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SEMI-ADDITIVE FUNCTIONALS AND COCYCLES IN THE CONTEXT OF SELF-SIMILARITY*

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Abstract

Kernel functions of stable, self-similar mixed moving averages are known to be related to nonsingular flows. We identify and examine here a new functional occuring in this relation and study its properties. To prove its existence, we develop a general result about semi-additive functionals related to cocycles. The functional we identify, is helpful when solving for the kernel function generated by a flow. Its presence also sheds light on the previous results on the subject.

Keywords and phrases: stable, self-similar processes with stationary increments, mixed moving averages, nonsingular flows, cocycles, semi-additive functionals.

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1. Introduction

Stable, self-similar mixed moving average processes form an important subclass of all stable (non-Gaussian), self-similar processes with stationary increments. Following the seminal approach of Rosiński (1995), we showed in Pipiras and Taqqu (2002*a*, 2002*b*) that self-similar mixed moving averages can be related to the so-called "nonsingular flows" and we classified these processes using the structure of the underlying flows.

The non-random kernel function which appears in the integral representation of the processes, determines their finite-dimensional distributions and plays an important role in the classification. The goal of this note is to better understand the structural properties of these kernel functions. The results we obtain are used in Pipiras and Taqqu (2007) to study the fine structure of the processes.

We shall not redefine stable, self-similar mixed moving averages here because their definition will not be used. Our results concern their kernel functions which are non-random objects. They are deterministic functions $G : X \times \mathbb{R} \mapsto \mathbb{R}$ such that

$$G_t(x,u) = G(x,t+u) - G(x,u), \quad x \in X, u \in \mathbb{R}, \tag{1.1}$$

satisfies $\{G_t\}_{t\in\mathbb{R}} \subset L^\alpha(X \times \mathbb{R}, \mu(dx)du)$, where $(X, \mathcal{X}, \mu)$ is a standard Lebesgue space and $\alpha \in (0,2)$ is the stability parameter of the associated stable processes. To avoid trivialities, it will be assumed throughout that

$$\operatorname{supp}\{G_t(x,u), t \in \mathbb{R}\} = X \times \mathbb{R} \quad \text{a.e. } \mu(dx)du \tag{1.2}$$

holds. By support $\operatorname{supp}\{G_t, t \in \mathbb{R}\}$ we mean a minimal (a.e.) set $A \subset X \times \mathbb{R}$ such that $m\{G_t(x,u) \neq 0, (x,u) \notin A\} = 0$ for every $t \in \mathbb{R}$, where $dm = \mu(dx)du$. When these stable, mixed moving averages are also self-similar, their kernel functions G may be related to flows in the following sense:

Definition 1.1 (Pipiras and Taqqu, 2002*a*)**.** A kernel function G associated with a self-similar mixed moving average representation is said to be *generated by a nonsingular measurable flow* $\{\psi_c\}_{c>0}$ on $(X, \mathcal{X}, \mu)$ if, for all $c > 0$,

$$c^{-(H-1/\alpha)}G(x,cu)$$

$$= b_c(x)\left\{\frac{d(\mu\circ\psi_c)}{d\mu}(x)\right\}^{1/\alpha} G\Big(\psi_c(x), u+g_c(x)\Big) + j_c(x), \tag{1.3}$$

$$\text{a.e. } \mu(dx)du,$$

where $\{b_c\}_{c>0}$ is a cocycle for the flow $\{\psi_c\}_{c>0}$ taking values in $\{-1,1\}$, $\{g_c\}_{c>0}$ is a semi-additive functional for the flow $\{\psi_c\}_{c>0}$, $j_c(x) : (0,\infty)\times X \to \mathbb{R}$ is some function, $\alpha\in(0,2)$ is the stability parameter of the associated stable process, and $H>0$ is its self-similarity parameter.

We shall now define the various terms used in Definition 1.1. A (multiplicative) *flow* $\{\psi_c\}_{c>0}$ is a collection of deterministic maps $\psi_c : X \to X$ such that

$$\psi_{c_1c_2}(x) = \psi_{c_1}(\psi_{c_2}(x)),\ \text{for all } c_1,c_2>0, x\in X, \tag{1.4}$$

and $\psi_1(x)=x$, for all $x\in X$. A flow is *nonsingular* if $\mu(A)=0$ implies $\mu(\psi_c^{-1}(A))=0$ for any $c>0$ and $A\in\mathcal{X}$. It is *measurable* if the map $\psi_c(x) : (0,\infty)\times X\mapsto X$ is measurable. A *cocycle* $\{b_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ is a measurable map $b_c(x) : (0,\infty)\times X\mapsto Y$ satisfying the relation

$$b_{c_1c_2}(x) = b_{c_1}(x)b_{c_2}(\psi_{c_1}(x)),\ \text{for all } c_1,c_2>0, x\in X. \tag{1.5}$$

In our context, we will have either $Y=\{-1,1\}$, $Y=(0,\infty)$ or $Y=\mathbb{R}\setminus\{0\}$. A *semi-additive functional* $\{g_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ is a measurable function $g_c(x) : (0,\infty)\times X\mapsto\mathbb{R}$ satisfying

$$g_{c_1c_2}(x) = \frac{g_{c_1}(x)}{c_2} + g_{c_2}(\psi_{c_1}(x)),\ \text{for all } c_1,c_2>0, x\in X. \tag{1.6}$$

All quantities entering in (1.3) thus obey a specific relation with the exception of the function $j_c(x)$. The main goal of this work is to show that, in fact, (1.3) also implies that this function obeys a special relation, namely,

$$j_{c_1c_2}(x)$$

$$(1.7) \qquad = c_2^{-(H-1/\alpha)} j_{c_1}(x) + b_{c_1}(x) \left\{ \frac{d(\mu \circ \psi_{c_1})}{d\mu}(x) \right\}^{1/\alpha} j_{c_2}(\psi_{c_1}(x)),$$

$$\text{for all } c_1, c_2 > 0, x \in X.$$

It is quite easy to show that $\{j_c\}_{c>0}$ satisfies (1.7) a.e. $\mu(dx)$, for all $c_1, c_2 > 0$ (see the proof of Theorem 3.1), in which case $\{j_c\}_{c>0}$ is called *an almost 2-semi-additive functional.* Having (1.7) for a.e. $\mu(dx)$, however, is often not enough or difficult to solve for $\{j_c\}_{c>0}$ given a specific flow $\{\psi_c\}_{c>0}$. Therefore, we need to show that $\{j_c\}_{c>0}$ has a version (an a.e. modification for fixed c) satisfying (1.7) for all $x \in X$, which entails the specification of an adequate version of the Radon-Nikodym derivative $d(\mu \circ \psi_c)/d\mu$. A functional $\{j_c\}_{c>0}$ satisfying (1.7) will be called *2-semi-additive functional* (Definition 3.1).

Since g_c plays a role parallel to j_c, a functional $\{g_c\}_{c>0}$ satisfying (1.6) will be called *1-semi-additive functional.* To find out which kernel G is related to a given flow ψ_c, one uses (1.7) to get the *2*-semi-additive functional j_c and then one uses (1.3) to obtain G. Thus, *2*-semi-additive and other functionals are important when solving the equation (1.3) for the kernel function G related to some specific flows (see, for example, Pipiras and Taqqu, 2007, Theorem 3.1 and its proof). *2*-semi-additive functionals also shed light on the structure of our previous results (see the discussion following the proof of Theorem 3.1 below and the remark following Example 4.3 below).

The proof of the existence of a version which makes (1.7) valid for all $x \in X$, is involved. We establish this result in a more general context, namely, that of *semi-additive functionals related to a cocycle.* After a simple transformation, *1*- and *2*-semi-additive functionals are examples of semi-additive functionals related to particular cocycles (see Examples 3.1 and 3.2).

We show not only the existence of a version but also obtain an expression for semi-additive functionals related to cocycles. We then use this expression to characterize *1*- and *2*-semi-additive functionals associated with cyclic flows.

The paper is organized as follows. Section 2 contains results on the semi-additive functionals related to cocycles. In Section 3, we prove that the function $j_c(x)$ in (1.3) can be taken as a *2*-semi-additive functional satisfying (1.7). We provide a number of examples in Section 4 which illustrate how one can derive the semi-additive functionals. Semi-additive functionals associated with cyclic flows are studied in Section 5.

2. SEMI-ADDITIVE FUNCTIONALS RELATED TO COCYCLES

We first define semi-additive functionals related to cocycles, and introduce their large class. See Section 1 for a motivation of these functionals.

