TOTAL OUTER-CONNECTED DOMINATION IN TREES

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Abstract

Let $G = (V, E)$ be a graph. Set $D \subseteq V(G)$ is a total outer-connected dominating set of $G$ if $D$ is a total dominating set in $G$ and $G[V(G) - D]$ is connected. The total outer-connected domination number of $G$, denoted by $\gamma_{tc}(G)$, is the smallest cardinality of a total outer-connected dominating set of $G$. We show that if $T$ is a tree of order $n$, then $\gamma_{tc}(T) \geq \lceil \frac{2n}{3} \rceil$. Moreover, we constructively characterize the family of extremal trees $T$ of order $n$ achieving this lower bound.

Keywords: total outer-connected domination number, domination number.

2010 Mathematics Subject Classification: 05C05, 05C69.

1. Introduction

Graph theory terminology not presented here can be found in [1, 5].

Let $G = (V, E)$ be a simple graph. The *neighbourhood* of a vertex $v$, denoted by $N_G(v)$, is the set of all vertices adjacent to $v$ in $G$ and the integer $d_G(v) = |N_G(v)|$ is the *degree* of $v$ in $G$. A vertex of degree one is called an *end-vertex*. A *support* is the unique neighbour of an end-vertex.

Let $P_n$ denotes the path of order $n$. For a vertex $v$ of $G$, we shall use the expression, *attach a $P_n$ at $v*$, to refer to the operation of taking the union of $G$ and a path $P_n$ and joining one of the end-vertices of this path to $v$ with an edge.

Set $D \subseteq V(G)$ is a *dominating set* in $G$ if $N_G(v) \cap D \neq \emptyset$ for every vertex $v \in V(G) - D$. The *domination number* of $G$, denoted $\gamma(G)$, is the cardinality of a minimum dominating set of $G$. 

Set $D \subseteq V(G)$ is a total dominating set of $G$ if each vertex of $V(G)$ has a neighbour in $D$. The cardinality of a minimum total dominating set in $G$ is the total domination number of $G$ and is denoted by $\gamma_t(G)$. Total domination in graphs is currently well studied in graph theory (for examples, see [2, 6]).

Set $D \subseteq V(G)$ is said to be a total outer-connected dominating set of $G$ if $D$ is a total dominating set and $G[V(G) - D]$ is connected. The cardinality of a minimum total outer-connected dominating set in $G$ is called the total outer-connected domination number of $G$ and is denoted by $\gamma_{tc}(G)$. Observe that every graph $G$ without isolates has a total outer-connected dominating set, since the set of all vertices of $G$ is a total outer-connected dominating set in $G$.

We will show that if $T$ is a tree of order $n$, then $\gamma_{tc}(T) \geq \left\lceil \frac{2n}{3} \right\rceil$. Moreover, we will constructively characterize the extremal trees $T$ of order $n \geq 3$ achieving this lower bound.

Similar bounds for various domination numbers in trees are given in [2, 6].

2. The Lower Bound

**Theorem 1.** If $T$ is a tree of order $n \geq 2$, then

$$\gamma_{tc}(T) \geq \left\lceil \frac{2n}{3} \right\rceil.$$  

**Proof.** The result is obvious for $n = 2$. Assume that $n \geq 3$ and let $D$ be a minimum total outer-connected dominating set of $T$. Let us denote by $S$ any component of $T[D]$. Since $T$ is a tree, no two vertices of $V(T) - D$ have a common neighbour in $S$. Hence $|N_T(S) \cap (V(T) - D)| \leq 1$. Moreover, $D$ is dominating in $T$ and isolate free, and thus

$$n(T) = |V(T) - D| + |D| \geq |V(T) - D| + 2|V(T) - D| \geq n - \gamma_{tc}(T) + 2n - 2\gamma_{tc}(T).$$

Finally, we have $\gamma_{tc}(T) \geq \frac{2}{3}n$, and so $\gamma_{tc}(T) \geq \left\lceil \frac{2n}{3} \right\rceil$. 

