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AN ASYMPTOTICALLY UNBIASED MOMENT ESTIMATOR OF A NEGATIVE EXTREME VALUE INDEX*

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In honour of Professor João Tiago Mexia on the occasion of his seventieth birthday

Abstract

In this paper we consider a new class of consistent semi-parametric estimators of a negative extreme value index, based on the set of the k largest observations. This class of estimators depends on a control or tuning parameter, which enables us to have access to an estimator with a null second-order component of asymptotic bias, and with a rather interesting mean squared error, as a function of k.

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We study the consistency and asymptotic normality of the proposed estimators. Their finite sample behaviour is obtained through Monte Carlo simulation.

Keywords: extreme value index, semi-parametric estimation; moment estimator.

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1. Introduction and outline

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics associated with the sequence X_i, $i = 1, \ldots, n$, of independent random variables with common distribution function (d.f.) F. Suppose that F belongs to the max-domain of attraction of a non-degenerate d.f. G and use the notation $F \in \mathcal{D}_{\mathcal{M}}(G)$. Then G is the Extreme Value distribution (Gnedenko, [5]):

$$G_\gamma(x) \equiv \exp(-(1+\gamma x)^{-1/\gamma}), \quad 1+\gamma x > 0, \quad \gamma \in \mathbb{R}. \tag{1}$$

The generalized shape parameter γ, also know as the *extreme value index* (EVI), is the parameter we want to estimate. A necessary and sufficient condition for $F \in \mathcal{D}_{\mathcal{M}}(G)$ is (de Haan, [10]):

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = D_\gamma(x) := \begin{cases} \dfrac{x^\gamma - 1}{\gamma}, & \gamma \neq 0 \\ \ln x, & \gamma = 0 \end{cases}, \quad \forall\, x > 0, \tag{2}$$

for some measurable positive function $a(t)$ and with $U(t)$ standing for the reciprocal quantile function defined by $U(t) := F^{\leftarrow}(1-1/t)$, $t \ge 1$ with $F^{\leftarrow}(x) := \inf\{y : F(y) \ge x\}$, the generalized inverse function of F.

Among the most popular EVI estimators, based on a set of positive upper order statistics, we refer the Hill estimator [12], defined by

$$\hat{\gamma}_n^H(k) \equiv M_n^{(1)}(k) := \frac{1}{k}\sum_{i=1}^{k} \left(\ln X_{n-i+1:n} - \ln X_{n-k:n}\right), \tag{3}$$

valid for the estimation of $\gamma > 0$. For a general $\gamma \in \mathbb{R}$, Dekkers *et al.* [2] proposed the Moment estimator with the functional expression

$$\hat{\gamma}_n^{MOM}(k) = \hat{\gamma}_n^{H}(k) + \hat{\gamma}_n^{NM}(k), \tag{4}$$

where

$$\hat{\gamma}_n^{NM}(k) := 1 - \frac{1}{2}\left(1 - \frac{\left(M_n^{(1)}(k)\right)^2}{M_n^{(2)}(k)}\right)^{-1} \tag{5}$$

and

$$M_n^{(\alpha)}(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\ln X_{n-i+1:n} - \ln X_{n-k:n}\right)^{\alpha}, \quad \alpha > 0. \tag{6}$$

The Moment estimator, in (4), combines two estimators: the Hill'S estimator, in (3), and the estimator in (5) which will be called, in this paper, the "negative Moment" estimator. For intermediate k, i.e., a sequence of integers $k = k_n$, $1 \le k < n$, such that

$$k = k_n \to \infty \quad \text{and} \quad k/n \to 0, \qquad \text{as} \quad n \to \infty, \tag{7}$$

it is well known that $\hat{\gamma}_n^{H}(k)$, $\hat{\gamma}_n^{MOM}(k)$ and $\hat{\gamma}_n^{NM}(k)$ in (3), (4) and (5), are consistent for $\gamma^+ := \max(0, \gamma)$, γ and $\gamma^- := \min(0, \gamma)$, respectively.