Definition 2.1. A measurable map $J_c(x) : (0,\infty) \times X \mapsto \mathbb{R}$ is called a *semi-additive functional related to a cocycle* $\{B_c\}_{c>0}$ (and a flow $\{\psi_c\}_{c>0}$) if, for all $c_1, c_2 \in \mathbb{R}$, $x \in X$,

$$J_{c_1c_2}(x) = J_{c_1}(x) + B_{c_1}(x)J_{c_2}(\psi_{c_1}(x)). \tag{2.1}$$

When $J_c(x)$ satisfies (2.1) a.e. $\mu(dx)$, for all $c_1, c_2 > 0$, it is called an *almost semi-additive functional related to a cocycle.*

Example 2.1. Let $J : X \mapsto \mathbb{R}$ be a function and set

$$J_c(x) = B_c(x)J(\psi_c(x)) - J(x). \tag{2.2}$$

By using the cocycle equation (1.5), we have for $c_1, c_2 > 0$,

$$J_{c_1c_2}(x) = B_{c_1}(x)B_{c_2}(\psi_{c_1}(x))J(\psi_{c_2}(\psi_{c_1}(x)))-J(x) = B_{c_1}(x)J(\psi_{c_1}(x))-J(x)$$

$$+B_{c_1}(x)\Big(B_{c_2}(\psi_{c_1}(x))J(\psi_{c_2}(\psi_{c_1}(x)))-J(\psi_{c_1}(x))\Big)=J_{c_1}(x)+B_{c_1}(x)J_{c_2}(\psi_{c_1}(x)).$$

Therefore, in view of (2.1), the functional $\{G_c\}_{c>0}$ in (2.2) is a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$.

Viewed from a different angle, (2.2) is a particular solution to the equation (2.1). It is not necessarily the unique solution. We will encounter in Section 4 a number of particular equations of the form (2.1), for which we derive the general solutions.

We now show that an *almost* semi-additive functional related to a cocycle has a version which is a semi-additive functional related to a cocycle. We say that $\{f_c\}_{c\in I} \subset L^0(S,\mathcal{S},m)$ is a *version* of $\{\widetilde{f_c}\}_{c\in I} \subset L^0(S,\mathcal{S},m)$, where I is an arbitrary index set, if $m(f_c \neq \widetilde{f_c}) = 0$ for all $c \in I$. The proof of this result uses the notion of a special flow $\widetilde{\psi}_c(y,u)$. Informally, the flow $\widetilde{\psi}_c(y,u)$ is defined on the set of points

$$\Omega = \{(y,u) : 1 \le u < p(y), y \in Y\} = Y \times [1, p(\cdot)),$$

where $p(y) > 1$ is a function. Plotting (y,u) in two dimensions, we can view the flow $\widetilde{\psi}_c$ as moving up vertically till it reaches the level $p(y)$, and then jumps back to a point $(y',1)$ before it renews its vertical climb, this time from the point y'. Thus, if one focuses only on the horizontal Y axis, the flow starting at y moves to $y' = Vy$, then to $V^2y,\dots,V^ny,\dots$ Since the flow $\widetilde{\psi}_c$ moves constantly, observe that it has no fixed points.

We shall apply Theorem 3.1 of Kubo (1969) to define $\widetilde{\psi}_c$ formally. According to that theorem, any (measurable, nonsingular) flow $\{\psi_c\}_{c>0}$ without fixed points on a standard Lebesgue space is null-isomorphic (mod 0) to some *special flow* $\{\widetilde{\psi}_c\}_{c>0}$ defined on the space $(Y \times [1,p(\cdot)), \mathcal{Y} \otimes \mathcal{B}([1,p(\cdot))), \tau(dy)du)$ by

$$\widetilde{\psi}_c(y,u) = (V^n y, cu/p_n(y)), \quad \text{for } 1 \le cu/p_n(y) < p(V^n y), \tag{2.3}$$

where $(Y,\mathcal{Y},\tau)$ is a standard Lebesgue space, V is a null-isomorphism of Y onto itself, $p > 1$ is a positive measurable function on Y satisfying $\prod_{k=-\infty}^{-1} p(V^k y) = \prod_{k=0}^{\infty} p(V^k y) = \infty$, and where $p_n(y) = \prod_{k=0}^{n-1} p(V^k y)$ if $n \ge 1$, $p_n(y) = 1$ if $n = 0$, and $p_n(y) = 1/\prod_{k=n}^{-1} p(V^k y)$ if $n \le -1$. For additional intuition and information on special flows, see Chapter 11 in Cornfeld, Fomin and Sinai (1982), or Appendix A in Pipiras and Taqqu (2004).

Remark. Note that special flows considered in the references above are not multiplicative as in (1.5) but additive instead. It is, however, easy to translate results from one framework to the other by considering either the logarithms or the exponentials.

Theorem 2.1. *Suppose that $\{J_c\}_{c>0}$ is an almost semi-additive functional related to a cocycle $\{B_c\}_{c>0}$ and to a measurable, nonsingular flow $\{\psi_c\}_{c>0}$ on a standard Lebesgue space $(X,\mathcal{X},\mu)$. Then, $\{J_c\}_{c>0}$ has a version which is a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$.*

Proof. We will extend the proof of Proposition 3.1 in Pipiras and Taqqu (2002*a*) and also use Proposition 1.2 in Kubo (1970). By using Remark 3.1 in Kubo (1969), it is enough to prove the proposition in the following two cases: *Case* 1 the flow $\{\psi_c\}_{c>0}$ is an identity flow, that is, $\psi_c(x) = x$ for all $c > 0, x \in X$, and *Case* 2 the flow $\{\psi_c\}_{c>0}$ is a special flow as described above with the function p satisfying $p(y) \geq \theta$ for some fixed $\theta > 1$.

Case 1. If the flow $\{\psi_c\}_{c>0}$ is the identity, then the cocycle B_c in (1.5) must satisfy $B_c(x) = 1$ for all $c > 0, x \in X$ (see, for example, Lemma 3.2 in Pipiras and Taqqu, 2002*b*). Relation (2.1) becomes

$$J_{c_1 c_2}(x) = J_{c_1}(x) + J_{c_2}(\psi_{c_1}(x)) \quad \text{a.e. } \mu(dx),$$

for all $c_1, c_2 > 0$, which shows that the almost semi-additive functional $\{J_c\}_{c>0}$ is also an almost cocycle taking values in $\mathbb{R}$ but with addition as a group operation (it is of the form (1.5) but with a sum instead of a product). Theorem B.9 in Zimmer (1984) implies that $\{J_c\}_{c>0}$ has a version which is a cocycle and hence, in our terminology, a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$.

Case 2. Suppose that $\{\psi_c\}_{c>0}$ is a special flow on $Y \times [1, p(\cdot))$ as defined above and hence satisfies (2.3). For notational convenience, we shall write $J(c, (y, u))$ instead of $J_c(y, u)$. Since $J(c, \cdot)$ is a semi-additive functional related to a cocycle B_c, we have that, for any $a, c > 0$,

$$J(ac, (y, u)) = J(a, (y, u)) + B_a(y, u)\, J(c, (V^n y, au/p_n(y)))$$

a.e. for (y, u) such that $p_n(y) \leq au < p_{n+1}(y)$. We can choose $u = u_0 \in (1, \theta)$ and use the Fubini's theorem to conclude that

$$J(ac, (y, u_0)) = J(a, (y, u_0)) + B_a(y, u_0)\, J(c, (V^n y, au_0/p_n(y))) \tag{2.4}$$

a.e. for (a, c, y) such that $p_n(y) \leq au_0 < p_{n+1}(y)$. Setting $n = 0$ and $au_0 = u$ in (2.4), we have

$$\begin{aligned} J(c, (y, u)) = {} & (B_{u/u_0}(y, u_0))^{-1} J(cu/u_0, (y, u_0)) \\ & -(B_{u/u_0}(y, u_0))^{-1} J(u/u_0, (y, u_0)) \end{aligned} \tag{2.5}$$

a.e. for (c, y, u) such that $1 \le u < p(y)$. We shall find expressions for the two J-terms on the right-hand side of (2.5). By using (2.4) with "a" and "c" indicated by the horizontal braces below, we have

$$J(cu/u_0, (y, u_0)) = J(\underbrace{p_n(y)v^{-1}}_{a}\underbrace{cuvu_0^{-1}p_n(y)^{-1}}_{c}, (y, u_0)) = J(p_n(y)/v, (y, u_0))$$
$$+ B_{p_n(y)/v}(y, u_0)J(cuvu_0^{-1}p_n(y)^{-1}, (V^m y, p_n(y)v^{-1}u_0 p_m(y)^{-1}))$$

a.e. for (c, y, u, v) such that

$$p_m(y) \le p_n(y)v^{-1}u_0 < p_{m+1}(y). \tag{2.6}$$

We can take $v = v_0 \in (1, u_0)$ for which the relation above holds a.e. for (c, y, u) so that we have

$$1 < v_0 < u_0 < \theta < p(y). \tag{2.7}$$

The inequality (2.6) implies that, for such v_0, we have $m = n$ and hence

$$\begin{aligned} J(cu/u_0, (y, u_0)) &= J(p_n(y)/v_0, (y, u_0)) \\ &+ B_{p_n(y)/v_0}(y, u_0)J(cuv_0u_0^{-1}p_n(y)^{-1}, (V^n y, u_0/v_0)) \end{aligned} \tag{2.8}$$

a.e. for (c, y, u). For the second term in (2.5), observe that, by (2.4) with $n = 0$,

$$J(\underbrace{uu_0^{-1}}_{c}\underbrace{v_0}_{a}, (y, \underbrace{u_0/v_0}_{u_0})) = J(v_0, (y, u_0/v_0)) + B_{v_0}(y, u_0/v_0)\, J(u/u_0, (y, u_0))$$

