ON VERTEX STABILITY WITH REGARD TO COMPLETE BIPARTITE SUBGRAPHS

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Abstract

A graph $G$ is called $(H;k)$-vertex stable if $G$ contains a subgraph isomorphic to $H$ even after removing any of its $k$ vertices. $Q(H;k)$ denotes the minimum size among the sizes of all $(H;k)$-vertex stable graphs. In this paper we complete the characterization of $(K_{m,n};1)$-vertex stable graphs with minimum size. Namely, we prove that for $m \geq 2$ and $n \geq m + 2$, $Q(K_{m,n};1) = mn + m + n$ and $K_{m,n} \ast K_1$ as well as $K_{m+1,n+1} - e$ are the only $(K_{m,n};1)$-vertex stable graphs with minimum size, confirming the conjecture of Dudek and Zwonek.

Keywords: vertex stable, bipartite graph, minimal size.

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1. Introduction

We deal with simple graphs without loops and multiple edges. We use the standard notation of graph theory, cf. [1]. The following notion was introduced in [2]. Let $H$ be any graph and $k$ a non-negative integer. A graph $G$ is called $(H;k)$-vertex stable if $G$ contains a subgraph isomorphic to $H$ even after removing any of its $k$ vertices. Then $Q(H;k)$ denotes minimum size among the sizes of all $(H;k)$-vertex stable graphs. Note that if $H$ does not have isolated vertices then after adding to or removing from a $(H;k)$-vertex

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stable graph any number of isolated vertices we still have a \((H; k)\)-vertex stable graph with the same size. Therefore, in the sequel we assume that no graph in question has isolated vertices.

There are two trivial examples of \((H; k)\)-vertex stable graphs, namely \((k + 1)H\) (a disjoint union of \((k + 1)\) copies of \(H\)) and \(H \ast K_k\) (a graph obtained from \(H \cup K_k\) by joining all the vertices of \(H\) to all the vertices of \(K_k\)). Therefore,

**Proposition 1.** \(Q(H; k) \leq \min \{ (k + 1)|E(H)|, |E(H)| + k|V(H)| + \binom{k}{2} \}.\)

On the other hand, the following is easily seen.

**Proposition 2.** Suppose that \(H\) contains \(k\) vertices which cover \(q\) edges. Then \(Q(H; k) \geq |E(H)| + q\).

Recall also the following

**Proposition 3** ([2]). Let \(\delta_H\) be a minimal degree of a graph \(H\). Then in any \((H; k)\)-vertex stable graph \(G\) with minimum size, \(\deg_G v \geq \delta_H\) for each vertex \(v \in G\).

The exact values of \(Q(H; k)\) are known in the following cases: \(Q(C_i; k) = i(k + 1), i = 3, 4, Q(K_4; k) = 5(k + 1), Q(K_n; k) = \binom{n+k}{2}\) for \(n\) large enough, and \(Q(K_{1,n}; k) = m(k + 1), Q(K_{n,n}; 1) = n^2 + 2n, Q(K_{n,n+1}; 1) = (n + 1)^2, n \geq 2,\) see [2, 3]. In this paper we complete the characterization of \((K_m,n; 1)\) vertex stable graphs with minimum size. Namely, we prove the following theorem and hence confirm Conjecture 1 formulated in [3].

**Theorem 1.** Let \(m, n\) be positive integers such that \(m \geq 2\) and \(n \geq m + 2\). Then \(Q(K_{m,n}; 1) = mn + m + n\) and \(K_{m,n} \ast K_1\) as well as \(K_{m+1,n+1} - e\), where \(e \in E(K_{m+1,n+1})\), are the only \((K_m,n; 1)\)-vertex stable graphs with minimum size.