Most of the classical EVI estimators have usually a high variance for small values of k and a high bias when k is large. This problem affects both the Hill'S and the Moment estimator and leads to a difficult choice of the "optimal" k, i.e., the value k that minimizes the asymptotic mean squared error. For heavy tails ($\gamma > 0$), the adequate accommodation of the bias of Hill's estimator has been extensively addressed by several authors. Recently, Caeiro *et al.* [1], Gomes *et al.* [7] and [6] introduced, in different ways, second-order minimum-variance reduced-bias estimators that reduce the bias of Hill's estimator without increasing the asymptotic variance, which is kept equal to γ^2.

As already noticed in Fraga Alves [4], when $\gamma > 0$, $\hat{\gamma}_n^{H}(k)$ has smaller asymptotic variance than $\hat{\gamma}_n^{MOM}(k)$, and when $\gamma < 0$, $\hat{\gamma}_n^{NM}(k)$ and $\hat{\gamma}_n^{MOM}(k)$ have the same asymptotic variance. This remark led us to study a semi-parametric class of consistent estimators for $\gamma < 0$, which generalizes the negative Moment estimator in (5). Such a class, given by

$$\hat{\gamma}_n^{NM(\theta)}(k) := \hat{\gamma}_n^{NM}(k) \;+\; \theta\, M_n^{(1)}(k), \quad \theta \in \mathbb{R}, \tag{8}$$

depends on a tuning parameter $\theta \in \mathbb{R}$, and we get the estimator in (5) for $\theta = 0$. With the appropriate choice of θ, $\hat{\gamma}_n^{NM(\theta)}(k)$ enables us to have access to an estimator of a negative EVI with a smaller asymptotic bias and the same asymptotic variance as the Moment estimator.

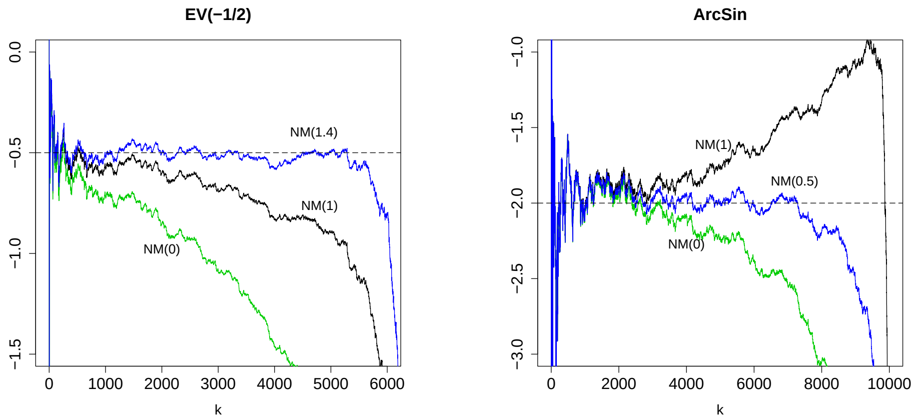

Figure 1. Sample paths of $\hat{\gamma}_n^{NM(\theta)}(k)$ in (8) against k, for one sample of size $n = 10000$ from the $G_{-0.5}$ (left) and Arcsin (right) models.

Figure 1 illustrates, for several values of θ, the behaviour of $\hat{\gamma}_n^{NM(\theta)}(k)$ versus k for a sample of size $n = 10000$ from the Extreme Value distribution in (1) with $\gamma = -0.5$ and the ArcSin distribution with d.f. $F(x) = (2/\pi)\arcsin(\sqrt{x})$, $0 < x < 1$, $(\gamma = -2)$.

In Section 2 of this paper, we state a few results already proved in the literature and derive the asymptotic properties of the new class of EVI estimators, in (8). Finally, in Section 3, we perform a small-scale Monte-Carlo simulation, in order to compare the behaviour of the estimators under study for finite samples.

