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# ON MAXIMAL IDEALS OF PSEUDO-BCK-ALGEBRAS

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**Abstract**

We investigate maximal ideals of pseudo-BCK-algebras and give some characterizations of them.

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## 1. Introduction

In 1958, C.C. Chang [1] introduced MV (Many Valued) algebras. In 1966, Y. Imai and K. Iséki [12] introduced the notion of BCK-algebra. In 1996, P. Hájek ([9], [10]) invented Basic Logic (BL for short) and BL-algebras, structures that correspond to this logical system. The class of BL-algebras contains the MV-algebras. G. Georgescu and A. Iorgulescu [5] (1999), and independently J. Rachůnek [20] introduced pseudo-MV-algebras which are a non-commutative generalization of MV-algebras. After pseudo-MV-algebras, the pseudo-BL-algebras [6] (2000), and the pseudo-BCK-algebras [7] (2001) were introduced and studied. The paper [7] contains basic properties of pseudo-BCK-algebras and their connections with pseudo-MV-algebras and with pseudo-BL-algebras. Y.B. Jun [17] obtained some characterizations of pseudo-BCK-algebras. A. Iorgulescu ([13], [14]) studied particular classes of pseudo-BCK-algebras.

K. Iséki and S. Tanaka ([16]) introduced the notion of ideals in BCK-algebras and investigated some interesting and fundamental results. R. Halaš and J. Kühr [11] applied this concept to pseudo-BCK-algebras. (They called ideals as deductive systems.) In this paper, we give some characterizations of maximal ideals in pseudo-BCK-algebras.

## 2. Preliminaries

The notion of pseudo-BCK-algebras is defined by Georgescu and Iorgulescu [7] as follows:

**Definition 2.1.** A *pseudo-BCK-algebra* is a structure $(A; \leq, *, \circ, 0)$, where "$\leq$" is a binary relation on a set $A$, "$*$" and "$\circ$" are binary operations on $A$ and "0" is an element of $A$, verifying the axioms: for all $x, y, z \in A$,

(pBCK-1) $(x * y) \circ (x * z) \leq z * y, \quad (x \circ y) * (x \circ z) \leq z \circ y,$

(pBCK-2) $x * (x \circ y) \leq y, \quad x \circ (x * y) \leq y,$

(pBCK-3) $x \leq x,$

(pBCK-4) $0 \leq x,$

(pBCK-5) $(x \leq y \text{ and } y \leq x) \;\Rightarrow\; x = y,$

(pBCK-6) $x \leq y \Leftrightarrow x * y = 0 \Leftrightarrow x \circ y = 0.$

Note that every pseudo-BCK-algebra satisfying $x * y = x \circ y$ for all $x, y \in A$ is a BCK-algebra.

**Proposition 2.2** ([7]). *Let $(A; \leq, *, \circ, 0)$ be a pseudo-BCK-algebra. Then for all $x, y, z \in A$ :*

(a) $x \leq y$ and $y \leq z \Rightarrow x \leq z;$

(b) $x * y \leq x, \quad x \circ y \leq x;$

(c) $(x * y) \circ z = (x \circ z) * y$;

(d) $x * 0 = x = x \circ 0$;

(e) $x \leq y \Rightarrow x * z \leq y * z, \quad x \circ z \leq y \circ z$.

If $(A; \leq, *, \circ, 0)$ is a pseudo-BCK-algebra, then $(A; \leqslant)$ is a poset by (pBCK-3), (pBCK-5), and Proposition 2.2 (a). The underlying order $\leqslant$ can be retrieved via (pBCK-6) and hence we may equivalently regard $(A; \leq, *, \circ, 0)$ to be an algebra $(A; *, \circ, 0)$. J. Kühr [18] showed that pseudo-BCK-algebras as algebras $(A; *, \circ, 0)$ of type $\langle 2, 2, 0\rangle$ form a quasivariety which is not a variety.

Throughout this paper $A$ will denote a pseudo-BCK-algebra. For $x, y \in A$ and $n \in \mathbb{N}_0$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) we define $x *^n y$ inductively

$$x *^0 y = x, \quad x *^{n+1} y = (x *^n y) * y \quad (n = 0, 1, \ldots).$$

$x \circ^n y$ is defined in the same way.