2. **Proof of the Main Result**

**Proof of Theorem 1.** Let \(m \geq 2\) and \(n \geq m + 2\) be positive integers. Define \(G_1 := K_{m,n} \ast K_1\) and \(G_2 := K_{m+1,n+1} - e\) where \(e \in E(K_{m+1,n+1})\). Let \(G = (V, E)\) be a \((K_{m,n}, 1)\)-vertex stable graph with minimum size. Thus, by Proposition 1, \(|E(G)| \leq mn + m + n\). Clearly \(G\) contains a subgraph
$H$ isomorphic to $K_{m,n}$. Let $H = (X,Y;E_H)$ with vertex bipartition sets $X,Y$ such that $|X| = m$ and $|Y| = n$. Let $v \in X$. Since $G$ is $(K_{m,n};1)$-vertex stable, $G - v$ contains a subgraph $H'$ isomorphic to $K_{m,n}$. Let $H' = (X',Y';E_{H'})$ with vertex bipartition sets $X',Y'$ such that $|X'| = m$ and $|Y'| = n$. We denote $x_1 = |X \cap X'|$, $x_2 = |X \cap Y'|$, $y_1 = |Y \cap X'|$, $y_2 = |Y \cap Y'|$. Hence $x_1 + x_2 \leq m - 1$, $y_1 + y_2 \leq n$, $y_1 \leq m$. One can see that $|E(G)| \geq 2mn - x_1y_2 - x_2y_1$. Consider the following linear programming problem with respect to $y_1$ and $y_2$

$$\begin{align*}
y_1 &\leq m \\
y_1 + y_2 &\leq n \\
y_1 &\geq 0 \\
y_2 &\geq 0 \\
c &= x_1y_2 + x_2y_1 \rightarrow \text{max}
\end{align*}$$

where $x_1$ and $x_2$ are parameters such that $x_1, x_2 \geq 0, x_1 + x_2 \leq m - 1$.

![Figure 1. Geometrical interpretation of the linear programming problem.](image)

The proof falls into two cases.

**Case 1.** $x_1 < x_2$. 
In this case \( y_1 = m, y_2 = n - m, c = x_2m + x_1(n - m) \) is the unique optimal solution of the above linear programming problem. This can be easily checked using a geometrical interpretation of the linear programming problem, see Figure 1. Thus \( |E(G)| \geq 2mn - x_2m - x_1(n - m) \) and the inequality is strict if \( y_1 \neq m \) or \( y_2 \neq n - m \). We assume that \( x_1 + x_2 = m - 1 \) because otherwise the size of \( G \) may only increase. Then

\[
|E(G)| \geq 2mn - m^2 + m + x_1(2m - n) := f(x_1).
\]

**Subcase 1a.** \( n > 2m \).

Then \( f(x_1) \) is decreasing. Furthermore, \( x_1 < \frac{m-1}{2} \) since \( x_1 < x_2 \). Thus

\[
|E(G)| > f\left(\frac{m-1}{2}\right) = \frac{3}{2}mn + \frac{1}{2}n \geq mn + m + n.
\]

Thus \( |E(G)| > mn + m + n \), a contradiction.

**Subcase 1b.** \( n < 2m \).

Then \( f(x_1) \) is increasing. Thus

\[
E(G) \geq f(0) = 2mn - m^2 + m \geq mn + m + n
\]

with equality if and only if \( m = 2 \) and \( n = 4 \), which is not possible in this subcase.

**Subcase 1c.** \( n = 2m \).

In this case

\[
E(G) \geq mn + m + n
\]

with equality if and only if \( m = 2, n = 4, y_1 = y_2 = 2 \). Recall that \( x_1 < x_2 \) whence \( x_1 = 0 \) and \( x_2 = 1 \). Let \( u \in Y' \setminus (X \cup Y) \). Thus \( |E(G)| \geq 12 + \deg u \).

Hence \( \deg u = 2 \) and \( |V(G)| = 7 \) because otherwise \( |E(G)| > mn + m + n \).

However, then \( G \) is not \((K_{2,4};1)\)-stable. Indeed let \( w \) be a neighbor of \( u \). Then \( G - w \) does not contain any subgraph isomorphic to \( K_{2,4} \) since \( G - w \) has 6 vertices and one of them has degree 1. Therefore Case 1 is not possible.

**Case 2.** \( x_1 \geq x_2 \).

In this case \( c = x_1n \) is the optimal solution of the above linear problem, see Figure 1. Therefore, \( |E(G)| \geq 2mn - x_1n \). If \( x_1 \leq m - 2 \) then \( |E(G)| \geq 2mn - (m - 2)n = mn + 2n > mn + m + n \). Hence we may assume that
$x_1 = m - 1$ and $x_2 = 0$. Thus there is only one vertex, say $u$, such that $u \in X' \setminus X$.

Subcase 2a. $y_2 = n$.

Thus, $u$ have $n$ neighbors in $Y$. Note that $|V(G)| \leq m + n + 2$. Indeed, otherwise by Proposition 3, $|E(G)| \geq mn + n + 2m - 1 > mn + m + n$.