2. MAIN RESULTS

2.1. Second order regular variation conditions

In order to derive the asymptotic behaviour of several semi-parametric EVI estimators, we need the following second order condition:

$$(9)\quad \lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} = H_{\gamma,\rho}(x) := \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma}\right),$$

for all $x > 0$, where $\rho \le 0$ is a second order parameter controlling the speed of convergence in (2) and $|A(t)| \in RV_\rho$, with RV_a standing for the class of regularly varying functions with an index of regular variation a, i.e. positive measurable functions g such that $\lim_{t\to\infty} g(tx)/g(t) = x^a$, for all $x > 0$. Since we need a second order condition for $\ln U(t)$, we state the following theorem:

Theorem 2.1 (Draisma *et al.*, [3], de Haan and Ferreira [11], Appendix. B3)**.** *Assume* $U(\infty) > 0$ *and* (9) *holds with* $\rho \le 0$ *and* $\gamma \ne \rho$. *Then*

$$\overline{A}(t) := \left(\frac{a(t)}{U(t)} - \gamma_+\right) \underset{t\to\infty}{\longrightarrow} 0$$

and with

$$l := \lim_{t\to\infty}\left(U(t) - \frac{a(t)}{\gamma}\right) \quad \textit{for} \;\; \gamma+\rho<0,$$

$$\frac{\overline{A}(t)}{A(t)} \underset{t\to\infty}{\longrightarrow} c = \begin{cases} 0, & \gamma < \rho \le 0; \\ \dfrac{\gamma}{\gamma+\rho}, & 0 \le -\rho < \gamma \vee (0 < \gamma < -\rho, \;\; l = 0); \\ \pm\infty, & \gamma = -\rho \vee (0<\gamma<-\rho, \; l \ne 0) \;\; \vee \rho < \gamma \le 0. \end{cases}$$

Furthermore, we have

$$\lim_{t\to\infty} \frac{\dfrac{\ln U(tx)-\ln U(t)}{\tilde a(t)} - \dfrac{x^{\gamma_-}-1}{\gamma_-}}{\widetilde A(t)} = H_{\gamma_-,\tilde\rho}(x), \tag{10}$$

for all $x>0$, *with* $\tilde a(t) := a(t)/U(t)$, H *defined in* (9),

$$\tilde\rho := \begin{cases} \gamma, & \rho<\gamma\le 0 \\ -\gamma, & 0<\gamma<-\rho,\ l\ne 0 \\ \rho, & (0<\gamma<-\rho\wedge l=0)\ \vee\ \gamma<\rho\le 0\ \vee\ \gamma\ge-\rho>0 \end{cases}$$

and

$$\widetilde A(t) := \begin{cases} A(t), & c=0 \\ \gamma_+ - \dfrac{a(t)}{U(t)}, & c=\pm\infty \\ \dfrac{\rho}{\gamma+\rho}A(t), & c=\dfrac{\gamma}{\gamma+\rho} \end{cases},$$

with $|\tilde A(t)| \in RV_{\tilde\rho}$.

Remark 2.1 (de Haan and Ferreira [11], Remark B.3.18)**.** It follows that

$$q_{\gamma,\rho} := \lim_{t\to\infty}\frac{\overline A(t)}{\widetilde A(t)} = \begin{cases} 0 & \gamma<\rho\le 0; \\ \dfrac{\gamma}{\rho} & (0<\gamma<-\rho\wedge l=0)\ \vee\ \gamma\ge-\rho>0; \\ -1 & (0<\gamma<-\rho,\ l\ne 0)\ \vee\ \rho<\gamma\le 0. \end{cases} \tag{11}$$

Since $\rho<0$ for a variety of models we add the following proposition with a similar proof to Proposition 1 from [8]:

Proposition 2.1. *Let us assume that* (9) *holds with* $\rho < 0$. *Then there exists* $\tilde{a}_0(\cdot)$ *and* $\widetilde{A}_0(\cdot)$ *such that*

$$\lim_{t\to\infty} \frac{\dfrac{\ln U(tx) - \ln U(t)}{\tilde{a}_0(t)} - \dfrac{x^{\gamma_-} - 1}{\gamma_-}}{\widetilde{A}_0(t)} = \frac{x^{\gamma_- + \tilde{\rho}} - 1}{\gamma_- + \tilde{\rho}}, \qquad \forall x > 0,$$

with $\widetilde{A}_0(t) = \frac{\widetilde{A}(t)}{\tilde{\rho}}$ *and* $\tilde{a}_0(t) = \tilde{a}(t)(1 - \widetilde{A}_0(t))$.

2.2. Auxiliary results

More generally than Lemma 3.5.5 in de Haan and Ferreira [11], but with a similar proof, we now state the following:

Lemma 2.1. *Under the second order framework in* (10) *and for intermediate* k, *i.e. whenever* (7) *holds we have for any* $\alpha > 0$, *and* $M_n^{(\alpha)}(k)$ *defined in* (6),

$$\frac{M_n^{(\alpha)}(k)}{(\tilde{a}(t))^\alpha} \stackrel{d}{=} \mu_{\alpha,\gamma} + \frac{\sigma_{\alpha,\gamma}}{\sqrt{k}} Z_k^{(\alpha)} + b_{\alpha,\gamma,\tilde{\rho}}\, \widetilde{A}(n/k)\,(1 + o_p(1))\,,$$

where with Y_i, $1 \le i \le n$ *a sequence of independent, identically distributed Pareto random variables with d.f.* $F_Y(x) = 1 - 1/x$, $x \ge 1$,

$$Z_k^{(\alpha)} := \frac{\sqrt{k}}{\sigma_{\alpha,\gamma}} \times \frac{1}{k}\sum_{i=1}^{k}\left(\left(\frac{Y_i^{\gamma_-} - 1}{\gamma_-}\right)^\alpha - \mu_{\alpha,\gamma}\right), \tag{12}$$

is a sequence of asymptotically standard normal random variables. Moreover,

$$\mu_{\alpha,\gamma} := E\left(\left(\frac{Y^{\gamma_-} - 1}{\gamma_-}\right)^\alpha\right) = \begin{cases} \dfrac{\Gamma(\alpha+1)\Gamma(-1/\gamma)}{(-\gamma)^{\alpha+1}\Gamma(\alpha+1-1/\gamma)}, & \gamma < 0; \\[2ex] \Gamma(\alpha+1), & \gamma \ge 0, \end{cases} \tag{13}$$

$$\sigma^2_{\alpha,\gamma} := V\left(\left(\frac{Y^{\gamma_-}-1}{\gamma_-}\right)^{\alpha}\right) = \mu_{2\alpha,\gamma} - \mu^2_{\alpha,\gamma} \tag{14}$$

and

$$b_{\alpha,\gamma,\tilde\rho} := E\left(\alpha\left(\frac{Y^{\gamma_-}-1}{\gamma_-}\right)^{\alpha-1} H_{\gamma_-,\tilde\rho}(Y)\right).$$

Remark 2.2. If $\tilde\rho < 0$ one can easily get an explicit expression for $b_{\alpha,\gamma,\tilde\rho}$, which holds for any $\alpha > 0$ and $\gamma \in \mathbb{R}$. It is given by

$$b_{\alpha,\gamma,\tilde\rho} = \begin{cases} \dfrac{\alpha}{\tilde\rho}\left(\dfrac{\mu_{\alpha-1,\gamma/(1-(\gamma+\tilde\rho))}}{(1-(\gamma+\tilde\rho))^{\alpha}(\gamma+\tilde\rho)} - \dfrac{\mu_{\alpha-1,\gamma}}{\gamma+\tilde\rho} - \mu_{\alpha,\gamma}\right), & \gamma < 0, \quad \tilde\rho < 0 \\ \\ \dfrac{\Gamma(\alpha+1)}{\tilde\rho}\left(\dfrac{1-(1-\tilde\rho)^{\alpha}}{\tilde\rho(1-\tilde\rho)^{\alpha}} - \alpha\right), & \gamma \geq 0, \quad \tilde\rho < 0 \end{cases},$$

with $\mu_{\alpha,\gamma}$ given in (13).