and hence

$$\begin{aligned} &J(u/u_0, (y, u_0)) \\ &= (B_{v_0}(y, u_0/v_0))^{-1}\Big(J(uu_0^{-1}v_0, (y, u_0/v_0)) - J(v_0, (y, u_0/v_0))\Big) \end{aligned} \tag{2.9}$$

a.e. for (y, u). Substituting (2.8) and (2.9) into (2.5), we obtain that

$$
\begin{aligned}
&J(c,(y,u))\\
&= (B_{u/u_0}(y,u_0))^{-1}B_{p_n(y)/v_0}(y,u_0)J(cuv_0u_0^{-1}p_n(y)^{-1},(V^n y,u_0/v_0))\\
&\quad - (B_{u/u_0}(y,u_0))^{-1}(B_{v_0}(y,u_0/v_0))^{-1}J(uv_0/u_0,(y,u_0/v_0))\\
&\quad + (B_{u/u_0}(y,u_0))^{-1}J(p_n(y)/v_0,(y,u_0))\\
&\quad + (B_{u/u_0}(y,u_0))^{-1}(B_{v_0}(y,u_0/v_0))^{-1}J(v_0,(y,u_0/v_0))\\
&=: J_1(c,(y,u)) + J_2(c,(y,u))
\end{aligned}
\tag{2.10}
$$

a.e. for (c,y,u), where J_1 and J_2 consist, respectively, of the first and last two terms in the sum (2.10). Since (2.8) holds for all $n \in \mathbb{Z}$, observe that the relation (2.10) does not depend on $n \in \mathbb{Z}$. We may therefore define $J_1(c,(y,u))$ and $J_2(c,(y,u))$ as above for $p_n(y) \le cu < p_{n+1}(y)$, $n \in \mathbb{Z}$.

By using the cocycle equation (1.5),

$$
\begin{aligned}
&(B_{u/u_0}(y,u_0))^{-1}B_{p_n(y)/v_0}(y,u_0)B_{cup_n(y)^{-1}u_0^{-1}}(V^n y,u_0)B_{v_0}(V^n y,u_0/v_0)\\
&= (B_{u/u_0}(y,u_0))^{-1}B_{p_n(y)}(y,u_0)B_{cup_n(y)^{-1}u_0^{-1}}(V^n y,u_0)\\
&= (B_{u/u_0}(y,u_0))^{-1}B_{cu/u_0}(y,u_0) = B_c(y,u),
\end{aligned}
$$

since $(y,u) \in Y \times [1,p(\cdot))$. Then, we have

$$
J_1(c,(y,u)) = B_c(y,u)\, J^*(\psi_c(y,u)) - J^*(y,u), \tag{2.11}
$$

where

$$
J^*(y,u) = (B_{u/u_0}(y,u_0))^{-1}(B_{v_0}(y,u_0/v_0))^{-1}J(uv_0/u_0,(y,u_0/v_0)).
$$

Example 2.1 shows that $J_1(c,\cdot)$, given by (2.11), is a semi-additive functional related to the cocycle B_c. We will now show that $J_2(c,\cdot)$ can be modified to a semi-additive functional related to the cocycle B_c.

It follows from the definition of the special flow that $\psi_c(y,u) = (y,cu)$ for $1 \le cu < p(y)$, and thus $\psi_{1/u_0}(y,u_0) = (y,1)$ and $\psi_{v_0}(y,u_0/v_0) = (y,u_0)$ for all $1 \le u_0 < p(y)$. Using the cocycle relation (1.5), we get $B_{u/u_0}(y,u_0) = B_{1/u_0}(y,u_0)B_u(y,1)$ and $B_{v_0/u_0}(y,u_0/v_0) = B_{v_0}(y,u_0/v_0)B_{1/u_0}(y,u_0)$.

Hence,

$$
\begin{aligned}
(2.12)\qquad & J_2(c,(y,u)) = (B_u(y,1))^{-1}\Big\{(B_{1/u_0}(y,u_0))^{-1}J(p_n(y)/v_0,(y,u_0)) \\
& +(B_{v_0/u_0}(y,u_0/v_0))^{-1}J(v_0,(y,u_0/v_0))\Big\} =: (B_u(y,1))^{-1}G_n(y).
\end{aligned}
$$

By Lemma 2.1 below,

$$
G_{n+m}(y) = G_n(y) + B_{p_n(y)}(y,1)\,G_m(V^n y) \quad \text{a.e. for } y. \tag{2.13}
$$

It follows that

$$
G_n(y) = \widetilde{G}_n(y) \text{ a.e. for } y,
$$

where

$$
\widetilde{G}_n(y) = \sum_{k\in[0,n)} B_{p_k(y)}(y,1)\,G_1(V^k y) \tag{2.14}
$$

and $\sum_{k\in[0,n)} = -\sum_{k\in[n,0)}$ for $n<0$. This can be checked by using

$$
\begin{aligned}
B_{p_{n+k}(y)}(y,1) =& B_{p_n(y)}(y,1)B_{p_k(V^n y)}(\psi_{p_n(y)}(y,1)) \\
&= B_{p_n(y)}(y,1)B_{p_k(V^n y)}(V^n y,1)
\end{aligned}
$$

to verify that $\widetilde{G}_n$ satisfies (2.13). By Lemma 2.2 below, the function

$$
\widetilde{J}_2(c,(y,u)) = (B_u(y,1))^{-1}\widetilde{G}_n(y), \tag{2.15}
$$

for $p_n(y) \le cu < p_{n+1}(y)$, is a semi-additive functional related to the cocycle B_c.

Since both J_1 and $\widetilde{J}_2$ are semi-additive functionals related to the cocycle B_c, so is the sum

$$
\widetilde{J}(c,(y,u)) = J_1(c,(y,u)) + \widetilde{J}_2(c,(y,u))
$$

and, since $J_2(c,(y,u)) = \widetilde{J}_2(c,(y,u))$ a.e. (c,y,u), we have

$$
J(c,(y,u)) = \widetilde{J}(c,(y,u)) \tag{2.16}
$$

a.e. (c,y,u).

To show that $\{\widetilde{J}(c,\cdot)\}_{c>0}$ is a version of $\{J(c,\cdot)\}_{c>0}$, it is enough to show that (2.16) holds also for all $c>0$, a.e. (y,u). The argument is standard. Set $\delta(c,(y,u)) = J(c,(y,u)) - \widetilde{J}(c,(y,u))$, $\Omega_c = \{(y,u) : \delta(c,(y,u)) = 0\}$ and also $\Omega_{a,c} = \{(y,u) : \delta(ac,(y,u)) = \delta(a,(y,u)) + B_a(y,u)\delta(c,\psi_a(y,u))\}$. Denoting the Lebesgue measure on $(0,\infty)$ by $\mathbb{L}$, we have $(\tau\otimes\mathbb{L})(\Omega_{a,c}^c) = 0$ for all $a,c>0$ but only $(\tau\otimes\mathbb{L})(\Omega_c^c) = 0$ a.e. for $c>0$. However, if $r>0$, then there are $a,c>0$ such that $ac=r$ and $(\tau\otimes\mathbb{L})(\Omega_a^c) = (\tau\otimes\mathbb{L})(\Omega_c^c) = 0$. We also have $(\tau\otimes\mathbb{L})((\psi_{1/a}\Omega_c)^c) = 0$ since ψ_a is one-to-one and onto. Then, $(\tau\otimes\mathbb{L})((\Omega_{a,c}\cap\Omega_a\cap\psi_{1/a}\Omega_c)^c) = 0$ and, for $(y,u)\in\Omega_{a,c}\cap\Omega_a\cap\psi_{1/a}\Omega_c$, we have $\delta(r,(y,u)) = \delta(a,(y,u)) + B_a(y,u)\delta(c,\psi_a(y,u)) = 0$. This shows that $(\tau\otimes\mathbb{L})\{J(r,(y,u)) \neq \widetilde{J}(r,(y,u)\} = 0$ for any $r>0$, that is, $\{\widetilde{J}(c,\cdot)\}_{c>0}$ is a version of $\{J(c,\cdot)\}_{c>0}$. ∎

The next corollary provides insight into the structure of semi-additive functionals related to a cocycle. Corollary 2.1 and the remark following it will be used in Propositions 5.1 and 5.2 below to deduce the forms of the *1*- and *2*-semi-additive functionals corresponding to a cyclic flow.

Corollary 2.1. *If the flow $\{\psi_c\}_{c>0}$ is given by its special representation* (2.3) *on a space $Y\times[1,p(\cdot))$, with the maps V and p, and $\{J_c\}_{c>0}$ is a semi-additive functional related to a cocycle $\{B_c\}_{c>0}$, then*

$$J_c(y,u) = J_c^{(1)}(y,u) + J_c^{(2)}(y,u), \tag{2.17}$$

where

$$J_c^{(1)}(y,u) \;=\; B_c(y,u)\,J(\psi_c(y,u)) - J(y,u), \tag{2.18}$$

$$J_c^{(2)}(y,u) \;=\; (B_u(y,1))^{-1}\sum_{k\in[0,n)} B_{p_k(y)}(y,1)J_1(V^k y), \tag{2.19}$$

for $p_n(y)\le cu < p_{n+1}(y)$, J, J_1 are some functions and $\sum_{k\in[0,n)} = -\sum_{k\in[n,0)}$ for $n<0$. Moreover, each of the functionals $\{J_c^{(1)}\}_{c>0}$ and $\{J_c^{(2)}\}_{c>0}$ is a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$.