Consider now a graph $G'' := G - w$ where $w \in Y$. Clearly $G - w$ contains a subgraph $H''$ isomorphic to $K_{m,n}$. Let $H'' = (X'',Y'';E_{H''})$ with vertex bipartition sets $X'',Y''$ such that $|X''| = m$ and $|Y''| = n$. Let $x'_1 = |X \cap X''|$, $y_1' = |Y \cap X''|$, $y_2' = |Y \cap Y''|$.

Suppose first that $|V(G)| = m + n + 2$ and $u, u_1 \in V(G) \setminus (X \cup Y)$. Since $|E(G)| \leq mn + m + n$, $\deg u_1 = m$ and $\deg u \leq n + 1$. In particular, $u_1 \notin X''$ and $u$ has no neighbor in $X$. Furthermore, $|E(G)| \geq mn + n + m + x' + y'_2$. Thus, $x'_1 = 0$ or $x'_2 = 0$, and $y'_1 = 0$ or $y'_2 = 0$. We distinguish two possibilities

1. $x'_1 = 0$. Then $y'_1 \neq 0$. Indeed, otherwise $X'' = \{u, u_1\}$, a contradiction with previous observation that $u_1 \notin X''$. Hence, $y'_2 = 0$. Thus, $x'_2 = m$ and $u, u_1 \in Y''$ (so $n = m + 2$). Consequently, $y'_1 = m$. However, then $G$ is not $(K_{m,m+2}; 1)$-stable. Indeed, let $w_1$ be a neighbor of $u_1, w_1 \in X'' \subset Y$. Then $G - w_1$ consists of a subgraph isomorphic to $K_{m+1,m+1}$ plus one vertex (namely $u_1$) and $m - 1$ edges incident to it. Therefore, $G - w_1$ does not contain any subgraph isomorphic to $K_{m,m+2}$.

2. $x'_1 \neq 0$. Then $x'_2 = 0$ and $u \notin Y''$. Consequently, $u_1 \in Y''$ and $y'_2 \neq 0$. Hence $y'_1 = 0$. It is easy to see now that $G \cong G_2$.

Assume now that $|V(G)| = m + n + 1$. Hence $x'_1 + x'_2 = m$ and $y'_1 + y'_2 = n - 1$. We have the next two possibilities.

3. $x'_1 + y'_1 = m$. Then $|E(G)| \geq mn + x'_1 x'_2 + y'_1 y'_2 + \deg u \geq mn + x'_1 x'_2 + y'_1 y'_2 + n + x'_1$. Hence

$|E(G)| \geq mn + (m - x'_1)(n - 1 - m + 2x'_1) + n + x'_1 =: f_1(x'_1), \ 0 \leq x'_1 \leq m$.

It is not difficult to see that $f_1(x'_1)$ obtains the smallest value for $x'_1 = 0$ or $x'_1 = m$ only. Thus, $|E(G)| \geq \min\{f_1(0), f_1(m)\}$. Note that $f_1(0) = 2mn + n - m - m^2 \geq mn + m + n$ with equality if and only if $n = m + 2$. However, then there is a vertex $y \in Y''$ such that $G - y \cong K_{m+1,m+1}$ so $G - y$ does not contain any subgraph isomorphic to $K_{m,m+2}$. Furthermore, $f(m) \geq mn + n + m$. Thus, $|E(G)| \geq mn + m + n$ with equality if and only $x_1 = m$. Then $G \cong G_1$. 

4. \( x_2^2 + y_2^2 = n \). Then \( |E(G)| \geq mn + x_1^2x_2^2 + y_1^2y_2^2 + n + x_2^2 \). Hence,
\[
|E(G)| \geq mn + (m-x_2^2)x_2^2 + (x_2^2-1)(n-x_2^2) + n + x_2^2 =: f_2(x_2), \quad 1 \leq x_2^2 \leq m.
\]
One can see that \( f_2(x_2^2) \) obtains the smallest value for \( x_2^2 = 1 \) or \( x_2^2 = m \) only. Thus, \( |E(G)| \geq \min \{ f_2(1), f_2(m) \} \). Note that \( f_2(1) = mn + n + m \).
Then \( G \cong G_1 \). On the other hand, \( f_2(m) = 2mn + 2m - m^2 > mn + m + n \).

Subcase 2b. \( y_2 < n \).