The next lemma follows closely Corollary 3.5.6 from [11].

Lemma 2.2. *Under the conditions of Lemma* 2.1,

$$\hat\gamma_n^{NM}(k) \stackrel{d}{=} \gamma_- + \frac{\sigma_{NM}}{\sqrt{k}} Z_k^{NM} + b_{NM}\ \widetilde{A}(n/k)\,(1+o_p(1))\,,$$

with

$$b_{NM} := \frac{(1-\gamma_-)(1-2\gamma_-)}{(1-\gamma_- -\tilde\rho)(1-2\gamma_- -\tilde\rho)},$$

$$\sigma^2_{NM} := \frac{(1-\gamma_-)^2(1-2\gamma_-)(1-\gamma_- + 6\gamma_-^2)}{(1-3\gamma_-)(1-4\gamma_-)}$$

and with $\overline{\sigma}_{\alpha,\gamma} := \sigma_{\alpha,\gamma}/\mu_{\alpha,\gamma}$*,* $\mu_{\alpha,\gamma}$ *and* $\sigma^2_{\alpha,\gamma}$ *given in* (13) *and* (14)*, respectively, and* $Z_k^{(\alpha)}$ *defined in* (12)*,*

$$Z_k^{NM} := (1-\gamma_-)(1-2\gamma_-)\big(\overline{\sigma}_{2,\gamma}Z_k^{(2)} - 2\overline{\sigma}_{1,\gamma}Z_k^{(1)}\big)$$

is an asymptotically standard normal random variable.

Consequently, if $\sqrt{k}\widetilde{A}(n/k) \underset{n\to\infty}{\longrightarrow} \lambda$ *finite, possibly non null,*

$$\sqrt{k}(\hat{\gamma}_n^{NM}(k) - \gamma_-) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N\left(\lambda\ b_{NM}, \sigma^2_{{}_{NM}}\right).$$

2.3. Asymptotic behaviour of the new class of estimators

Theorem 2.2. *Under the conditions of Lemma* 2.1

$$\hat{\gamma}_n^{NM(\theta)}(k) \overset{d}{=} (\gamma_- + \theta\gamma_+) + \frac{1}{\sqrt{k}}\left(\sigma_{{}_{NM}}Z_k^{NM} + \gamma_+\sigma_{1,\gamma}Z_k^{(1)}\right)$$

(15)

$$+\left(b_{NM} + \theta\gamma_+ b_{1,\gamma,\tilde{\rho}} + \frac{\theta\ q_{\gamma,\rho}}{1-\gamma_-}\right)\widetilde{A}(n/k)\,(1+o_p(1))\,,$$

with $q_{\gamma,\rho}$ *defined in* (11)*.*

Proof. Using Lemma 2.1,

$$M_n^{(1)}(k) \overset{d}{=} \tilde{a}(t)\left(\frac{1}{1-\gamma_-} + \frac{\sigma_{1,\gamma}}{\sqrt{k}}Z_k^{(1)} + b_{1,\gamma,\tilde{\rho}}\,\widetilde{A}(n/k)\,(1+o_p(1))\right).$$

Next, since $\tilde{a}(t) = \gamma_+ + q_{\gamma,\rho}\widetilde{A}(t)$ and $\gamma_+/(1-\gamma_-) = \gamma_+$,

$$M_n^{(1)}(k) \stackrel{d}{=} \gamma_+ + \frac{\gamma_+ \sigma_{1,\gamma}}{\sqrt{k}} Z_k^{(1)} + \left(\gamma_+ b_{1,\gamma,\tilde\rho} + \frac{q_{\gamma,\rho}}{1-\gamma_-}\right) \widetilde{A}(n/k)(1+o_p(1)).$$