Proof. The corollary follows from the proof of Theorem 2.1 by replacing "a.e." by "for all" conditions. See, in particular, (2.10), (2.11) and (2.12) together with (2.14). ∎

Remark. It is sometimes convenient to represent a special flow $\{\psi_c\}_{c>0}$ on a different space $Y \times [0, r(\cdot))$ as

$$\psi_c(y, w) = (V^n y, w + \ln c - r_n(y)),$$
$$\text{for } 0 \le w + \ln c - r_n(y) < r(V^n y), \tag{2.20}$$

where r is a positive function, $r_n(y) = \sum_{k=0}^{n-1} r(V^k y)$ if $n \ge 1$, $r_n(y) = 0$ if $n = 0$, and $r_n(y) = -\sum_{k=n}^{-1} r(V^k y)$ if $n \le -1$. (Here, $\sum_{k=-\infty}^{-1} r(V^k y) = \sum_{k=0}^{\infty} r(V^k y) = \infty$.) For a special flow $\{\psi_c\}_{c>0}$ having the representation (2.20), one can easily show that a semi-additive functional $\{J_c\}_{c>0}$ related to a cocycle $\{B_c\}_{c>0}$, has the form (2.17) with u replaced by w, where

$$J_c^{(1)}(y, w) = B_c(y, w)\, J(\psi_c(y, w)) - J(y, w),$$

$$J_c^{(2)}(y, w) = (B_{e^w}(y, 0))^{-1} \sum_{k \in [0,n)} B_{e^{r_k(y)}}(y, 0) J_1(V^k y),$$

for some functions J, J_1.

The following two auxiliary lemmas were used in the proof of Theorem 2.1.

Lemma 2.1. *Let $G_n(y)$ be defined by* (2.12). *Then, for $n, m \in \mathbb{Z}$,*

$$G_{n+m}(y) = G_n(y) + B_{p_n(y)}(y, 1) G_m(V^n y) \ \ a.e.\ y. \tag{2.21}$$

Proof. Recall from (2.12) that

$$G_n(y) = (B_{1/u_0}(y, u_0))^{-1} J(p_n(y)/v_0, (y, u_0))$$
$$+ (B_{v_0/u_0}(y, u_0/v_0))^{-1} J(v_0, (y, u_0/v_0)) \tag{2.22}$$

where u_0 and v_0 satisfy (2.7). To show (2.21), we shall use the relation

$$p_{n+m}(y) = p_n(y) p_m(V^n y), \tag{2.23}$$

which is easy to verify by using the definition of $p_n(y)$. Observe that, for $n, m \in \mathbb{Z}$, by using (2.23) and (2.4),

(2.24)

$$J(p_{n+m}(y)/v_0,(y,u_0)) = J(\underbrace{p_n(y)v_0^{-1}}_{a}\underbrace{p_m(V^n y)}_{c},(y,u_0))$$

$$= J(p_n(y)/v_0,(y,u_0)) + B_{p_n(y)/v_0}(y,u_0)\, J(p_m(V^n y),(V^n y,u_0/v_0))$$

$$= J(p_n(y)/v_0,(y,u_0)) + B_{p_n(y)/v_0}(y,u_0)\, J(\underbrace{v_0}_{a}\underbrace{p_m(V^n y)v_0^{-1}}_{c},(V^n y,u_0/v_0))$$

$$= J(p_n(y)/v_0,(y,u_0)) + B_{p_n(y)/v_0}(y,u_0)\, J(v_0,(V^n y,u_0/v_0))$$

$$+B_{p_n(y)/v_0}(y,u_0)B_{v_0}(V^n y,u_0/v_0)\, J(p_m(V^n y)/v_0,(V^n y,u_0)).$$

To compute G_{n+m} we use (2.22) and substitute (2.24) for $J(p_{n+m}(y)/v_0,$ $(y,u_0))$. This yields

(2.25)

$$G_{n+m}(y) = (B_{1/u_0}(y,u_0))^{-1}J(p_n(y)/v_0,(y,u_0))$$

$$+(B_{v_0/u_0}(y,u_0/v_0))^{-1}J(v_0,(y,u_0/v_0))$$

$$+(B_{1/u_0}(y,u_0))^{-1}B_{p_n(y)/v_0}(y,u_0)B_{v_0}(V^n y,u_0/v_0)J(p_m(V^n y)/v_0,(V^n y,u_0))$$

$$+(B_{1/u_0}(y,u_0))^{-1}B_{p_n(y)/v_0}(y,u_0)\, J(v_0,(V^n y,u_0/v_0))$$

$$= G_n(y) + (B_{1/u_0}(y,u_0))^{-1}B_{p_n(y)/v_0}(y,u_0)B_{v_0}(V^n y,u_0/v_0)B_{1/u_0}(V^n y,u_0)\cdot$$

$$\cdot(B_{1/u_0}(V^n y,u_0))^{-1}J(p_m(V^n y)/v_0,(V^n y,u_0))$$

$$+(B_{1/u_0}(y,u_0))^{-1}B_{p_n(y)/v_0}(y,u_0)B_{v_0/u_0}(V^n y,u_0/v_0)\cdot$$

$$\cdot(B_{v_0/u_0}(V^n y,u_0/v_0))^{-1}J(v_0,(V^n y,u_0/v_0))$$

$$= G_n(y) + (B_{1/u_0}(y,u_0))^{-1}B_{p_n(y)/v_0}(y,u_0)B_{v_0/u_0}(V^n y,u_0/v_0)\, G_m(V^n y)$$

a.e. y, where to obtain the last identity, we used the relation $B_{v_0}(V^n y,u_0/v_0)B_{1/u_0}(V^n y,u_0) = B_{v_0/u_0}(V^n y,u_0/v_0)$. Since $B_{1/u_0}(y,u_0)$ $B_{u_0}(y,1) = B_{u_0/u_0}(y,1) = B_0(y,1) = 1$, we get $(B_{1/u_0}(y,u_0))^{-1} = B_{u_0}(y,1)$ and, since $B_{u_0}(y,1)B_{p_n(y)/v_0}(y,u_0)B_{v_0/u_0}(V^n y,u_0/v_0) = B_{p_n(y)v_0^{-1}u_0}(y,1)$ $B_{v_0/u_0}(V^n y,u_0/v_0) = B_{p_n(y)}(y,1)$, we see that (2.25) reduces to (2.21). ■

Lemma 2.2. *If $J_c(y,u)$ is defined by the right-hand side of* (2.15), *together with* (2.14), *then $J_c(y,u)$ is a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$.*

Proof. Fix $a, c > 0$ and $(y,u) \in Y \times [1, p(\cdot))$. We are interested in $J_a(y,u) = (B_u(y,1))^{-1}\widetilde{G}_n(y)$, where n is such that $p_n(y) \le au < p_{n+1}(y)$, and in $J_{ac}(y,u) = (B_u(y,1))^{-1}\widetilde{G}_{n+m}(y)$, where, in addition, m is such that $p_{n+m}(y) \le acu < p_{n+m+1}(y)$, an inequality equivalent to $p_m(V^n y) \le acu/p_n(y) < p_{m+1}(V^n y)$ by (2.23). Then, by using the fact that $\widetilde{G}_n$ satisfies (2.21), we obtain that

$$B_a(y,u)J_c(\psi_a(y,u)) = B_a(y,u)J_c(V^n y, au/p_n(y))$$

$$= B_a(y,u)(B_{au/p_n(y)}(V^n y,1))^{-1}\widetilde{G}_m(V^n y)$$

$$= B_a(y,u)(B_{au/p_n(y)}(V^n y,1))^{-1}(B_{p_n(y)}(y,1))^{-1}\Big(\widetilde{G}_{n+m}(y) - \widetilde{G}_n(y)\Big)$$

$$= B_a(y,u)(B_{au}(y,1))^{-1}\Big(\widetilde{G}_{n+m}(y) - \widetilde{G}_n(y)\Big)$$

$$= (B_u(y,1))^{-1}\Big(\widetilde{G}_{n+m}(y) - \widetilde{G}_n(y)\Big) = J_{ac}(y,u) - J_a(y,u).$$

This concludes the proof. ∎

3. 2-SEMI-ADDITIVE FUNCTIONALS

In this section, we apply Theorem 2.1 to show that the function $j_c(x)$ in the relation (1.3) can be chosen, without loss of generality, as a *2*-semi-additive functional satisfying (1.7). Observe first that because of the properties of the Radon-Nikodym derivatives, one has for all $c_1, c_2 > 0$,

$$\begin{aligned} \frac{d(\mu\circ\psi_{c_1c_2})}{d\mu}(x) &= \frac{d(\mu\circ\psi_{c_1})}{d\mu}(x)\frac{d(\mu\circ\psi_{c_2}\circ\psi_{c_1})}{d(\mu\circ\psi_{c_1})}(x) \\ &= \frac{d(\mu\circ\psi_{c_1})}{d\mu}(x)\frac{d(\mu\circ\psi_{c_2})}{d\mu}(\psi_{c_1}(x)) \end{aligned} \tag{3.1}$$

a.e. $\mu(dx)$, which is the relation (1.5) defining a cocycle but valid only a.e. $\mu(dx)$ and not for all $x \in X$. We start with the following lemma which shows that the collection of the Radon-Nikodym derivatives $\{d(\mu\circ\psi_c)/d\mu\}_{c>0}$ has a version which is a cocycle for all $x \in X$.

