is clear that $\int_D T_o(ds)f(s)$ is a well defined element of Y. Hence, as a consequence of Hahn-Banach theorem, for every such f, there exists a $y_f^* \in \partial B_1(Y^*)$ such that

$$
\begin{aligned}
\Big\| \int_D T_o(ds)f(s) \Big\| &= < y_f^*, \int_D T_o(ds)f(s) >_{Y^*,Y} \\
&= < y_f^*, \int_D \big(T_o(ds) - T_n(ds)\big) f(s) > + < y_f^*, \int_D T_n(ds)f(s) > .
\end{aligned} \tag{9}
$$

Since T_n converges weakly to T_o in $M_{ba}(\Sigma, \mathcal{L}(X,Y))$, it follows from (9) that, for every $\varepsilon > 0$, there exists a positive integer N_ε such that

$$
\Big\| \int_D T_o(ds)f(s) \Big\| \leq \varepsilon + | < y_f^*, \int_D T_n(ds)f(s) > | \leq \varepsilon + |T_n|_s \parallel f \parallel_\infty \tag{10}
$$

for all $n \geq N_\varepsilon$. By assumption (i), Γ is bounded and so there exists a finite positive number γ such that $\sup\{\hat{T} \equiv |T|_s, T \in \Gamma\} \leq \gamma$. Hence it follows from the above inequality that

$$
\Big\| \int_D T_o(ds)f(s) \Big\| \leq \varepsilon + \gamma \parallel f \parallel_\infty . \tag{11}
$$

Since $\varepsilon > 0$ is arbitrary and the above inequality holds for every $f \in \mathcal{S}(D,X)$, it follows from the definition of semivariation that T_o has finite semivariation. For countable additivity of T_o (in the strong operator topology), first note that, for every $x \in X$, it follows from (6) that μ^x is the weak limit of the sequence of countably additive measures $\{\mu_n^x\} \in M_{ca}(\Sigma, Y)$. Using this fact and countable additivity of T_n in the strong operator topology, it is easy to verify that μ^x is weakly countably additive and therefore by Pettis theorem [2, Theorem IV.10.1, p. 318], μ^x is countably additive. Hence for any sequence of disjoint Σ-measurable sets $\{\sigma_i\}$, with $\bigcup \sigma_i \in \Sigma$, we have

$$
T_o\Big(\bigcup \sigma_i\Big)x \equiv \mu^x\Big(\bigcup \sigma_i\Big) = \sum \mu^x(\sigma_i) \equiv \sum T_o(\sigma_i)x
$$

verifying countable additivity of T_o in the strong operator topology. Thus we have $T_o \in M_{casbsv}(\Sigma, \mathcal{L}(X,Y))$, and it is evident that $T_o \in \bar{\Gamma}$. That the limit is unique follows from the fact that, for each $f \in B_\infty(D,X)$, μ^f given by (5) is unique. Thus, we have proved that conditional weak compactness of K^f in $M_{ca}(\Sigma, Y)$, for each $f \in B_\infty(D,X)$, implies conditional weak compactness of Γ in $M_{casbsv}(\Sigma, \mathcal{L}(X,Y)) \subset M_{ba}(\Sigma, \mathcal{L}(X,Y))$. In other words, if conditions (i) and (ii) hold, then Γ is conditionally weakly compact. The reverse implication is transparent. Indeed, if Γ is a conditionally weakly compact subset of $M_{casbsv}(D, \mathcal{L}(X,Y))$, it is necessarily bounded, that is, $\sup\{\hat{T} \equiv |T|_s, T \in \Gamma\} < \infty$. Thus (i) holds. Clearly, conditional weak compactness of Γ implies that for each $f \in B_\infty(D,X)$ the set $\Gamma(f) \equiv K^f$ is a conditionally weakly compact subset of $M_{ca}(\Sigma, Y)$. Thus it follows from Brook's theorem once again that condition (iia) or equivalently condition (ii) is necessary. Hence conditions (i) and (ii) are necessary. This completes the proof. ∎

Our main result is based on the weak compactness result of Brooks [3] in the space $M_{ba}(\Sigma, Y)$. The original result was proved under the assumption that Y is reflexive. This result was later generalized by Brooks and Lewis [7, Theorem 3.1, p. 152] (see also [1, Corollary 6, p. 106]) to cover spaces Y which, along with their duals Y^*, satisfy Radon-Nikodym property (RNP). It may be tempting to use this general result and extend our main result (Theorem 1) to cover these cases. But, unfortunately this seems to be impossible without additional assumptions since the unit ball $B_1(\mathcal{L}(X,Y))$ is compact in the weak operator topology if, and only if, Y is reflexive [2]. This fact is crucial in our proof of Theorem 1.