\hfill \blacksquare
3. The Characterization of the Extremal Trees

For \( n \geq 2 \), let \( T_n = \{ T \mid T \) is a tree of order \( n \) such that \( \gamma_{tc}(T) = \lceil \frac{2n}{3} \rceil \}, T = \bigcup_{n \geq 2} T_n \). We will present a constructive characterization of the family \( T \). For this purpose, we define a type (1) operation on a tree \( T \) as attaching \( P_3 \) at \( v \) where \( v \) is a vertex of \( T \) not belonging to some minimum total outer-connected dominating set of \( T \), and a type (2) operation as attaching \( P_1 \) at \( v \) where \( v \) belongs to some minimum total outer-connected dominating set of \( T \).

We now define families of trees as follows. Let \( C_n = \{ T \mid T \) is a tree of order \( n \) which can be obtained from the path \( P_3 \) by a finite sequence of operations of type (1) and (2), where the operation of type (2) appears in the sequence exactly \( n \) (mod 3) times, \( n \geq 3 \), and \( C_2 = \{ P_2 \} \).

We shall establish:

**Theorem 2.** For \( n \geq 2 \), \( T_n = C_n \).

We prove Theorem 2 by establishing eight lemmas.

**Lemma 3.** If \( D \) is a minimum total outer-connected dominating set of a tree \( T \) of order at least 6 and \( T \not\subseteq T \) and \( \Omega(T) \), then every end-vertex of \( T \) and every support of \( T \) belongs to \( D \).

**Lemma 4.** If \( T \not\subseteq T \), then \( |\Omega(T)| \leq |S(T)| + 2 \), where \( \Omega(T) \) is the set of all end-vertices of \( T \) and \( S(T) \) is the set of all supports of \( T \).

**Proof.** Let \( D \) be a minimum total outer-connected dominating set of a tree \( T \) belonging to \( T \). Then for some positive integer \( n \) we have \( T \in T_n \) and \( |D| = \lceil \frac{2n}{3} \rceil \). Suppose \( |\Omega(T)| = |S(T)| + t \), \( t > 2 \). Denote by \( s_1, \ldots, s_m \) the supports of \( T \) and by \( l_1, \ldots, l_m, l_{m+1}, \ldots, l_{m+t} \) the end-vertices of \( T \), where \( l_i \in N_T(s_i), 1 \leq i \leq m \). Notice that \( D - \{ l_{m+1}, l_{m+2}, l_{m+3} \} \) is a total outer-connected dominating set of a tree \( T' = T - \{ l_{m+1}, l_{m+2}, l_{m+3} \} \). Hence \( \gamma_{tc}(T') \leq |D| - 3 = \lceil \frac{2n-9}{3} \rceil \). On the other hand, by Theorem 1, we have \( \gamma_{tc}(T') \geq \lceil \frac{2(n-3)}{3} \rceil \) and consequently \( \lceil \frac{2(n-3)}{3} \rceil \leq \gamma_{tc}(T') \leq \lceil \frac{2n-9}{3} \rceil \), which is impossible.

Thus we have what follows.

**Corollary 1.** If \( T \in T \), then exactly one of the following conditions holds:
(i) every support of $T$ is a neighbour of exactly one end-vertex;
(ii) exactly one support of $T$ is a neighbour of exactly two end-vertices, while every other support is a neighbour of exactly one end-vertex;
(iii) exactly one support of $T$ is a neighbour of three end-vertices, while every other support is a neighbour of exactly one end-vertex or exactly two supports of $T$ are the neighbours of exactly two end-vertices, while every other support is a neighbour of exactly one end-vertex.

Lemma 5. If $T \in \mathcal{T}_n$, $n \geq 3$, and $T'$ is obtained from $T$ by a type (1) operation, then $T' \in \mathcal{T}_{n+3}$.