Lemma 3.1. *Suppose that the relations* (1.3) *and* (1.2) *hold. Then, the Radon-Nikodym derivatives* $\{d(\mu\circ\psi_c)/d\mu\}_{c>0}$ *have a version which is*

1) *jointly measurable in* (c,x),
2) *a cocycle mapping* $(0,\infty)\times X\to(0,\infty)$,
3) *a Radon-Nikodym derivative* $d(\mu\circ\psi_c)/d\mu$ *for all* $c>0$.

Proof. We first show that the collection $\{d(\mu\circ\psi_c)/d\mu\}_{c>0}$ has a jointly measurable version. By using the notation (1.1), relation (1.3) implies that, for any $t\in\mathbb{R}$ and $c>0$,

$$G_{ct}(x,cu)=c^{H-1/\alpha}b_c(x)\left\{\frac{d(\mu\circ\psi_c)}{d\mu}(x)\right\}^{1/\alpha}G_t(\psi_c(x),u+g_c(x)) \tag{3.2}$$

a.e. $\mu(dx)du$.

If $\mathrm{supp}\{G_t(x,u)\}=X\times\mathbb{R}$ a.e. $\mu(dx)du$ for some fixed $t\in\mathbb{R}$, we have for $c>0$,

$$\frac{d(\mu\circ\psi_c)}{d\mu}(x)=\left\{\frac{c^{1/\alpha-H}G_{ct}(x,cu)}{b_c(x)G_t(\psi_c(x),u+g_c(x))}\right\}^{\alpha}$$

a.e. $\mu(dx)du$. Hence, since the right-hand side of the expression above is jointly measurable, we may conclude that $\{d(\mu\circ\psi_c)/d\mu\}_{c>0}$ has a jointly measurable version.

Consider now the general case when $\mathrm{supp}\{G_t(x,u)\}=X\times\mathbb{R}$ a.e. $\mu(dx)du$ may possibly not hold for any $t\in\mathbb{R}$. Let $\mathcal{G}=\mathrm{Sp}\{G_t,t\in\mathbb{R}\}$ be the linear span of G_t, $t\in\mathbb{R}$, and $\overline{\mathcal{G}}$ be the closure of $\mathcal{G}$ in the space $L^\alpha(X\times\mathbb{R},\mu(dx)du)$. Since $\{G_t,t\in\mathbb{R}\}\subset\overline{\mathcal{G}}$, the assumption (1.2) implies that $\mathrm{supp}\{\overline{\mathcal{G}}\}=X\times\mathbb{R}$ a.e. $\mu(dx)du$. By Lemma 3.2 in Hardin (1981), there is a function $G^*\in\overline{\mathcal{G}}$ such that $\mathrm{supp}\{G^*(x,u)\}=X\times\mathbb{R}$ a.e. $\mu(dx)du$. Since $G^*\in\overline{\mathcal{G}}$, there are functions $G^{(n)}(x,u)=\sum_i a_{ni}G_{t_{ni}}(x,u)\in\mathcal{G}$, $n\ge1$, $a_{ni},t_{ni}\in\mathbb{R}$, such that $G^{(n)}(x,u)\to G^*(x,u)$ a.e. $\mu(dx)du$.

Let also $G_c^{(n)}(x,u)=\sum_i a_{ni}G_{ct_{ni}}(x,cu)$. Relation (3.2) implies that, for any $c>0$ and $n\ge1$,

$$G_c^{(n)}(x,u)=c^{H-1/\alpha}b_c(x)\left\{\frac{d(\mu\circ\psi_c)}{d\mu}(x)\right\}^{1/\alpha}G^{(n)}(\psi_c(x),u+g_c(x)) \tag{3.3}$$

a.e. $\mu(dx)du$. For any $c > 0$, the right-hand side of (3.3) converges to

$$c^{H-1/\alpha} b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha} G^*(\psi_c(x), u + g_c(x))$$

a.e. $\mu(dx)du$, as $n \to \infty$. Since the right-hand side of (3.3) converges, the left-hand side of (3.3) converges to some function $G_c^*(x,u)$. Hence, for any $c > 0$,

$$G_c^*(x,u) = c^{H-1/\alpha} b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha} G^*(\psi_c(x), u + g_c(x))$$

or, since $\mathrm{supp}\{G^*\} = X \times \mathbb{R}$ a.e.,

$$\frac{d(\mu \circ \psi_c)}{d\mu}(x) = \left\{ \frac{c^{1/\alpha - H} G_c^*(x,u)}{b_c(x) G^*(\psi_c(x), u + g_c(x))} \right\}^{\alpha} \tag{3.4}$$

a.e. $\mu(dx)du$. Observe that $G_c^*(x,u)$ is jointly measurable in (c,x,u) because it is the a.e. limit of functions jointly measurable in (c,x,u). Since $G^*(x,u)$ is measurable in (x,u), $\psi_c(x), g_c(x)$ and $b_c(x)$ are measurable in (c,x), the function $c^{H-1/\alpha} b_c(x)^{1/\alpha} G^*(\psi_c(x), u+g_c(x))$ is measurable in (c,x,u). Hence, the right-hand side of (3.4) is jointly measurable which is to say that $\{d(\mu \circ \psi_c)/d\mu\}_{c>0}$ has a jointly measurable version.

Suppose then, without loss of generality, that $d(\mu \circ \psi_c)/d\mu(x)$ is jointly measurable. We still need to show that $\{d(\mu \circ \psi_c)/d\mu\}_{c>0}$ has a version which is a cocycle. Since the flow $\{\psi_c\}_{c>0}$ is nonsingular, the measures $\mu \circ \psi_c$ and μ are equivalent and hence we may suppose that $(d(\mu \circ \psi_c)/d\mu)(x) : (0,\infty) \times X \to \mathbb{R} \setminus \{0\}$. By (3.1), $\{d(\mu \circ \psi_c)/d\mu\}_{c>0}$ is an *almost cocycle* for the flow $\{\psi_c\}_{c>0}$ where "almost" refers to the fact that the relation (3.1) holds a.e. $\mu(dx)$ for $c_1, c_2 > 0$, in contrast to (1.5) which holds for all $x \in X$ and $c_1, c_2 > 0$. By Theorem B.9 in Zimmer (1984) (see also Theorem A.1 in Kolodyński and Rosiński, 2000) and since $(d(\mu \circ \psi_c)/d\mu)(x)$ is measurable in (c,x), $\{d(\mu \circ \psi_c)/d\mu\}_{c>0}$ has a version which is a cocycle for the flow $\{\psi_c\}_{c>0}$ taking values in $(0,\infty)$. Property 3) follows from the definition of "version". ∎

Remark. The version specified in Lemma 3.1 which satisfies Conditions 1), 2) and 3) is not unique. Suppose, for instance, that $X = \mathbb{R}^2 = \{(x_1, x_2)\}$, $\mu(dx) = dx_1 dx_2$ and $\psi_c(x_1, x_2) = (x_1, x_2 + \ln c)$. Then,

$$\frac{d(\mu \circ \psi_c)}{d\mu}(x_1, x_2) \equiv 1$$

is a version of the Radon-Nikodym derivatives satisfying Conditions 1), 2) and 3). On the other hand, let $b : \mathbb{R} \mapsto (0, \infty)$ be an arbitrary function and $x_1^* \in \mathbb{R}$ be fixed. Then,

$$\frac{d(\mu \circ \psi_c)}{d\mu}(x_1, x_2) = \begin{cases} 1, & x_1 \neq x_1^*, \\ \dfrac{b(x_2 + \ln c)}{b(x_2)}, & x_1 = x_1^*, \end{cases}$$

is also a version of the Radon-Nikodym derivatives satisfying Conditions 1), 2) and 3). Indeed, it is jointly measurable and also, for fixed $c > 0$, it is still a Radon-Nikodym derivative since it was modified on the set $\{(x_1, x_2) : x_1 = x_1^*\}$ of a μ-measure zero. It satisfies a cocycle equation for all $x \in \mathbb{R}^2$, $c_1, c_2 > 0$, because it does so on the disjoint subsets $\{(x_1, x_2) : x_1 \neq x_1^*\}$ and $\{(x_1, x_2) : x_1 = x_1^*\}$ of $\mathbb{R}^2$ which are invariant under the flow. Observe that the two versions of the Radon-Nikodym derivatives above are different when $b \not\equiv 1$.

We can now give a precise definition of *2*-semi-additive functional.

Definition 3.1. A measurable function $j_c(x) : (0, \infty) \times X \to \mathbb{R}$ is a *2-semi-additive functional* for a flow $\{\psi_c\}_{c>0}$ and a cocycle $\{b_c\}_{c>0}$ if the relation (1.7) holds, where $d(\mu \circ \psi_c)/d\mu$ satisfies Conditions 1), 2) and 3) of Lemma 3.1. A semi-additive functional $\{g_c\}_{c>0}$ satisfying (1.6) is called a *1-semi-additive functional*.