Remark 2.

(1) It will be interesting to extend our result to the finitely additive case characterizing relatively weakly compact sets in $M_{ba}(\Sigma, \mathcal{L}(X,Y))$.

(2) Extend our result to cover cases where the unit ball $B_1(\mathcal{L}(X,Y))$ is not compact in the weak operator topology.

Remark 3. We observe that every $B \in M_{casbsv}(\Sigma, \mathcal{L}(X,Y))$ gives rise to a bounded linear operator, say L_B, from $B_\infty(D,X)$ to Y through the integral expression

$$L_B(f) \equiv \int_D B(ds) f(s),$$

where the integral is understood in the sense of Dobrakov [5, 6]. Thus we have

$$M_{casbsv}(\Sigma, \mathcal{L}(X,Y)) \hookrightarrow \mathcal{L}(B_\infty(D,X), Y).$$

The question is: does the reverse inclusion hold ? In other words, is every bounded linear operator $L \in \mathcal{L}(B_\infty(D,X),Y)$ has the integral representation through an operator valued measure from $M_{casbsv}(\Sigma,\mathcal{L}(X,Y))$. The answer seems to be unknown. However according to the following proposition the following isometric isomorphism

$$M_{fabsv}(\Sigma,\mathcal{L}(X,Y)) \cong \mathcal{L}(B_\infty(D,X),Y)$$

holds. Hence the answer to the previous question is in the negative since obviously

$$M_{casbsv}(\Sigma,\mathcal{L}(X,Y)) \subsetneq M_{fabsv}(\mathcal{L}(X,Y)).$$

Proposition 4. *Let D be a locally compact Hausdorff space and Σ an algebra of subsets of the set D, and $\{X,Y\}$ a pair of Banach spaces and $\mathcal{L}(X,Y)$ the space of bounded linear operators from X to Y. Then the following isometric isomorphism holds*

$$M_{fabsv}(\Sigma,\mathcal{L}(X,Y)) \cong \mathcal{L}(B_\infty(D,X),Y).$$

Proof. First we show that the inclusion $M_{fabsv}(\Sigma,\mathcal{L}(X,Y)) \hookrightarrow \mathcal{L}(B_\infty(D,X),Y)$ holds. Indeed, for any $B \in M_{fabsv}(\Sigma,\mathcal{L}(X,Y))$, it is easy to see that the operator

$$L_B(f) \equiv \int_D B(ds)f(s)$$

is a well defined bounded linear operator on $S_\infty(D,X) \subset B_\infty(D,X)$ to Y. Since B has bounded semivariation and $S_\infty(D,X)$ is dense in $B_\infty(D,X)$, the operator L_B admits a continuous extension to a bounded linear operator from $B_\infty(D,X)$ to Y which we denote by L_B again. This verifies the embedding as stated. Clearly

$$\| L_B \|_{\mathcal{L}(B_\infty(D,X),Y)} \leq |B|_s.$$

For the reverse inclusion, $\mathcal{L}(B_\infty(D,X),Y) \hookrightarrow M_{fabsv}(\Sigma,\mathcal{L}(X,Y))$, let $L \in \mathcal{L}(B_\infty(D,X),Y)$ and let $x \in X$ and χ_σ the characteristic function of $\sigma \in \Sigma$. Then define the operator valued set function B by

$$B(\sigma)x \equiv L(\chi_\sigma x), \sigma \in \Sigma, x \in X.$$

It is easy to see that B is finitely additive. Further, for any $f \in S_\infty(D,X)$ given by $f \equiv \sum \chi_{\sigma_i} x_i$ for $\sigma_i \in \Sigma$, $\sigma_i \cap \sigma_j = \emptyset$, for $i \neq j$ and $x_i \in X$, $n \in N$, we have

$$L(f) = L(\sum_{i=1}^{n} \chi_{\sigma_i} x_i) = \sum_{i=1}^{n} B(\sigma_i) x_i.$$