Thus, there is a vertex \( z \in Y' \) such that \( z \in V(G) \setminus (X \cup Y) \). This clearly forces \( m - 1 \) neighbors of \( z \) in \( X \setminus \{v\} \). Consider now a graph \( G - v_1 \), \( v \neq v_1 \in X \). We repeat all preceding arguments to the graph \( G - v_1 \).

Consequently, \( G \cong G_i \), \( i = 1, 2 \), or there is a vertex \( z_1 \in V(G) \setminus (X \cup Y) \) which has \( m - 1 \) neighbors in \( X \setminus \{v_1\} \). If \( z = z_1 \) then \( z \) has \( m \) neighbors in \( X \) and \( G \cong G_1 \) if \( u \in Y \) or \( G \cong G_2 \) otherwise. If \( z \neq z_1 \) then either \( \deg z + \deg z_1 \geq 2m + 1 \) if both vertices \( z \) and \( z_1 \) are involved in a \( K_{m,n} \) contained in \( G - v \) or \( G - v_1 \), or \( \deg u \geq n + 1 \) otherwise. Thus, \( |E(G)| \geq mn + 2m - 1 + n > mn + m + n \).

3. **Concluding Remarks**

In [2, 3] it is proved that \( Q(K_{1,n}; k) = (k + 1)n \). However, for \( n \geq 3 \) the extremal graphs are not characterized.

**Proposition 4.** Let \( G \) be a \((K_{1,n}; k)\)-vertex stable graph with minimum size, \( n \geq 3 \). Then \( G = (k + 1)K_{1,n} \).

**Proof.** The proof is by induction on \( k \). The thesis is obvious for \( k = 0 \). Assume that \( k > 0 \). Let \( G \) be a \((K_{1,n}; k)\)-vertex stable graph with minimum size. Hence, \( |E(G)| = (k+1)n \). Note that each \((K_{1,n}; k)\)-vertex stable graph contains \( k + 1 \) vertices of degree at least \( n \). Let \( v \in V(G) \) with \( \deg v \geq n \).

Thus, \( G - v \) is \((K_{1,n}; k-1)\)-vertex stable graph with \( |E(G-v)| \leq kn \). Hence, \( |E(G-v)| = kn \) and \( \deg v = n \). By the induction hypothesis \( G - v = kK_{1,n} \).

Note that \( v \) is not a neighbor of any vertex of degree \( n \). Suppose on the contrary, that \( uv \in E(G) \) and \( \deg u = n \). Then \( G - u \) contains only \( k - 1 \) vertices of degree greater than or equal to \( n \) whence is not \((K_{1,n}; k-1)\)-vertex stable, a contradiction. Thus, \( G \) contains \( k + 1 \) independent vertices of degree exactly \( n \). We will show that these vertices have pairwise disjoint sets of neighbors. Indeed, otherwise let \( x \) be a common neighbor of two
vertices of degree $n$. Thus, again, $G - x$ has only $k - 1$ vertices of degree greater than or equal to $n$, a contradiction.

In the following table we present the complete characterization of $(K_{m,n}; 1)$-vertex stable graphs with minimum size.

<table>
<thead>
<tr>
<th>$m; n$</th>
<th>$Q(K_{m,n}; 1)$</th>
<th>All $(K_{m,n}; 1)$-vertex stable graphs with minimum size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 1, n = 1$</td>
<td>2</td>
<td>$2K_{1,1}$ [3]</td>
</tr>
<tr>
<td>$m = 1, n = 2$</td>
<td>4</td>
<td>$K_{2,2}, 2K_{1,2}$ [3]</td>
</tr>
<tr>
<td>$m = 1, n \geq 3$</td>
<td>$2n$</td>
<td>$2K_{1,n}$</td>
</tr>
<tr>
<td>$m = 2, n = 2$</td>
<td>8</td>
<td>$K_{2,2} \ast K_1, K_{3,3} - e, 2K_{2,2}$ [3]</td>
</tr>
<tr>
<td>$m \geq 2, n = m + 1$</td>
<td>$(m + 1)^2$</td>
<td>$K_{m+1,m+1}$ [3]</td>
</tr>
<tr>
<td>$m \geq 3, n = m$</td>
<td>$m^2 + 2m$</td>
<td>$K_{m,m} \ast K_1, K_{m+1,m+1} - e$ [3]</td>
</tr>
<tr>
<td>$m \geq 2, n \geq m + 2$</td>
<td>$mn + m + n$</td>
<td>$K_{m,n} \ast K_1, K_{m+1,n+1} - e$</td>
</tr>
</tbody>
</table>

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References


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