In the following examples, we show that after multiplication by a suitable factor, the *1*-semi-additive functional $\{g_c\}_{c>0}$ in (1.6) and the *2*-semi-additive functionals $\{j_c\}_{c>0}$ in (1.7) become semi-additive functionals related to a cocycle, and we identify these cocycles.

Example 3.1. If $\{g_c\}_{c>0}$ is a *1*-semi-additive functional satisfying (1.6), then $J_c(x) = c g_c(x)$ satisfies

$$J_{c_1 c_2}(x) = J_{c_1}(x) + c_1 J_{c_2}(\psi_{c_1}(x)), \quad \text{for all } c_1, c_2 > 0, x \in X.$$

But $B_c(x) = c$ is a cocycle for the flow $\{\psi_c\}_{c>0}$ (it satisfies (1.5)). Therefore, $\{J_c\}_{c>0}$ is a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$.

Example 3.2. If $\{j_c\}_{c>0}$ is a *2*-semi-additive functional satisfying (1.7), then $J_c(x) = c^{H-1/\alpha} j_c(x)$ satisfies

$$J_{c_1c_2}(x) = J_{c_1}(x) + B_{c_1}(x) J_{c_2}(\psi_{c_1}(x)), \quad \text{for all } c_1, c_2 > 0, x \in X$$

with

$$B_c(x) = c^{H-1/\alpha} b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha}. \tag{3.5}$$

Since $\{b_c\}_{c>0}$ is a cocycle taking values in $\{-1, 1\}$ and $\{d(\mu \circ \psi_c)/d\mu\}_{c>0}$ is a cocycle taking values in $\mathbb{R} \setminus \{0\}$, it is easy to check that $\{B_c\}_{c>0}$ is also a cocycle taking values in $\mathbb{R} \setminus \{0\}$. Thus, $\{J_c\}_{c>0}$ is a semi-additive functional related to the cocycle (3.5).

Remark. The preceding examples can be used in the following way. Suppose that $\{g_c\}_{c>0}$ and $\{j_c\}_{c>0}$ are only *almost* semi-additive functionals. Then $\{J_c\}_{c>0}$ would also be almost semi-additive functionals. Since Theorem 2.1 applies to $\{J_c\}_{c>0}$, these have a version which is a semi-additive functional. In view of the expressions relating J_c to g_c and j_c, it follows that $\{g_c\}_{c>0}$ and $\{j_c\}_{c>0}$ have also a version which is a semi-additive functional. This type of argument is used in the proof of the following theorem.

Theorem 3.1. *The function $j_c(x)$ in relation* (1.3) *can be taken to be a 2-semi-additive functional.*

Proof. We need first to show that the function $j_c(x)$ in (1.3) is an almost semi-additive functional. Observe that it equals

$$j_c(x) = c^{-(H-1/\alpha)} G(x, cu) - \widetilde{b}_c(x)\, G(\psi_c(x), u + g_c(x)), \tag{3.6}$$

a.e. $\mu(dx)du$, for any $c > 0$, where

$$\widetilde{b}_c(x) = b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha}. \tag{3.7}$$

By Lemma 3.1 above, the Radon-Nikodym derivative $d(\mu\circ\psi_c)/d\mu$ and hence its $1/\alpha$-power have a version which is a cocycle taking values in $(0,\infty)$. Since $b_c(x)$ is a cocycle, the product $\widetilde{b}_c(x)$ also has a version which is a cocycle taking values in $\mathbb{R}\backslash\{0\}$. We may therefore suppose without loss of generality that $\{\widetilde{b}_c\}_{c>0}$ is a cocycle in (3.6), that is,

$$\widetilde{b}_{c_1c_2}(x) = \widetilde{b}_{c_1}(x)\widetilde{b}_{c_2}(\psi_{c_1}(x)). \tag{3.8}$$

By using (3.6), we have for $c_1, c_2 > 0$,

$$j_{c_1c_2}(x) = (c_1c_2)^{-(H-1/\alpha)}G(x, c_1c_2u) - \widetilde{b}_{c_1c_2}(x)\, G(\psi_{c_1c_2}(x), u + g_{c_1c_2}(x))$$

a.e. $\mu(dx)du$. By using (3.8) and $g_{c_1c_2}(x) = c_2^{-1}g_{c_1}(x) + g_{c_2}(\psi_{c_1}(x))$, we conclude that

(3.9)
$$\begin{aligned}
&j_{c_1c_2}(x)\\
&= c_2^{-(H-1/\alpha)}\Big\{c_1^{-(H-1/\alpha)}G(x, c_1(c_2u)) - \widetilde{b}_{c_1}(x)\, G(\psi_{c_1}(x), c_2u + g_{c_1}(x))\Big\}\\
&\quad + \widetilde{b}_{c_1}(x)\Big\{c_2^{-(H-1/\alpha)}G\Big(\psi_{c_1}(x), c_2(u + c_2^{-1}g_{c_1}(x))\Big)\\
&\qquad - \widetilde{b}_{c_2}(\psi_{c_1}(x))G\Big(\psi_{c_2}(\psi_{c_1}(x)), u + c_2^{-1}g_{c_1}(x) + g_{c_2}(\psi_{c_1}(x))\Big)\Big\}\\
&= c_2^{-(H-1/\alpha)}j_{c_1}(x) + \widetilde{b}_{c_1}(x)j_{c_2}(\psi_{c_1}(x))
\end{aligned}$$

a.e. $\mu(dx)$. Hence, $\{j_c\}_{c>0}$ is an almost semi-additive functional.

Multiplying (3.9) by $(c_1c_2)^{H-1/\alpha}$ and setting $J_c(x) = c^{H-1/\alpha}j_c(x)$, $B_c(x) = c^{H-1/\alpha}\widetilde{b}_c(x)$, we obtain that

$$J_{c_1,c_2}(x) = J_{c_1}(x) + B_{c_1}(x)J_{c_2}(\psi_{c_1}(x)) \quad \text{a.e. } \mu(dx). \tag{3.10}$$

Since $\{B_c\}_{c>0}$ is also a cocycle for the flow $\{\psi_c\}_{c>0}$, relation (3.10) shows that $\{J_c\}_{c>0}$ is an almost semi-additive functional related to the cocycle $\{B_c\}_{c>0}$ in the sense of Definition 2.1 above. By Theorem 2.1, $\{J_c\}_{c>0}$ has a version which is a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$. But $j_c(x) = c^{-(H-1/\alpha)}J_c(x)$. Hence, when multiplied by $c^{-(H-1/\alpha)}$, this version is a *2*-semi-additive functional which is a version of $\{j_c\}_{c>0}$. ■

The corresponding result for g_c was proved in Proposition 3.1 of Pipiras and Taqqu (2002*a*), pages 421–426. (In that proposition, "semi-additive functional" means "*1*-semi-additive-functional".) The proof of that Proposition 3.1 also follows from the more general setting of the present paper. In fact, it reduces, at this stage, to the argument in Example 3.1 and the remark preceding the statement of Theorem 3.1.

Theorem 3.1 is useful when solving for the kernel function G generated by a given flow. One such case, studied in Proposition 3.1 of Pipiras and Taqqu (2007), concerns kernel functions related to cyclic flows. *2*-semi-additive functionals could have also been used in Theorem 5.1 of Pipiras and Taqqu (2002*b*), where kernels related to identity flows of Example 4.1 below are considered. In particular, the argument used in Theorem 5.1 of that paper involving an almost everywhere version of the Cauchy functional equation would not be necessary anymore. We explain this in greater detail in the remark following Example 4.3 below.

4. Examples

In the following examples, we consider *1*- and *2*-semi-additive functionals $\{j_c\}_{c>0}$ for identity, dissipative and cyclic flows $\{\psi_c\}_{c>0}$, and related cocycles $\{b_c\}_{c>0}$.

Example 4.1. Consider the *identity* flow $\{\psi_c\}_{c>0}$ on (X, μ) such that

$$\psi_c(x) = x \tag{4.1}$$

for all $c > 0$ and $x \in X$. We can take

$$\frac{d(\mu \circ \psi_c)}{d\mu}(x) = \frac{d\mu}{d\mu}(x) \equiv 1 \tag{4.2}$$

as a cocycle for the identity flow $\{\psi_c\}_{c>0}$. By Lemma 3.2 in Pipiras and Taqqu (2002*b*),

$$b_c(x) = 1 \tag{4.3}$$

for the identity flow $\{\psi_c\}_{c>0}$. The *2*-semi-additive functional $\{j_c\}_{c>0}$ in (1.7) therefore satisfies

$$j_{c_1 c_2}(x) = c_2^{-(H-1/\alpha)} j_{c_1}(x) + j_{c_2}(x), \tag{4.4}$$

for all $x \in X$, $c_1, c_2 > 0$. Relation (4.4) is an equation for the functional $j_c(x)$, which we shall now solve.