Since L is a bounded linear operator, it follows from the above expression that there exists a positive constant c such that

$$\left\| \sum B(\sigma_i) x_i \right\|_Y = \| Lf \|_Y \leq c \| f \|,$$

where $\| f \| = \sup\{|x_i|_X, 1 \leq i \leq n, n \in N\}$. This is true for all $f \in S_\infty(D,X)$ and hence by density it is true for all $f \in B_\infty(D,X)$. Thus it follows from the above inequality and the definition of semivariation that $\hat{B} \equiv |B|_s \leq c$. Choosing the smallest c, for which the above inequality holds, we have the isometry, that is

$$|B|_s = \| L \|_{\mathcal{L}(B_\infty(D,X),Y)} .$$

Thus we have proved that, corresponding to each $L \in \mathcal{L}(B_\infty(D,X),Y)$, there exists a $B \in M_{fabsv}(\Sigma, \mathcal{L}(X,Y))$ such that $L(f) = \int_D B(ds) f(s)$ for every $f \in B_\infty(D,X)$. The reader can easily verify that B is determined uniquely by L alone. This proves the isometric isomorphism as stated. ■

It is well known that any closed bounded convex subset $\mathcal{K} \subset \mathcal{L}(B_\infty(D,X),Y)$ is compact in the weak operator topology if and only if Y is reflexive. Thus, given that Y is reflexive, any net $\{L_\alpha\} \in \mathcal{K}$ has a subnet, relabeled as the original net, and an element $L_o \in \mathcal{K}$ such that $L_\alpha \xrightarrow{\tau_{wo}} L_o$. It follows from the above proposition that corresponding to the net $\{L_\alpha\} \subset \mathcal{K}$ there exists a net $B_\alpha \in M_{fabsv}(\Sigma, \mathcal{L}(X,Y))$ such that

$$L_\alpha(f) = \int_D B_\alpha(ds) f(s)$$

for every $f \in B_\infty(D,X)$; and corresponding to L_o there exists a $B_o \in M_{fabsv}(\Sigma, \mathcal{L}(X,Y))$ such that

$$L_o(f) = \int_D B_o(ds) f(s).$$

As seen above, for $B \in M_{fabsv}(\Sigma, \mathcal{L}(X,Y))$, the operator L_B given by $L_B(f) = \int_D B(ds)f(s)$ is a bounded linear operator from $B_\infty(D,X)$ to Y. Define the set

$$\Gamma \equiv \{B \in M_{fabsv}(\Sigma, \mathcal{L}(X,Y)) : L_B \in \mathcal{K}\}.$$

Since compactness is preserved under isomorphism, we can conclude that Γ is a weakly compact subset of $M_{fabsv}(\Sigma, \mathcal{L}(X,Y)$ if and only if $\mathcal{K}$ is a weakly compact subset of $\mathcal{L}(B_\infty(D,X),Y)$.

Remark 5. Given that Y is a reflexive Banach space, it is trivial but interesting to note that any operator $T \in \mathcal{L}(B_\infty(D,X),Y)$ is weakly compact.

4. Some applications

Example 1. Consider minimizing the functional

$$\Phi(B) \equiv G\Big(\int_D < g_1(s), B(ds)f_1(s) >, \cdots, \int_D < g_m(s), B(ds)f_m(s) > \Big)$$

where $f_i \in B_\infty(D,X)$, $g_i \in B_\infty(D,Y^*)$ and $G : R^m \longrightarrow R$ is a lower semicontinuous function and there exists an $r \in R$ such that $G(z) \geq r$ for all $z \in R^m$. Let $\Gamma \subset M_{casbsv}(D, \mathcal{L}(X,Y))$ weakly conditionally compact and also closed so weakly compact. Then $\Phi(B)$ attains its infimum on Γ.

Example 2. Consider the functional

$$\Phi(B) \equiv F\Big(\int_D Tr(L_1(t)B(dt)), \cdots, \int_D Tr(L_m(t)B(dt))\Big)$$

with $L_i \in B_\infty(D, \mathcal{L}_1(Y,X)), 1 \leq i \leq m$, fixed and $B \in M_{casbsv}(\Sigma, \mathcal{L}(X,Y))$. Using the representation of nuclear operators on Banach spaces, it is easy to verify that the map

$$B \longrightarrow \int_D Tr(L_i(t)B(dt))$$

is a weakly continuous linear functional on $M_{casbsv}(\Sigma, \mathcal{L}(X,Y))$. Lower semicontinuity of F on R^m implies w.l.s.c of Φ on $M_{casbsv}(\Sigma, \mathcal{L}(X,Y))$. Thus it follows from weak lower semicontinuity of Φ and weak compactness of Γ that Φ attains its infimum on Γ.