Proof. By definition of a type (1) operation on a tree $T$, there exists a minimum total outer-connected dominating set of $T$ such that adding a new end-vertex of $T'$ and a new support of $T'$ to it produces a total outer-connected dominating set of $T'$. Hence, since $T \in \mathcal{T}_n$, $\gamma_{tc}(T') \leq \gamma_{tc}(T) + 2 = \left\lceil \frac{2n+6}{3} \right\rceil$. However, $T'$ is a tree of order $n + 3$, and so, by Theorem 1, $\gamma_{tc}(T') \geq \left\lceil \frac{2(n+3)}{3} \right\rceil$. Consequently, $\gamma_{tc}(T') = \left\lceil \frac{2(n+3)}{3} \right\rceil$, and hence $T' \in \mathcal{T}_{n+3}$.

Notice that $C_3 = \{P_3\} = T_3$. Hence an immediate consequence of Lemma 5 now follows.

Lemma 6. If $n \geq 3$ and $n \equiv 0 \pmod{3}$, then $C_n \subseteq \mathcal{T}_n$.

We will now prove the inverse inclusion.

Lemma 7. If $n \geq 3$ and $n \equiv 0 \pmod{3}$, then $\mathcal{T}_n \subseteq C_n$.

Proof. We proceed by induction on $n \geq 3$. Since $T_3 = \{P_3\} = C_3$, the result is true for $n = 3$. Let $n \geq 6$ satisfy $n \equiv 0 \pmod{3}$ and assume that $\mathcal{T}_k \subseteq C_k$ for all integers $k \equiv 0 \pmod{3}$, where $3 \leq k < n$. Let $T \in \mathcal{T}_n$. We show that $T \in C_n$. Let $D$ be a minimum total outer-connected dominating set of $T$. Let $P = (v_1, v_2, \ldots, v_m)$ be a longest path in $T$. By Lemma 3, \(\{v_1, v_2, v_{m-1}, v_m\} \subseteq D\).

We will show that $d_T(v_2) \equiv 2$ and $\{v_3, v_4\} \cap D = \emptyset$. Suppose that $v_2$ is adjacent to two end-vertices, say $v_1$ and $l_1$. Then $D' = D - \{l_1\}$ is a total outer-connected dominating set of $T' = T - l_1$. Hence, since $T \in \mathcal{T}_n$, $\gamma_{tc}(T') \leq \left\lceil \frac{2n}{3} \right\rceil - 1 = \frac{2n}{3} - 1$. However, $T'$ is a tree of order $n - 1 \equiv 2 \pmod{3}$, and so, by Theorem 1, $\gamma_{tc}(T') \geq \left\lceil \frac{2(n-1)}{3} \right\rceil = \frac{2n}{3}$, a contradiction.
Suppose now $v_3 \in D$. Then the set $D' = D - \{v_1\}$ is a total outer-connected dominating set of $T' = T - v_1$ and $\frac{2n}{3} \leq \gamma_{tc}(T') \leq \frac{2n}{3} - 1$ — a contradiction. Hence $d_T(v_2) = 2$ and $v_3 \notin D$. From Lemma 3 and from the fact that $V(T) - D$ is a tree we conclude that $m \geq 6$ and $v_4 \notin D$.

We will now prove that $d_T(v_3) = 2$. Since $v_3 \notin D$, $v_3$ is not a support. Suppose there exists a path $P' = (u_1, u_2, v_3)$ in $T$ such that $u_2 \notin \{v_2, v_4\}$. By Lemma 4, $\{u_1, u_2\} \subseteq D$. Moreover $D' = D - \{u_1, u_2\}$ is a total outer-connected dominating set of $T' = T - \{u_1, u_2\}$. Hence $\gamma_{tc}(T') \leq \gamma_{tc}(T) - 2 = \frac{2n}{3} - 2$, which contradicts the fact that (by Theorem 1) $\gamma_{tc}(T') \geq \left[\frac{2(n-2)}{3}\right]$.

Let us consider tree $T' = T - \{v_1, v_2, v_3\}$. The set $D' = D - \{v_1, v_2\}$ is a total outer-connected dominating set of $T'$. Hence $\gamma_{tc}(T') \leq \left[\frac{2n}{3}\right] - 2 = \left[\frac{2n-6}{3}\right]$. Moreover by Theorem 1, $\gamma_{tc}(T') \geq \left[\frac{2(n-3)}{3}\right]$ and so $T' \in T_{n-3}$. Thus, by the inductive hypothesis, $T' \in C_{n-3}$. Since $v_4$ does not belong to some minimum total outer-connected dominating set of $T'$, namely $D'$, $T$ is constructed from $T'$ by a type (1) operation. Hence $T \in C_n$. 