Finally, using Lemma 2.2, (15) follows. ∎

Remark 2.3. If $\gamma < 0$, and $\sqrt{k}\widetilde{A}(n/k) \underset{n\to\infty}{\longrightarrow} \lambda$, finite,

$$\sqrt{k}(\hat\gamma_n^{NM(\theta)}(k) - \gamma) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N\left(\lambda \left(b_{NM} + \frac{\theta\ q_{\gamma,\rho}}{1-\gamma}\right), \sigma^2_{{}_{NM}}\right).$$

Remark 2.4. With the adequate choice of $\theta_0 = -(1-\gamma) b_{NM}/q_{\gamma,\rho}$, if $q_{\gamma,\rho} \neq 0$, we have access to an asymptotically unbiased second-order EVI estimator. The adaptively choice of the "optimal" θ is outside the scope of this work.

Remark 2.5. We advise not to choose blindly the value of θ in (8). At this stage we think sensible to draw a few sample paths of $\hat\gamma_n^{NM(\theta)}(k)$, for a few values of θ, and elect the value θ_0 which provides higher stability in the region of k values, for which we get admissible estimates. However, we hope to be able to adaptively choose θ on the basis of the bootstrap methodology and the consideration of an auxiliary statistic like $\hat\gamma_n^{NM(\theta)}(k) - \hat\gamma_n^{NM(\theta)}\big([k/2]\big)$, of the type of the one used in Gomes and Oliveira [9] for the adaptive choice of k through the Hill estimator in (3). This is however outside the scope of this paper.

3. Finite sample properties of the new class of EVI-estimators

We now study the finite sample behaviour of the extreme value index estimator $\hat\gamma_n^{NM(\theta)}(k)$ and compare it with the classical Moment estimator. We have chosen two different values of θ: one of them is $\theta = 0$ and the other one was chosen by *trial* until we had a stable sample path of the estimates (for small up to moderate values of k). We generated 5000 pseudorandom samples of size n, with $n \in \{50, 100, 200, 500, 1000, 2000, 5000, 10000\}$, from the following distributions:

- The EV_γ distribution in (1), with $\gamma = -0.5, -0.9$ $(\rho = -1, \tilde{\rho} = \gamma)$;
- The Arcsin distribution with d.f. $F(x) = (2/\pi)\arcsin(\sqrt{x})$, $0 < x < 1$, for which we have $\gamma = \rho = \tilde{\rho} = -2$;
- The Half-normal distribution, i.e., the absolute value of a standard normal distribution $(\gamma = \rho = \tilde{\rho} = 0)$.

Although the asymptotic results, from the previous section, exclude the case $\gamma = \rho$, we have decided to include the Arcsin and Half-normal distributions in the simulation study. To ilustraste the finite sample behaviour of the EVI estimators for such models, we present, in Figure 3, the simulated mean values (E) and root mean squared error (RMSE) patterns of the above mentioned estimators, as functions of k, for a sample size $n = 200$, 1000 and 5000.