Example 3. Consider the semilinear system on the Banach space X

$$dx = Axdt + B(dt)y + f(x)dt, x(0) = \xi \tag{12}$$

$$y = Lx + \eta \ \ \text{(output)}, t \in I \equiv [0,T]. \tag{13}$$

The space X (state space) is a reflexive B-space and Y(output space) is any Banach space. The objective functional is given by

$$J(B) \equiv \int_0^T \ell(t, x(t))dt + |B|_s. \tag{14}$$

The admissible structural controls are the operator valued measures

$$\{B\} \in \Gamma \subset M_{casbsv}(\Sigma_I, \mathcal{L}(Y,X)).$$

The objective is to find a $B \in \Gamma$ that minimizes this functional.

Theorem 4. *Suppose $A \in \mathcal{G}_0(M, \omega)$ is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, compact for $t > 0$, Γ a weakly compact subset of $M_{casbsv}(\Sigma_I, \mathcal{L}(Y,X))$, f locally Lipschitz with at most linear growth, $L \in \mathcal{L}(X,Y)$, $\eta \in B_\infty(I,Y)$. There exists $\nu \in M^+_{cabv}(\Sigma_I)$ such that $|B|_s(\sigma) \leq \nu(\sigma)$ for $\sigma \in \Sigma_I$ uniformly w.r.t $B \in \Gamma$. The cost integrand ℓ is measurable in t and lower semicontinuous in x on X and there exists $\alpha \in L^+_1(I)$ and $\beta \geq 0$ satisfying*

$$|\ell(t,x)| \leq \alpha(t) + \beta |x|^p_X, \ \textit{for any } p \in (0, \infty).$$

Then, there exists a $B_o \in \Gamma$ at which J attains its minimum.

Proof. (Outline) For $\xi \in X$ fixed, and $B \in \Gamma$, let $x(B)(\cdot) \in B_\infty(I,X)$ denote the mild solution of the system (12)(13). Under the given assumptions, it is easy to verify that there exists a ball $B_r \subset X$ of radius $r \in (0,\infty)$, centered at the origin, such that $x(B)(t) \in B_r$ for all $t \in I$ and all $B \in \Gamma$. We show that $B \longrightarrow J(B)$ is weakly lower semicontinuous on $M_{casbsv}(\Sigma_I, \mathcal{L}(Y,X))$. Let $B_n \xrightarrow{w} B_o$ in $M_{casbsv}(\Sigma_I, \mathcal{L}(Y,X))$ and let $\{x_n, x_o\}$ denote the corresponding mild solutions of the feedback system

$$\begin{aligned} dx &= Axdt + B(dt)Lx + f(x)dt + B(dt)\eta(t), \\ x(0) &= \xi. \end{aligned} \tag{15}$$

Let λ denote the lebesgue measure on I and $\tilde{M} \equiv \sup\{\| S(t) \|, t \in I\}$, and define the measure

$$\mu(\sigma) \equiv \tilde{M}[K_r \lambda(\sigma) + \| L \|_{\mathcal{L}(X,Y)} \nu(\sigma)], \sigma \in \Sigma_I,$$

where K_r is the Lipschitz constant of f on $B_r \subset X$. This is a c.a. non-negative bounded measure on $\Sigma_I \equiv \sigma(I)$. Now using the integral equations associated with the pair $\{x_n, x_o\}$ and taking the difference and carrying out some elementary algebraic operations, it is easy to verify that for all $t \in I$,

$$|x_o(t) - x_n(t)|_X \leq |e_n(t)|_X + \int_0^t |x_o(s) - x_n(s)|_X \mu(ds), \tag{16}$$

where

$$e_n(t) \equiv \int_0^t S(t-s)(B_o(ds) - B_n(ds))[Lx_o(s) + \eta(s)].$$

Since $S(t), t > 0$, is a compact semigroup and $B_n \xrightarrow{w} B_o$ and $Lx_o + \eta \in B_\infty(I, Y)$, it is not difficult to verify that

$$e_n(t) \xrightarrow{s} 0 \text{ in } X \ \forall \ t \in I.$$

It follows from a generalized Gronwall inequality [Ahmed, 11] applied to (16) that

$$|x_o(t) - x_n(t)|_X \leq |e_n(t)|_X + e^{\|\mu\|_v} \int_0^t |e_n(s)|_X \mu(ds), \tag{17}$$