**Lemma 8.** If $T \in T_n$, $n \geq 3$, and $n \not\equiv 2 \pmod{3}$, then a tree $T'$ obtained from $T$ by a type (2) operation belongs to $T_{n+1}$.

**Proof.** By definition of a type (2) operation on a tree $T$, there exists a minimum total outer-connected dominating set of $T$ such that adding to it the new end-vertex of $T'$ produces a total outer-connected dominating set of $T'$. Hence, since $T \in T_n$ and $n \not\equiv 2 \pmod{3}$, $\gamma_{tc}(T') \leq \gamma_{tc}(T) + 1 = \left[\frac{2n+2}{3}\right] = \left[\frac{2n+2}{3}\right]$. However, $T'$ is a tree of order $n + 1$, and so, by Theorem 1, $\gamma_{tc}(T') \geq \left[\frac{2(n+1)}{3}\right]$. Consequently, $\gamma_{tc}(T') = \left[\frac{2n+2}{3}\right]$ and $T' \in T_{n+1}$. 

**Lemma 9.** If $n \geq 4$ and $n \not\equiv 0 \pmod{3}$, then $C_n \subseteq T_n$.

**Proof.** We proceed by induction on $n \geq 4$. The base case is true since $C_4 = \{K_{1,3}, P_3\} \subseteq T_4$ and $C_5 = \{K_{1,4}, P_5, T_1\} \subseteq T_5$, where $T_1$ is a tree obtained from a star $K_{1,3}$ by subdivision of exactly one of its edges.

Assume now that the result is true for $k \not\equiv 0 \pmod{3}$, $4 \leq k < n$. Let $T$ be a tree belonging to the family $C_n$. Thus $T$ can be obtained from a tree $T'$ by either one operation of type (1) or one operation of type (2). If $T$ is constructed from $T'$ as a result of operation of type (1), then $T'$ is a tree of order $n - 3$ and by our induction hypothesis $T' \in T_{n-3}$. Therefore, by Lemma 5, $T \in T_n$.

If $T$ is obtained from $T'$ by one operation of type (2), then $T'$ is a tree of order $n - 1$. We consider two cases:
Case 1. If \( n = 1 \) \((\text{mod} \; 3)\), then the construction of \( T' \) is accomplished by using only type \((1)\) operations starting with the path \( P_3 \) and thus \( T' \in C_{n-1} \). From Lemma 6 we conclude that \( T' \in T_{n-1} \). Hence, by Lemma 8, \( T \in T_n \).

Case 2. If \( n \equiv 2 \) \((\text{mod} \; 3)\), then \( T' \in C_{n-1} \) and by our induction hypothesis \( T' \in T_{n-1} \). Finally, by Lemma 8, \( T \in T_n \).

**Lemma 10.** If \( n \geq 4 \) and \( n \not\equiv 0 \) \((\text{mod} \; 3)\), then \( T_n \subseteq C_n \).

**Proof.** We proceed by induction on \( n \geq 4 \). Since \( P_4 = \{P_4, K_{1,3}\} = C_4 \) and \( P_5 = \{K_{1,4}, P_3, T_1\} = C_5 \), where \( T_1 \) is a tree obtained from a star \( K_{1,3} \) by subdivision of exactly one of its edges, the result is true for \( n = 4 \) and \( n = 5 \).

