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A NOTE ON ANDERSON'S NOTE ON A STATIONARY AUTOREGRESSIVE PROCESS

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Abstract

A form of the covariance matrix of a weakly stationary first-order autoregressive process is established.

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In a short note [1] Anderson has commented an earlier result of Nguyen [2] concerning a characterization of a vector variate normal distribution. Anderson has restated this result in a context of the autoregressive process

$$\mathbf{X}_t = \mathbf{B}\mathbf{X}_{t-1} + \mathbf{U}_t \tag{1}$$

and has given very simple and elegant proof that its stationarity implies that $\rho(\mathbf{B}) < 1$, where $\rho(\mathbf{B})$ denotes the spectral radius of $\mathbf{B}$. This statement corresponds to the conclusion (a) of Nguyen's Theorem 2.1.

The aim of this note is to establish, also in a simple way, the form of covariance matrix of $\mathbf{X}_t$, which is a crucial point in the conclusion (b) of Nguyen's Theorem 2.1.

Theorem. *Let $\{\mathbf{U}_t\}$ be a sequence of i.i.d. random vectors with zero mean and the common non-singular covariance matrix, $C(\mathbf{U}_t) = \mathbf{\Sigma}$, and let $\mathbf{X}_t$ be a random vector related to $\mathbf{X}_{t-1}$ and $\mathbf{U}_t$ by (1), where $\mathbf{X}_{t-1}$ and $\mathbf{U}_t$ are independent, while $\mathbf{B}$ is a fixed $p \times p$ matrix. If this autoregressive process is stationary, i.e. $C(\mathbf{X}_t) = \mathbf{\Gamma}$ for all t, then*

$$\mathbf{\Gamma} = \sum_{s=0}^{\infty} \mathbf{B}^s \mathbf{\Sigma} \mathbf{B}'^s. \tag{2}$$

Proof. From the assumption of stationarity we have

$$\mathbf{\Gamma} = \mathbf{B}\mathbf{\Gamma}\mathbf{B}' + \mathbf{\Sigma}, \tag{3}$$

which can be written as

$$(\mathbf{I} - (\mathbf{B} \otimes \mathbf{B}))\text{vec}\mathbf{\Gamma} = \text{vec}\mathbf{\Sigma},$$

where $\otimes$ denotes the Kronecker product while the vec(.) operator stacks the columns of the matrix argument one under the other (for details see e.g.[3]). Following the result of Anderson [1], (3) implies that $\rho(\mathbf{B}) < 1$. But $\rho(\mathbf{B}) < 1$ implies that $\rho(\mathbf{B} \otimes \mathbf{B}) < 1$, which assures that the matrix $(\mathbf{I} - (\mathbf{B} \otimes \mathbf{B}))$ is non-singular and its inverse is the sum of an appropriate Neumann series,

$$\sum_{s=0}^{\infty} (\mathbf{B} \otimes \mathbf{B})^s = (\mathbf{I} - (\mathbf{B} \otimes \mathbf{B}))^{-1}.$$

Therefore

$$\text{vec}\mathbf{\Gamma} = \sum_{s=0}^{\infty} (\mathbf{B}^s \otimes \mathbf{B}^s)\text{vec}\mathbf{\Sigma},$$

which leads to (2). ■

Finally note that in case when $\mathbf{B}$ admits a decomposition $\mathbf{B} = \mathbf{C}\mathbf{\Lambda}\mathbf{C}^{-1}$, with $\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_p)$, the solution of (3) can be established directly omitting any iteration. First, transform (3) to the form

$$\mathbf{A} = \mathbf{\Lambda}\mathbf{A}\mathbf{\Lambda} + \mathbf{S},$$

where $\mathbf{A} = \mathbf{C}^{-1}\mathbf{\Gamma}(\mathbf{C}^{-1})' = (a_{ij})$ and $\mathbf{S} = \mathbf{C}^{-1}\mathbf{\Sigma}(\mathbf{C}^{-1})'$. Therefore,

$$a_{ij} = \lambda_i \lambda_j a_{ij} + s_{ij}, \quad i, j = 1, 2, ..., p,$$

or $\mathbf{A} = (s_{ij}/(1 - \lambda_i \lambda_j))$, since $|\lambda_i| < 1$ for all i. In result $\mathbf{\Gamma} = \mathbf{CAC}'$ provides the solution of (3) which exactly coincides with that given by Anderson [4, Section 5.3].

References

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