**Example 2.3** ([11], Example 2.4)**.** Let $A = \{0, a, b, c\}$ and define binary operations "$*$" and "$\circ$" on $A$ by the following tables:

| $*$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | 0 | 0 |
| $b$ | $b$ | $b$ | 0 | 0 |
| $c$ | $c$ | $b$ | $b$ | 0 |

| $\circ$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | 0 | 0 |
| $b$ | $b$ | $b$ | 0 | 0 |
| $c$ | $c$ | $c$ | $a$ | 0 |

Then $(A; *, \circ, 0)$ is a pseudo-BCK-algebra.

**Example 2.4.** Let $(M; \oplus, ^-, ^\sim, 0, 1)$ be a pseudo-MV-algebra and we put $x \odot y = (y^- \oplus x^-)^\sim$ $(= (y^\sim \oplus x^\sim)^-$ by Proposition 1.7 (1) of [8]). Define

$$x * y = x \odot y^- \quad \text{and} \quad x \circ y = y^\sim \odot x.$$

By 4.1.3 of [18], $(M; *, \circ, 0)$ is a pseudo-BCK-algebra.

## 3. Ideals

**Definition 3.1.** A subset $I$ of a pseudo-BCK-algebra $A$ is called an *ideal* of $A$ if it satisfies for all $x, y \in A$:

(I1) $0 \in I$,

(I2) if $x * y \in I$ and $y \in I$, then $x \in I$.

We will denote by $\mathrm{Id}(A)$ the set of all ideals of $A$.

**Proposition 3.2.** *Let $I \in \mathrm{Id}(A)$. Then for any $x, y \in A$, if $y \in I$ and $x \leq y$, then $x \in I$.*

***Proof.*** Straightforward.

**Proposition 3.3.** *Let $I$ be a subset of $A$. Then $I$ is an ideal of $A$ if and only if it satisfies conditions* (I1) *and*

(I2$'$) *for all $x, y \in A$, if $x \circ y \in I$ and $y \in I$, then $x \in I$.*

***Proof.*** It suffices to prove that if (I2) is satisfied, then (I2$'$) is also satisfied. The proof of the converse of this implication is analogous. Suppose that $x \circ y \in I$ and $y \in I$. From (pBCK-2) we know that $x * (x \circ y) \leq y$. Then, by Proposition 3.2, $x * (x \circ y) \in I$. Hence, since $x \circ y \in I$, (I2) shows that $x \in I$. ∎

For every subset $X \subseteq A$, we denote by $(X]$ the ideal of $A$ generated by $X$, that is, $(X]$ is the smallest ideal containing $X$. If $X = \{a\}$, we write $(a]$ for $(\{a\}]$. By Lemma 2.2 of [11], $(\emptyset] = \{0\}$ and for every $\emptyset \neq X \subseteq A$,

$$\begin{aligned} (X] &= \{x \in A : (\cdots(x * a_1) * \cdots) * a_n = 0 \text{ for some } a_1, \ldots, a_n \in X\} \\ &= \{x \in A : (\cdots(x \circ a_1) \circ \cdots) \circ a_n = 0 \text{ for some } a_1, \ldots, a_n \in X\}. \end{aligned}$$

**Definition 3.4.** An ideal $I$ of $A$ is called *normal* if it satisfies the following condition:

(N) for all $x, y \in A$, $x * y \in I \Leftrightarrow x \circ y \in I$.

**Example 3.5.** Let $A$ be the pseudo-BCK-algebra from Example 2.3. Ideals of $A$ are $\{0\}, \{0, a\}, A$; $\{0, a\}$ is not normal, because $c \circ b = a \in I$ while $c * b = b \notin I$.

**Example 3.6** ([2], see also [15], 430)**.** Let $A = \{(1, y) \in \mathbb{R}^2 : y \geqslant 0\} \cup \{(2, y) \in \mathbb{R}^2 : y \leqslant 0\}$ and $\mathbf{0} = (1, 0), \mathbf{1} = (2, 0)$. For any $(a, b), (c, d) \in A$, we define operations $\oplus, ^-, ^\sim$ as follows:

$$(a, b) \oplus (c, d) = \begin{cases} (ac, bc + d) & \text{if } ac < 2 \text{ or } (ac = 2 \text{ and } bc + d < 0) \\ (2, 0) & \text{otherwise,} \end{cases}$$

$$(a, b)^- = \left(\frac{2}{a}, \frac{-b}{a}\right), \quad (a, b)^\sim = \left(\frac{2}{a}, \frac{-2b}{a}\right).$$