Let \( n \geq 7 \) satisfy \( n \not\equiv 0 \) \((\text{mod} \; 3)\), and assume that \( T_k \subseteq C_k \) for all integers \( k \not\equiv 0 \) \((\text{mod} \; 3)\), where \( 4 \leq k < n \). Let \( T \in T_n \) and let \( D \) be a minimum total outer-connected dominating set of \( T \). Let \( P = (v_1, v_2, \ldots, v_m) \) be the longest path in \( T \). By Lemma 3, \( \{v_1, v_2, v_{m-1}, v_m\} \subseteq D \). We consider two cases:

**Case 1.** One of the vertices \( v_2 \) or \( v_{m-1} \) is adjacent to at least two endvertices. Without loss of generality, we can assume that \( |N_T(v_2) \cap \Omega(T)| \geq 2 \). Let \( l_1 \in N_T(v_2) \cap \Omega(T) \), \( l_1 \neq v_1 \). In this case \( D' = D - \{l_1\} \) is a total outer-connected dominating set of \( T' = T - l_1 \) and hence \( \gamma_{tc}(T') \leq \gamma_{tc}(T) - 1 = \left\lceil \frac{2n-3}{3} \right\rceil = \left\lceil \frac{2n-2}{3} \right\rceil \). Thus, Theorem 1 implies \( \gamma_{tc}(T') = \left\lceil \frac{2n-2}{3} \right\rceil \). Depending on whether \( n \equiv 1 \) \((\text{mod} \; 3)\) or \( n \equiv 2 \) \((\text{mod} \; 3)\) we have \( T' \in C_{n-1} \) from Lemma 7 or by our induction hypothesis, respectively. Hence we obtain \( T \in C_n \).

**Case 2.** The vertices \( v_2 \) and \( v_{m-1} \) have degree 2. Suppose that \( v_3 \) or \( v_{m-2} \), say \( v_3 \), belongs to \( D \). Then for tree \( T' = T - v_1 \) and for \( D' = D - \{v_1\} \), similarly to Case 1, we have that \( T \in C_n \). Hence we can assume that \( \{v_3, v_{m-2}\} \cap D = \emptyset \). Thus from connectivity of \( V(T) - D \) we have \( \{v_4, v_{m-3}\} \cap D = \emptyset \).

We will now show that \( v_3 \) or \( v_{m-2} \) is of degree two. Suppose to the contrary, that neither \( v_3 \) nor \( v_{m-2} \) is of degree 2. Let \( y \) be the neighbour of \( v_3, y \neq v_2 \) and \( y \neq v_4 \), and let \( z \) be the neighbour of \( v_{m-2}, z \neq v_{m-1} \) and \( z \neq v_{m-3} \). Then neither \( y \) nor \( z \) is not an end-vertex – otherwise we would have \( v_3 \in D \) or \( v_{m-2} \in D \). From that and from our choice of path \( (v_1, v_2, \ldots, v_m) \) it is straightforward that \( y \) and \( z \) are supports and \( A = N_T(y) - \{v_3\} \subseteq \Omega(T) \), \( B = N_T(z) - \{v_{m-2}\} \subseteq \Omega(T) \). We also have that \( D - (A \cup B \cup \{y, z\}) \) is a total outer-connected dominating set of \( T' = T - (A \cup B \cup \{y, z\}) \), and so \( \left\lceil \frac{2(n-2)|A|-|B|}{3} \right\rceil \leq \gamma_{tc}(T') \leq \gamma_{tc}(T) - 2 - |A| - |B| \leq \left\lceil \frac{2n}{3} \right\rceil - 2 - |A|-|B| \), which
is impossible. Therefore, without the loss of generality, we may assume that $\deg_T(v_3) = 2$.

Let us consider $T' = T - \{v_1, v_2, v_3\}$. The set $D' = D - \{v_1, v_2\}$ is a total outer-connected dominating set of $T'$, and hence $\gamma_{tc}(T') \leq \lceil \frac{2n}{3} \rceil - 2 = \lceil \frac{2n-6}{3} \rceil$. Moreover, by Theorem 1, $\gamma_{tc}(T') \geq \lceil \frac{2(n-3)}{3} \rceil$ and so $T' \in \mathcal{T}_{n-3}$. Therefore, by the inductive hypothesis, $T' \in \mathcal{C}_{n-3}$. However, $T$ is constructed from $T'$ by a type (1) operation. Hence $T \in \mathcal{C}_n$.

Theorem 2 now follows immediately from Lemmas 6, 7, 9 and 10.

References


Received 18 March 2009
Revised 27 July 2009
Accepted 17 August 2009