If $H \neq 1/\alpha$, by subtracting $j_{c_1c_2}(x) = c_1^{-(H-1/\alpha)} j_{c_2}(x) + j_{c_1}(x)$ from (4.4), we obtain that

$$(1 - c_2^{-(H-1/\alpha)}) j_{c_1}(x) = (1 - c_1^{-(H-1/\alpha)}) j_{c_2}(x).$$

By fixing $c_2 \neq 1$, we conclude that

$$j_c(x) = j(x)(1 - c^{-(H-1/\alpha)}),$$

where $j(x)$ is some function. If $H = 1/\alpha$, then

$$j_{c_1c_2}(x) = j_{c_2}(x) + j_{c_1}(x)$$

and by using Lemma 1.1.6 in Bingham *et al.,* (1987), we have

$$j_c(x) = j(x) \ln c,$$

where $j(x)$ is some function.

Example 4.2. Consider the flow

$$\psi_c(y, u) = (y, u + \ln c) \tag{4.5}$$

on the space $(X, \mu) = (Y \times \mathbb{R}, \nu(dy)du)$. By Krengel's theorem (see, for example, Theorem 3.1 in Pipiras and Taqqu (2002*b*), any *dissipative* flow on (X, μ) is null-isomorphic to a flow $\{\psi_c\}_{c>0}$ of the form (4.5). Let $\mathbb{L}$ denote the Lebesgue measure on $\mathbb{R}$. We can take

$$\frac{d((\nu \otimes \mathbb{L}) \circ \psi_c)}{d(\nu \otimes \mathbb{L})}(y, u) = \frac{d\nu(y)}{d\nu(y)} \frac{d(u + \ln c)}{du} \equiv 1$$

as a cocycle for the flow $\{\psi_c\}_{c>0}$. By Lemma 3.1 in Pipiras and Taqqu 2002*b*), for the dissipative flow $\{\psi_c\}_{c>0}$, the cocycle b_c is given by

$$b_c(y, u) = \frac{b(\psi_c(y, u))}{b(y, u)} \tag{4.6}$$

with some function b taking values in $\{-1,1\}$. Hence, a *2*-semi-additive functional $\{j_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ in (1.7) satisfies

$$j_{c_1c_2}(y,u) = c_2^{-(H-1/\alpha)} j_{c_1}(y,u) + \frac{b(\psi_c(y,u))}{b(y,u)} j_{c_2}(\psi_{c_1}(y,u))$$

for all $(y,u) \in Y \times \mathbb{R}$ and $c_1, c_2 > 0$.

To solve this equation for j_c, set $\widetilde{j}_c(y,u) = b(y,u)j_c(y,u)$ so that

$$\widetilde{j}_{c_1c_2}(y,u) = c_2^{-(H-1/\alpha)} \widetilde{j}_{c_1}(y,u) + \widetilde{j}_{c_2}(y, u + \ln c_1).$$

Substituting $u = 0$ into this relation and setting $\ln c_1 = v$ so that $c_1 = e^v$, $c_2 = c$ and $\widetilde{j}(y,s) = \widetilde{j}_{e^s}(y,0)$, we obtain that $c_1c_2 = e^{v+\ln c}$ and

$$\widetilde{j}(y, v + \ln c) = c^{-(H-1/\alpha)} \widetilde{j}(y,v) + \widetilde{j}_c(y,v).$$

Hence,

$$j_c(y,u) = (b(y,u))^{-1}\widetilde{j}_c(y,u) = \frac{\widetilde{j}(\psi_c(y,u))}{b(y,u)} - c^{-(H-1/\alpha)} \frac{\widetilde{j}(y,u)}{b(y,u)}$$

$$= b_c(y,u) j(\psi_c(y,u)) - c^{-(H-1/\alpha)} j(y,u),$$

by (4.6), where $j(y,u) = \widetilde{j}(y,u)/b(y,u)$ is some function.

Example 4.3. In view of (1.6), a *1*-semi-additive functional $\{g_c\}_{c>0}$ for the identity flow (4.1) satisfies $g_{c_1c_2}(x) = c_2^{-1} g_{c_1}(x) + g_{c_2}(x)$ whose solution is

$$g_c(x) = (c^{-1} - 1) g(x)$$

for some function $g : X \mapsto \mathbb{R}$ (Lemma 3.2 in Pipiras and Taqqu, 2002*b*). The corresponding equation for the dissipative flow (4.5) is $g_{c_1c_2}(y,u) = c_2^{-1} g_{c_1}(y,u) + g_{c_2}(y, u + \ln c_2)$ where solution is

$$g_c(y,u) = g(y, u + \ln c) - c^{-1} g(y,u),$$

for some function $g : Y \times \mathbb{R} \mapsto \mathbb{R}$ (Lemma 3.1 in Pipiras and Taqqu, 2002*b*).

Remark. Consider relation (1.3) where the flow $\{\psi_c\}_{c>0}$ is the identity. Observe that, by using the relations (4.2) and (4.3) and Examples 4.1 and 4.3, the equation (1.3) becomes: for any $c > 0$,

$$c^{-(H-1/\alpha)}G(x, cu) = G\Big(x, u + (c^{-1} - 1)g(x)\Big) + (1 - c^{-(H-1/\alpha)})j(x)$$

a.e. $\mu(dx)du$, when $H \neq 1/\alpha$, and

$$G(x, cu) = G\Big(x, u + (c^{-1} - 1)g(x)\Big) + (\ln c)j(x)$$

a.e. $\mu(dx)du$, when $H = 1/\alpha$. These equations were also obtained in the proof of Theorem 5.1 in Pipiras and Taqqu (2002*b*) and then used to solve for the function G. The arguments in that theorem, leading to the two equations were quite involved but as we see here, the equations follow easily once *2*-semi-additive functionals are used.

Example 4.4. For $v \in \mathbb{R}$ and $a > 0$, let

$$[v]_a = \max\{n \in \mathbb{Z} : na \leq v\}, \quad \{v\}_a = v - a[v]_a. \tag{4.7}$$

By Theorem 2.1 in Pipiras and Taqqu (2004), any *cyclic flow* is null-isomorphic to the flow

$$\psi_c(z, v) = (z, \{v + \ln c\}_{q(z)}) \tag{4.8}$$

on the space $(X, \mu) = (Z \times [0, q(\cdot)), \sigma(dz)dv)$, where $q(z) > 0$ is some function. Observe that $\{v + \ln c\}_{q(z)}$, as a function of v, has the shape of a seesaw. Relation (1.6) for the *1*-semi-additive functional $\{g_c\}_{c>0}$ of the flow (4.8) becomes $g_{c_1c_2}(z, v) = c_2^{-1}g_{c_1}(z, v) + g_{c_2}(z, \{v + \ln c_2\}_{q(z)})$. The solution to this equation, which is given in Proposition 5.1 below, is as follows:

$$g_c(z, v) = g(z, \{v + \ln c\}_{q(z)}) - c^{-1}g(z, v), \tag{4.9}$$

for some function $g : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$.

Example 4.5. We now consider *2*-semi-additive functionals for cyclic flows (4.8). By Lemma 8.2 in Pipiras and Taqqu (2007), the cocycle $b_c(z, v)$ in (1.3) for the flow $\{\psi_c\}_{c>0}$ can be expressed as

$$b_c(z,v) = b_1(z)^{[v+\ln c]_{q(z)}} \frac{b(\psi_c(z,v))}{b(z,v)} \tag{4.10}$$

for some functions $b_1 : Z \mapsto \{-1,1\}$ and $b : Z \times [0,q(\cdot)) \mapsto \{-1,1\}$. The Radon-Nikodym derivatives

$$\frac{d((\sigma \otimes \mathbb{L}) \circ \psi_c)}{d(\sigma \otimes \mathbb{L})}(z,v) \equiv 1$$

because $d\{v + \ln c_2\}_{q(z)}/dv = 1$ for almost all v since $q(z)$ does not affect the slope. These Radon-Nikodym derivatives can be taken as a cocycle for the flow $\{\psi_c\}_{c>0}$. The *2*-semi-additive functional $\{j_c\}_{c>0}$ in (1.7) therefore satisfies

$$j_{c_1c_2}(z,v) = c_2^{-(H-1/\alpha)} j_{c_1}(z,v) + b_1(z)^{[v+\ln c]_{q(z)}} \frac{b(\psi_c(z,v))}{b(z,v)} j_{c_2}(z,v).$$

The solution to this equation, which is given in Proposition 5.2 below, is as follows:

$$\begin{aligned} j_c(z,v) =& b_1(z)^{[v+\ln c]_{q(z)}} \frac{b(\psi_c(z,v))}{b(z,v)} j(\psi_c(z,v)) - c^{-(H-1/\alpha)} j(z,v) \\ &+ \frac{j_1(z)}{b(z,v)} [v + \ln c]_{q(z)} 1_{\{b_1(z)=1\}} 1_{\{H=1/\alpha\}}, \end{aligned} \tag{4.11}$$

for some functions $j_1 : Z \mapsto \mathbb{R}$ and $j : Z \times [0,q(\cdot)) \mapsto \mathbb{R}$.

5. Semi-additive functionals for cyclic flows

In this section, we solve the *1*- and *2*-semi-additive functional equations (1.6) and (1.7) for the cyclic flows $\{\psi_c\}_{c>0}$ of the form (4.8). The results are used in Pipiras and Taqqu (2007) to obtain a general form for the kernel G of a mixed moving average generated by a cyclic flow.