Then $(A, \oplus, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ is a pseudo-MV-algebra. For $x, y \in A$, we set

$$x * y = (y \oplus x^\sim)^- \quad \text{and} \quad x \circ y = (x^- \oplus y)^\sim.$$

Therefore $(A; *, \circ, \mathbf{0})$ is a pseudo-BCK-algebra (see Example 2.4). We have

$$(a, b) * (c, d) = \left((c, d) \oplus \left(\frac{2}{a}, \frac{-2b}{a}\right)\right)^-$$

and hence

$$(a, b) * (c, d) = \begin{cases} \left(\frac{a}{c}, \frac{b - d}{c}\right) & \text{if } a = 2c \text{ or } (a = c \text{ and } d < b) \\ (1, 0) & \text{otherwise.} \end{cases}$$

Similarly,

$$(a,b)\circ(c,d)=\begin{cases}\left(\dfrac{a}{c},b-\dfrac{ad}{c}\right) & \text{if } a=2c \text{ or } (a=c \text{ and } d<b)\\ (1,0) & \text{otherwise.}\end{cases}$$

It is easy to see that $I=\{(1,y): y\geqslant 0\}$ is an ideal of $A$. Observe that $I$ is normal. Indeed,

$$(a,b)*(c,d)\notin I \Leftrightarrow a=2c \Leftrightarrow (a,b)\circ(c,d)\notin I.$$

**Lemma 3.7.** *Let $I$ be a normal ideal of $A$. Then*

$$x *^n a \in I \Leftrightarrow x \circ^n a \in I$$

*for all $x, a \in A$ and $n \in \mathbb{N}$.*

***Proof.*** The proof is by induction on $n$. ■

Following [18] (see also [19], p. 357), for any normal ideal $I$ of $A$, we define the congruence on $A$:

$$x \sim_I y \Leftrightarrow x*y\in I \text{ and } y*x\in I.$$

We denote by $x/I$ the congruence class of an element $x\in A$ and on the set $A/I=\{x/I : x\in A\}$ we define the operations:

$$x/I * y/I = (x*y)/I,\ \ x/I\circ y/I=(x\circ y)/I$$

($*$ and $\circ$ are well defined on $A/I$, because $\sim_I$ is a congruence on $A$). The resulting quotient algebra $(A/I;*,\circ,I)$ becomes a pseudo-BCK-algebra (see Proposition 2.2.4 of [18]), called the *quotient algebra of $A$ by the normal ideal $I$*. It is clear that

$$x/I=0/I\Leftrightarrow x\in I. \tag{1}$$

**Proposition 3.8.** *Let $I$ be a normal ideal of $A$ and let $J\subseteq A/I$. Then $J\in \mathrm{Id}(A/I)$ if and only if $J=I_0/I$ for some $I_0\in\mathrm{Id}(A)$ such that $I\subseteq I_0$.*

***Proof.*** Suppose that $J \in \mathrm{Id}(A/I)$. Let $I_0 = \{x \in A : x/I \in J\}$. By (1), $I \subseteq I_0$. Observe that $I_0$ is an ideal of $A$. Indeed, $0 \in I_0$ and let $x * y, y \in I_0$. Then $(x*y)/I \in J$ and $y/I \in J$. Hence $x/I \in J$ and therefore $x \in I_0$. Thus $I_0 \in \mathrm{Id}(A)$. It is easy to see that $J = I_0/I$.

Conversly, let $J = I_0/I$ for some $I_0 \in \mathrm{Id}(A)$ such that $I \subseteq I_0$. Of course, $0/I \in J$. Let $x/I * y/I, y/I \in J$. Then $x * y \in I_0$ and $y \in I_0$. Since $I_0$ is an ideal of $A$, we see that $x \in I_0$, hence that $x/I \in J$. Consequently, $J \in \mathrm{Id}(A/I)$. ∎

**Proposition 3.9.** *Let $I$ be a normal ideal of $A$ and let $a \in A$. Denote by*

$$I_a = \{x \in A : x *^n a \in I \text{ for some } n \in \mathbb{N}\}.$$

*Then $I_a = (I \cup \{a\}]$.*

***Proof.*** We first show that

$$I_a \subseteq (I \cup \{a\}]. \tag{2}$$

Let $x *^n a \in I$ for some $n \in \mathbb{N}$. We have $(x *^n a) * (x *^n a) = 0$. Thus

$$((\cdots((x * b_1) * b_2) * \cdots) * b_n) * b_{n+1} = 0,$$

where $b_1 = \cdots = b_n = a$ and $b_{n+1} = x *^n a \in I$. Thus $x \in (I \cup \{a\}]$. This gives (2).