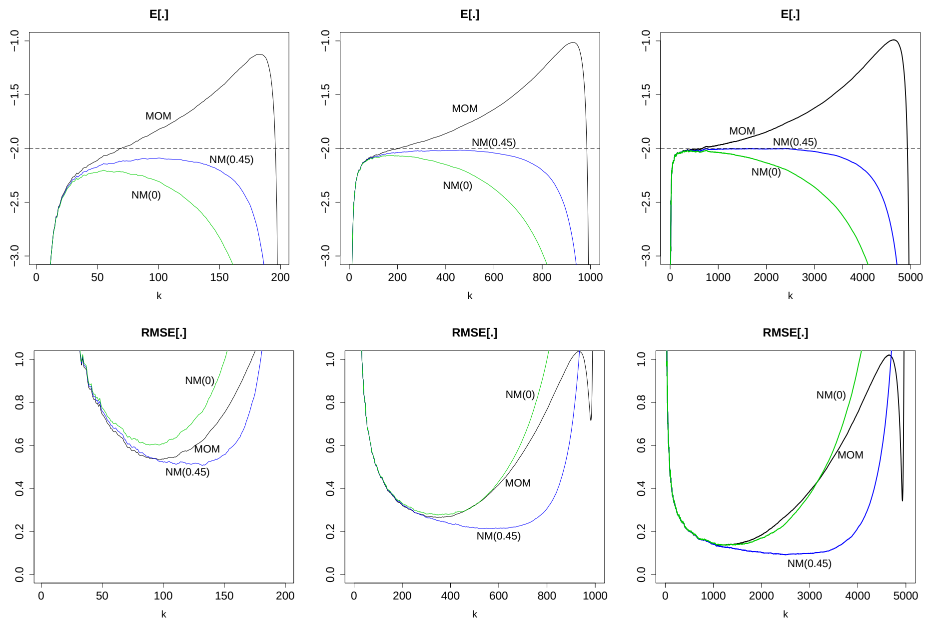

Figure 2. Simulated mean values (above) and root mean squared errors (below) of MOM, NM(0) and NM(0.45) extreme value index estimators, for samples of size $n = 200$ (left), 1000 (center) and 5000 (right) from the Arcsin distribution.

Figures 3 and 4 have the same simulated quantities for the $EV_{-0.5}$ and Half-normal distribution with $n = 1000$.

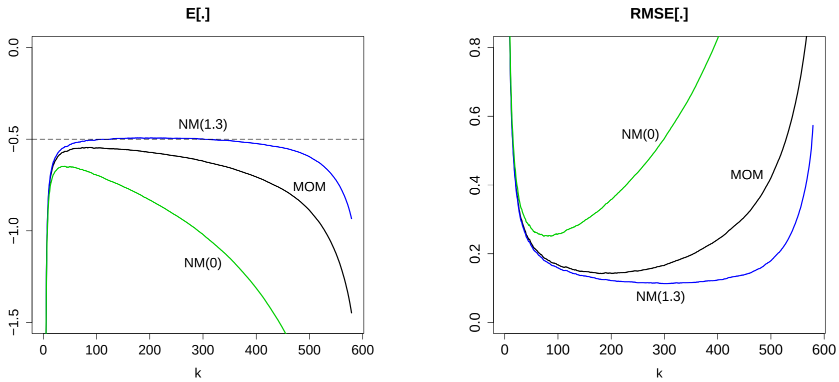

Figure 3. Simulated mean values (left) and root mean squared errors (right) of MOM, NM(0) and NM(1.3) extreme value index estimators, for samples of size $n = 1000$ from the EV distribution with $\gamma = -0.5$.

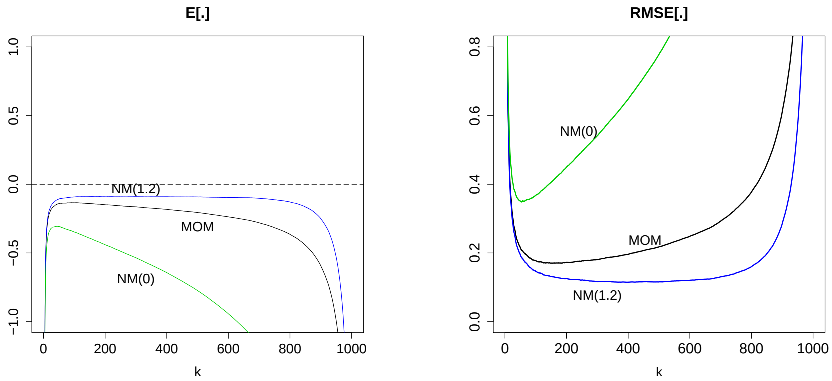

Figure 4. Simulated mean values (left) and root mean squared errors (right) of MOM, NM(0) and NM(1.2) extreme value index estimators, for samples of size $n = 1000$ from the Half-normal distribution ($\gamma = 0$).