where $\| \mu \|_v$ denotes the variation of the measure μ on I. From the expression for e_n and the assumption on Γ, we have

$$|e_n(t)|_X \leq 2M \int_0^t |Lx_o(s) + \eta(s)|_Y \nu(ds), t \in I. \tag{18}$$

Since ν is a countably additive bounded positive measure and $Lx_o + \eta \in B_\infty(I, Y)$ there exists a finite positive number b such that $\sup\{|e_n(t)|_X, t \in I\} \leq b$. Thus it follows from Lebesgue bounded convergence theorem applied to (17) that

$$x_n(t) \xrightarrow{s} x_o(t) \text{ in } X$$

for each $t \in I$ and even uniformly on I. Now focusing on the cost integrand for $J(B)$, it follows from the lower semicontinuity of ℓ in the second argument that

$$\ell(t, x_o(t)) \leq \underline{\lim}\, \ell(t, x_n(t)) a.a \ \ t \in I.$$

Hence it follows from the property of ℓ and Fatou's lemma, that we have

$$\int_I \ell(t, x_o(t))dt \leq \underline{\lim} \int_I \ell(t, x_n(t))dt.$$

Using the definition of semi variation and Hahn-Banach theorem, it is easy to verify that

$$|B_o|_s \leq \underline{\lim}\, |B_n|_s$$

whenever $B_n \stackrel{w}{\longrightarrow} B_o$ in $M_{casbsv}(\Sigma_I, \mathcal{L}(X,Y))$. Combining the above facts we obtain

$$J(B_o) \leq \underline{\lim}\, J(B_n).$$

This shows that J is weakly lower semicontinuous on $M_{casbsv}(\Sigma_I, \mathcal{L}(Y,X))$. Since Γ is compact with respect to the weak topology, J attains its minimum on Γ. This completes the proof. ■

In regards to the assumption on compactness of the semigroup, we note that the topology used here for the class of operator valued measures representing structural feedback controls is much weaker than the topology used in our previous paper [8, Definition 3.6, p. 108]. Thus the assumption on compactness of the semigroup $S(t), t > 0$, cannot be relaxed unless a stronger condition (such as strong compactness) is imposed on the set of admissible operator valued measures Γ.

Here we indicate one such topology and relax the compactness assumption on the semigroup. Suppose $M_{casbsv}(\Sigma, \mathcal{L}(Y,X))$ is endowed with the topology of set wise convergence on Σ in the strong operator topology on $\mathcal{L}(Y,X)$ and denote this topology by τ_{swso}. Then a set $\Gamma_s \subset M_{casbsv}(\Sigma, \mathcal{L}(Y,X))$ is said to be τ_{swso} compact if every sequence $\{B_k\} \in \Gamma_s$ has a subsequence $\{B_{k_n}\}$ and a $B_o \in \overline{\Gamma}_s$ such that

$$\lim_{n \to \infty} |(B_o(\sigma)y - B_{k_n}(\sigma)y| = 0$$

for every $\sigma \in \Sigma$ and $y \in Y$. The fact that the limit $B_o \in \overline{\Gamma}$ follows from Vitali-Hahn-Saks-Nikodym theorem.

Using this stronger topology we can relax compactness assumption of the semigroup. This is stated in the following theorem.

Theorem 5. *Suppose $A \in \mathcal{G}_0(M,\omega)$ is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, Γ_s a compact subset of the topological space $(M_{casbsv}(\Sigma_I, \mathcal{L}(Y,X)), \tau_{swso})$, f locally Lipschitz with at most linear growth, $L \in \mathcal{L}(X,Y)$, $\eta \in B_\infty(I,Y)$. There exists $\nu \in M^+_{cabv}(\Sigma_I)$ such that $|B|_s(\sigma) \leq \nu(\sigma)$ for $\sigma \in \Sigma_I$ uniformly w.r.t $B \in \Gamma_s$. The cost integrand ℓ is measurable in t and lower semicontinuous in x on X and there exists $\alpha \in L_1^+(I)$ and $\beta \geq 0$ satisfying*

$$|\ell(t,x)| \leq \alpha(t) + \beta |x|_X^p, \; \textit{for any } p \in (0,\infty).$$

Then, there exists a $B_o \in \Gamma_s$ at which J attains its minimum.

Remark 6. It would be interesting to relax the assumption on the semivariations of the admissible set of operator valued measures being dominated by a countably additive nonnegative scalar measure.

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