Since $a * a = 0 \in I$, we see that $a \in I_a$. Let $x \in I$. Then $x * a \in I$, because $x * a \leqslant x$. Therefore $x \in I_a$ and hence $I_a$ contains $I$. Suppose now that $x * y \in I_a$ and $y \in I_a$. It follows that there exist $k, l \in \mathbb{N}$ such that $(x * y) *^k a \in I$ and $y *^l a \in I$. By Lemma 3.7, $(x * y) \circ^k a \in I$. Applying Proposition 2.2 (c) we conclude that

$$(x * y) \circ^k a = ((x \circ a) * y) \circ^{k-1} a = ((x \circ^2 a) * y) \circ^{k-2} a = \cdots = (x \circ^k a) * y.$$

Therefore $b := (x \circ^k a) * y \in I$. Then $((x \circ^k a) * y) \circ b = 0$ and hence $((x \circ^k a) \circ b) * y = 0$. Thus $(x \circ^k a) \circ b \leqslant y$. By Proposition 2.2 (e), $((x \circ^k a) \circ b) *^l a \leqslant y *^l a \in I$. Consequently, $((x \circ^k a) \circ b) *^l a \in I$.

According to Proposition 2.2 (c) we have $((x \circ^k a) *^l a) \circ b \in I$. Since $b \in I$, we see that $(x \circ^k a) *^l a \in I$. Lemma 3.7 now shows that $x *^{k+l} a \in I$, that is, $x \in I_a$. This proves that $I_a$ is an ideal of $A$. Thus

$$(I \cup \{a\}] \subseteq I_a. \tag{3}$$

From (2) and (3) we obtain $I_a = (I \cup \{a\}]$. ■

Proposition 3.9 and Lemma 3.7 give.

**Corollary 3.10.** *Let $I$ be a normal ideal of $A$ and let $a \in A$. Then*

$$\begin{aligned}(I \cup \{a\}] &= \{x \in A : x *^n a \in I \text{ for some } n \in \mathbb{N}\} \\ &= \{x \in A : x \circ^n a \in I \text{ for some } n \in \mathbb{N}\}.\end{aligned}$$

**Corollary 3.11.** *Let $a \in A$. Then $(a] = \{x \in A : x *^n a = 0$ for some $n \in \mathbb{N}\}$.*

***Proof.*** This follows from Proposition 3.9 when we put $I = \{0\}$. ■

Let $A$ and $B$ be pseudo-BCK-algebras and let $f : A \to B$ be a homomorphism. The *kernel* of $f$ is the set

$$\mathrm{Ker} f := \{x \in A : f(x) = 0\},$$

that is, $\mathrm{Ker} f = f^{\leftarrow}(\{0\})$, where $f^{\leftarrow}(X)$ denote the *f-inverse image* of $X \subseteq B$. It is easy to see that the next lemma holds.

**Lemma 3.12.** *Let $f : A \to B$ be a homomorphism and let $x, y \in A$. If $f(x) = f(y)$, then $x * y, y * x \in \mathrm{Ker} f$.*

**Proposition 3.13.** *Let $f : A \to B$ be a homomorphism and let $I \in \mathrm{Id}(B)$. Then $f^{\leftarrow}(I) \in \mathrm{Id}(A)$.*

***Proof.*** The proof is straightforward. ■

**Proposition 3.14.** *Let $f : A \to B$ be a surjective homomorphism and let $I$ be an ideal of $A$ containing* $\mathrm{Ker}f$. *Then* $f(I) \in \mathrm{Id}(B)$.

***Proof.*** Obviously, $0 \in f(I)$. Let $x \in B$, $y \in f(I)$, and let $x * y \in f(I)$. Then there are $a, b \in I$ such that $y = f(a)$ and $x * y = f(b)$. Since $f$ is surjective, $x = f(c)$ for some $c \in A$. We have $f(b) = f(c) * f(a) = f(c * a)$ and hence, by Lemma 3.12, $(c * a) * b \in \mathrm{Ker}f \subseteq I$. Since $a, b \in I$, we conclude that $c \in I$. Therefore $x = f(c) \in f(I)$. Consequently, $f(I) \in \mathrm{Id}(B)$. ■

## 4. Maximal ideals

**Definition 4.1.** Let $I$ be a proper ideal of $A$ (i.e., $I \neq A$).

(a) $I$ is called *prime* if, for all $I_1, I_2 \in \mathrm{Id}(A)$, $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.