In Tables 1 and 2 we present the simulated mean values and root mean squared errors of the estimators under study, at their simulated optimal levels.

Table 1. Simulated mean vlues at optimal levels.

E	50	100	200	500	1000	2000	5000	10000
				Arcsin ($\gamma = -2$)				
MOM	*−1.876*	−1.827	−1.844	−1.879	−1.904	−1.923	−1.964	−1.960
NM(.45)	−2.385	*−2.180*	*−2.128*	*−2.052*	*−2.033*	*−2.031*	*−2.010*	*−2.009*
NM(0)	−2.717	−2.375	−2.262	−2.172	−2.117	−2.087	−2.058	−2.036
				$G_{-0.5}$				
MOM	−0.922	−0.724	−0.651	−0.589	−0.565	−0.545	−0.534	−0.525
NM(1.3)	*−0.779*	*−0.621*	*−0.549*	*−0.512*	*−0.499*	*−0.497*	*−0.496*	*−0.498*
NM(0)	−1.313	−1.023	−0.857	−0.730	−0.667	−0.626	−0.597	−0.579
				$G_{-0.9}$				
MOM	−1.546	−1.252	−1.134	−1.043	−1.019	−0.989	−0.962	−0.948
NM(1.75)	*−1.287*	*−1.066*	*−0.970*	*−0.923*	*−0.910*	*−0.903*	*−0.900*	*−0.898*
NM(0)	−1.806	−1.475	−1.240	−1.130	−1.065	−1.022	−0.993	−0.964
				Half-Normal ($\gamma = -0$)				
MOM	−0.342	−0.251	−0.207	−0.168	−0.140	−0.123	−0.107	−0.098
NM(1.2)	*−0.220*	*−0.149*	*−0.115*	*−0.098*	*−0.092*	*−0.086*	*−0.079*	*−0.070*
NM(0)	−0.785	−0.583	−0.467	−0.365	−0.310	−0.274	−0.239	−0.214

Table 2. Simulated root mean squared errors at optimal levels.

RMSE	50	100	200	500	1000	2000	5000	10000
				Arcsin				
MOM	*1.222*	*0.761*	0.533	0.352	0.265	0.199	0.137	0.102
NM(.45)	1.303	0.776	*0.507*	*0.306*	*0.213*	*0.151*	*0.096*	*0.069*
NM(0)	1.486	0.898	0.601	0.379	0.276	0.202	0.137	0.101
				$G_{-0.5}$				
MOM	0.911	0.508	0.327	0.201	0.143	0.105	0.071	0.053
NM(1.3)	*0.839*	*0.451*	*0.282*	*0.164*	*0.113*	*0.080*	*0.050*	*0.036*
NM(0)	1.195	0.735	0.499	0.333	0.251	0.195	0.143	0.114
				$G_{-0.9}$				
MOM	1.349	0.741	0.481	0.299	0.216	0.162	0.112	0.086
NM(1.75)	*1.215*	*0.630*	*0.390*	*0.226*	*0.157*	*0.110*	*0.069*	*0.049*
NM(0)	1.554	0.887	0.589	0.373	0.278	0.209	0.148	0.113
				Half-Normal				
MOM	0.526	0.359	0.268	0.203	0.169	0.146	0.123	0.110
NM(1.2)	*0.450*	*0.286*	*0.199*	*0.140*	*0.116*	*0.100*	*0.088*	*0.080*
NM(0)	0.922	0.677	0.531	0.410	0.349	0.302	0.259	0.231

Based on the simulated results here presented, we may draw the following conclusions:

1. With the proper choice of θ, the new class of estimators, $\hat{\gamma}_n^{NM(\theta)}(k)$, appears to be asymptotically unbiased. The variance is kept unchanged and is equal to the variance of the Moment estimator.

2. The new class of EVI estimators has, in general, reasonably stable sample paths, which makes less troublesome the choice of the level k.

3. Although outside of the theoretical framework, we get the same properties for the Arcsin and Half-normal distribution.

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