By (4.7), one has $\psi_c(z,v) = (z, \{\ln c + v\}_{q(z)}) = (z, \ln c + v - nq(z))$ when $nq(z) \le \ln c + v < (n+1)q(z)$ and hence these flows have the special representation (2.20) with

$$V(z) = z, \quad r_n(z) = nq(z). \tag{5.1}$$

This representation is convenient when applying Remark following Corollary 2.1 to characterize semi-additive functionals as in the following propositions.

Proposition 5.1. *Let $\{\psi_c\}_{c>0}$ be a cyclic flow on the space $Z \times [0, q(\cdot))$ given by* (4.8), *and let $\{g_c\}_{c>0}$ be a 1-semi-additive functional for the flow $\{\psi_c\}_{c>0}$ satisfying* (1.6). *Then, the solution to the equation* (1.6) *is given by* (4.9).

Proof. Example 3.1 shows that $J_c(z,v) = cg_c(z,v)$ is a semi-additive functional in the sense of (2.1) for the cyclic flow $\{\psi_c\}_{c>0}$ and the cocycle $\{B_c\}_{c>0}$ defined by $B_c(z,v) = c$. Since $\psi_c(z,v)$ has a special representation (2.20) with V and r_n defined in (5.1), Remark following Corollary 2.1 shows that the semi-additive functional $\{J_c\}_{c>0}$ can be expressed as the sum of two semi-additive functionals. After substituting $J_c(z,v) = cg_c(z,v)$ into their expressions, one gets

$$g_c(z,v) = g_c^{(1)}(z,v) + g_c^{(2)}(z,v),$$

where

$$g_c^{(1)}(z,v) = c^{-1}B_c(z,v)g^{(1)}(z, \{v + \ln c\}_{q(z)}) - c^{-1}g^{(1)}(z,v),$$

$$g_c^{(2)}(z,v) = c^{-1}(B_{e^v}(z,0))^{-1} \sum_{k\in[0,n)} B_{e^{r_k(z)}}(z,0)g_1(V^k z),$$

if $r_n(z) \le \ln c + v < r_{n+1}(z)$, for some measurable functions $g^{(1)} : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$ and $g_1 : Z \mapsto \mathbb{R}$.

The function $g_c^{(1)}(z,v)$ has the form (4.9) since $c^{-1}B_c(z,v) = 1$. Consider now the function $g_c^{(2)}(z,v)$. Since $r_n(z) = nq(z)$ by (5.1), we have $r_n(z) \le \ln c + v < r_{n+1}(z)$ when $n = [v + \ln c]_{q(z)}$ using (4.7). By using $B_c(z,v) = c$ and (5.1), we obtain that

$$\begin{aligned} &g_c^{(2)}(z,v) \\ &= g_1(z)\, e^{-v-\ln c} \sum_{k\in[0,[v+\ln c]_{q(z)})} e^{kq(z)} = g_0(z)\, e^{-v-\ln c}\Big(e^{[v+\ln c]_{q(z)}q(z)} - 1\Big), \end{aligned}$$

where $g_0(z) = g_1(z)/(e^{q(z)} - 1)$. Applying (4.7), we get

$$g_c^{(2)}(z,v) = g_0(z)(e^{-\{v+\ln c\}_{q(z)}} - c^{-1}e^{-v}),$$

which has the form (4.9) with $g(z,v) = g_0(z)e^{-v}$. Thus, $g_c^{(1)} + g_c^{(2)}$ has the form (4.9). ■

Proposition 5.2. *Let $\{\psi_c\}_{c>0}$ be a cyclic flow on the space $Z\times[0,q(\cdot))$ given by* (4.8)*. Let also $\{j_c\}_{c>0}$ be a 2-semi-additive functional for the flow $\{\psi_c\}_{c>0}$ satisfying* (1.7)*, and choose a version of the Radon-Nikodym derivatives*

$$\frac{d((\sigma\otimes\mathbb{L})\circ\psi_c)}{d(\sigma\otimes\mathbb{L})}(z,v) \equiv 1$$

which is a cocycle for the flow $\{\psi_c\}_{c>0}$, where $\sigma(dz)$ is a measure on Z and $\mathbb{L}$ denotes the Lebesgue measure on $\mathbb{R}$. Then, the solution to (1.7) *is given by* (4.11).

Proof. Example 3.2 shows that $J_c(z,v) = c^{H-1/\alpha}j_c(z,v)$ is a semi-additive functional related to the cocycle $B_c(z,v) = c^{H-1/\alpha}b_c(z,v)$. Since $\psi_c(z,v)$ has a special representation (2.20) with V and r_n given in (5.1), we can apply Remark following Corollary 2.1 to express $\{J_c\}_{c>0}$ by the sum of two functionals. By substituting $J_c(z,v) = c^{H-1/\alpha}j_c(z,v)$ into these expressions, we get

$$j_c(z,v) = j_c^{(1)}(z,v) + j_c^{(2)}(z,v),$$

where

$$j_c^{(1)}(z,v) = c^{-(H-1/\alpha)}B_c(z,v)j^{(1)}(z,\{v+\ln c\}_{q(z)}) - c^{-(H-1/\alpha)}j^{(1)}(z,v),$$

$$j_c^{(2)}(z,v) = c^{-(H-1/\alpha)}(B_{e^v}(z,0))^{-1}\sum_{k\in[0,n)} B_{e^{r_k(z)}}(z,0)j_1(V^kz),$$

if $r_n(z) \le \ln c + v < r_{n+1}(z)$, for some measurable functions $j^{(1)} : Z\times[0,q(\cdot)) \mapsto \mathbb{R}$ and $j_1 : Z \mapsto \mathbb{R}$.

Since $c^{-(H-1/\alpha)}B_c(z,v) = b_c(z,v)$ and b_c is given by (4.10), the function $j_c^{(1)}(z,v)$ has the form of the first two terms of (4.11). Consider now the function $j_c^{(2)}(z,v)$. Observe that $r_n(z) \le \ln c + v < r_{n+1}(z)$ when $n = [v+\ln c]_{q(z)}$, since $r_n(z) = nq(z)$. Since $B_c(z,v) = c^{H-1/\alpha}b_c(z,v)$, $V^nz = z$ and $r_n(z) = nq(z)$, we obtain that

$$j_c^{(2)}(z,v) =$$

$$j_1(z)c^{-(H-1/\alpha)}e^{-(H-1/\alpha)v}(b_{e^v}(z,0))^{-1}\sum_{k\in[0,[v+\ln c]_{q(z)})} e^{(H-1/\alpha)kq(z)}b_{e^{kq(z)}}(z,0).$$

By using the expression (4.10) of $b_c(z,v)$, we have

$$b_{e^v}(z,0) = b_1(z)^{[v]_{q(z)}}\frac{b(z,\{v\}_{q(z)})}{b(z,0)} = \frac{b(z,v)}{b(z,0)},$$

since $(z,v)\in Z\times[0,q(\cdot))$ and $b_1\in\{-1,1\}$. We also get that $b_{e^{kq(z)}}(z,0) = b_1(z)^k$. Hence,

$$j_c^{(2)}(z,v) =$$

$$j_0(z)(b(z,v))^{-1}e^{-(H-1/\alpha)(v+\ln c)}\sum_{k\in[0,[v+\ln c]_{q(z)})} e^{(H-1/\alpha)kq(z)}b_1(z)^k,$$

where $j_0(z)=j_1(z)b(z,0)$. If $H\neq 1/\alpha$ or $b_1(z)\neq 1$, then $e^{(H-1/\alpha)q(z)}b_1(z)\neq 1$. Hence, by using (4.7) and (4.10),

$$j_c^{(2)}(z,v) =$$

$$j(z)(b(z,v))^{-1}e^{-(H-1/\alpha)(v+\ln c)}\Big(e^{(H-1/\alpha))q(z)[v+\ln c]_{q(z)}}b_1(z)^{[v+\ln c]_{q(z)}} - 1\Big)$$

$$= j(z)\left(\frac{e^{-(H-1/\alpha)\{v+\ln c\}_{q(z)}}}{b(z,v)}b_1(z)^{[v+\ln c]_{q(z)}} - \frac{e^{-(H-1/\alpha)(v+\ln c)}}{b(z,v)}\right)$$

$$= j(z)\left(b_c(z,v)\frac{e^{-(H-1/\alpha)\{v+\ln c\}_{q(z)}}}{b(z,\{v+\ln c\}_{q(z)})} - c^{-(H-1/\alpha)}\frac{e^{-(H-1/\alpha)v}}{b(z,v)}\right),$$

where $j(z) = j_0(z)/(e^{(H-1/\alpha)q(z)}b_1(z) - 1)$. Thus $j_c^{(2)}(z,v)$ with $H \neq 1/\alpha$ or $b_1(z) \neq 1$ has the form of the first two terms of (4.11) (as did $j_c^{(1)}(z,v)$). If $H = 1/\alpha$ and $b_1(z) = 1$, then

$$j_c^{(2)}(z,v)$$
$$= j_0(z)(b(z,v))^{-1} \sum_{k\in[0,[v+\ln c]_{q(z)})} 1 = j_0(z)(b(z,v))^{-1}[v+\ln c]_{q(z)},$$

which has the form of the last term of (4.11). ■

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