(b) $I$ is *maximal* iff whenever $J$ is an ideal such that $I \subseteq J \subseteq A$, then either $J = I$ or $J = A$.

Next lemma is obvious and its proof will be omitted.

**Lemma 4.2.** *Every proper ideal of $A$ can be extended to a maximal ideal.*

**Lemma 4.3.** *If* $I \in \mathrm{Id}(A)$ *is maximal, then $I$ is prime.*

***Proof.*** Let $I$ be a maximal ideal of $A$ and let $I = I_1 \cap I_2$ for some $I_1, I_2 \in \mathrm{Id}(A)$. Then $I \subseteq I_1$ and $I \subseteq I_2$. Suppose that $I \neq I_1$. Since $I$ is maximal, we conclude that $I_1 = A$ and hence $I = A \cap I_2 = I_2$. By definition, $I$ is prime. ■

**Theorem 4.4.**

(i) *For each $t \in T$, let $I_t$ be an ideal of the pseudo-BCK-algebra $(A_t; *_t, \circ_t, 0_t)$. Then $I := \prod_{t \in T} I_t$ is an ideal of $A := \prod_{t \in T} A_t$. Conversely, if $I$ is an ideal of $A$, then $I_t := \pi_t(I)$, where $\pi_t$ is the t-th projection of $A$ onto $A_t$, is an ideal of $A_t$, and $I = \prod_{t \in T} I_t$.*

(ii) *An ideal $I := \prod_{t \in T} I_t$ is maximal in $A := \prod_{t \in T} A_t$ if and only if there is an unique index $s \in T$ such that $I_s$ is a maximal ideal of $A_s$ and $I_t = A_t$ for any $t \neq s$.*

***Proof.***

(i) The first part of the assertion is obvious. Suppose now that $I$ is an ideal of $A$ and let $I_t = \pi_t(I)$. Then $0_t = \pi_t(0) \in I_t$. Let $x_t *_t y_t \in I_t$ and $y_t \in I_t$. We define $x, y \in A$ by:

$$x(s) = \begin{cases} x_t & \text{for} \quad s = t \\ 0_s & \text{for} \quad s \neq t \end{cases} \quad \text{and } y(s) = \begin{cases} y_t & \text{for} \quad s = t \\ 0_s & \text{for} \quad s \neq t. \end{cases}$$

Since $I_t = \pi_t(I)$, there exists an element $z \in I$ such that $\pi_t(z) = x_t *_t y_t$. We have $(x * y)(t) = x(t) *_t y(t) = x_t *_t y_t = z(t)$ and $(x * y)(s) = 0_s *_s 0_s = 0_s \leqslant z(s)$ for any $s \neq t$. Therefore $x * y \leqslant z$ which implies that $x * y \in I$. Similarly there is an element $v \in I$ such that $\pi_t(v) = y_t \in I_t$. Obviously, $y \leqslant v$ and hence $y \in I$. This means that $I_t$ is an ideal of $A_t$. Since $\pi_t(I) = I_t$ for all $t \in T$, we see that $I = \prod_{t \in T} I_t$.

(ii) Let $I = \prod_{t \in T} I_t$ be a maximal ideal of $A$. It is easily seen that there is at least one index $t$ such that $I_t$ is a maximal ideal of $A_t$. Assume that there are two indices $t_1$ and $t_2$ such that $I_{t_1}$ and $I_{t_2}$ are proper ideals of $A_{t_1}$ and $A_{t_2}$, respectively. Then $J := \prod_{t \in T} I'_t$, where $I'_t = I_t$ if $t \neq t_1$ and $I'_{t_1} = A_{t_1}$, is a proper ideal of $A$ containing $I$, which contradicts the maximality of $I$. Suppose that $I = \prod_{t \in T} I_t$, where $I_s$ is a maximal ideal of $A_s$ and $I_t = A_t$ for all $t \neq s$. By (i), $I \in \mathrm{Id}(A)$. Observe that $I$ is maximal. Indeed, let $K \in \mathrm{Id}(A)$ and $K \supset I$. Then $\pi_s(K) \supset I_s$ and $\pi_t(K) = A_t$ for all $t \neq s$. Since $I_s$ is maximal in $A_s$, we see that $\pi_s(K) = A_s$, and therefore $\pi_t(K) = A_t$ for all $t \in T$. Thus $K = A$ and consequently, $I$ is a maximal ideal of $A$. ■

The following two theorems give the homomorphic properties of maximal ideals.

**Theorem 4.5.** *Let $f : A \to B$ be a surjective homomorphism and let $I$ be a maximal ideal of $A$ containing* Ker$f$. *Then $f(I)$ is a maximal ideal of $B$.*

***Proof.*** By Proposition 3.14, $f(I) \in \mathrm{Id}(B)$. Let $x \in A - I$ and suppose that $f(I) = B$. Then $f(x) = f(y)$ for some $y \in I$. Applying Lemma 3.12 we conclude that $x * y \in I$, and hence $x \in I$, a contradiction. Therefore $f(I) \neq B$. We take a proper ideal $J$ of $B$ such that $J \supseteq f(I)$. From Proposition 3.13 we deduce that $f^{\leftarrow}(J) \in \mathrm{Id}(A)$. It is easy to see that $I \subseteq f^{\leftarrow}(J) \subset A$. Since $I$ is maximal, $f^{\leftarrow}(J) = I$. Consequently, $f(I) = f(f^{\leftarrow}(J)) = J$. Thus $f(I)$ is a maximal ideal of $B$. ■

**Theorem 4.6.** *Let $f : A \to B$ be a surjective homomorphism and let $J$ be a maximal ideal of $B$. Then $f^{\leftarrow}(J)$ is a maximal ideal of $A$.*

***Proof.*** From Proposition 3.13 it follows that $I := f^{\leftarrow}(J) \in \mathrm{Id}(A)$. It is easily seen that $I \neq A$. By Lemma 4.2 there is a maximal ideal $I'$ of $A$ containing $I$. We have

$$I = f^{\leftarrow}(J) \supseteq f^{\leftarrow}(\{0\}) = \mathrm{Ker} f.$$

Since $I' \supseteq I \supseteq \mathrm{Ker} f$, Theorem 4.5 shows that $f(I')$ is a maximal ideal of $B$. Obviously, $f(I') \supseteq f(f^{\leftarrow}(J)) = J$ and hence $f(I') = J$. Then $I' \subseteq f^{\leftarrow}(f(I')) = f^{\leftarrow}(J) = I \subseteq I'$, that is, $f^{\leftarrow}(J) = I'$. Thus $f^{\leftarrow}(J)$ is a maximal ideal of $A$. ■

**Theorem 4.7.** *For every proper normal ideal $I$ of a pseudo-BCK-algebra $A$, the following conditions are equivalent:*

(a) *$I$ is a maximal ideal of $A$;*

(b) *for any $x \in A$, $y \in A - I$, $x *^n y \in I$ for some $n \in \mathbb{N}$;*

(c) *for any $x \in A$, $y \in A - I$, $x \circ^n y \in I$ for some $n \in \mathbb{N}$;*

(d) $|\mathrm{Id}(A/I)| = 2$.

***Proof.*** (a) $\Rightarrow$ (b): Let $x \in A$. Suppose that $I$ is a maximal ideal of $A$ and let $y \in A - I$. Then $(I \cup \{y\}] = A$ and hence $x \in (I \cup \{y\}]$. By Proposition 3.9, $x *^n y \in I$ for some $n \in \mathbb{N}$.

(b) $\Leftrightarrow$ (c): The equivalence of (b) and (c) follows from the fact that $I$ is a normal ideal.

(c) $\Rightarrow$ (a): Let $J$ be an ideal of $A$ containing $I$. Suppose that $J \neq I$ and let $y \in J - I$. For every $x \in A$, by assumption, $x \circ^n y \in I$ for some $n \in \mathbb{N}$. Then $x \circ^n y \in J$ and hence $x \in J$, because $y \in J$. Therefore $J = A$.

(a) $\Rightarrow$ (d): Let $I$ be a normal and maximal ideal of $A$, and let $J$ be an ideal of $A/I$. By Proposition 3.8, $J = I_0/I$ for some $I_0 \in \mathrm{Id}(A)$ such that $I \subseteq I_0$. Since $I$ is maximal, $I_0 = I$ or $I_0 = A$. Consequently, $J = \{0/I\}$ or $J = A/I$.

(d) $\Rightarrow$ (a): Let $I_0$ be a proper ideal of $A$ containing $I$. From Proposition 3,8 it follows that $J = I_0/I$ is an ideal of $A/I$. Therefore $J = \{0/I\}$, that is, $I_0 = I$, which proves that $I$ is maximal. ■

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