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RANDOM PROCEDURES FOR DOMINATING SETS IN BIPARTITE GRAPHS

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Abstract

Using multilinear functions and random procedures, new upper bounds on the domination number of a bipartite graph in terms of the cardinalities and the minimum degrees of the two colour classes are established.

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We consider finite, undirected and simple graphs without isolated vertices. The domination number $\gamma = \gamma(G)$ of a graph $G = (V, E)$ is the minimum cardinality of a set $D \subseteq V$ of vertices such that every vertex in $V \setminus D$ has a neighbour in D. This parameter is one of the most well-studied in graph theory, and the two volume monograph [12, 13] provides an impressive account of the research related to this concept.

Here we establish upper bounds on the domination number of a bipartite graph. Note that the decision problem DOMINATION remains NP-complete if the instance is restricted to bipartite graphs (e.g., see [7]).

Many random procedures constructing dominating sets essentially yield a bound on the domination number in terms of a multilinear function depending on the involved probabilities. For instance, if we use an individual probability x_i for every vertex $v_i \in V = \{v_1, ..., v_n\}$ of the graph G in the procedure of Alon and Spencer [1], then the expected cardinality of the resulting dominating set equals $\sum_{i=1}^{n} \left(x_i + \prod_{v_j \in N_G[v_i]}(1 - x_j)\right)$. This is in fact a multilinear function, i.e., fixing all but one variable results in a linear function.

To obtain a compact expression as a bound, one often sets all values of x_i equal to some x and solves the arising one-dimensional optimization problem over $x \in [0, 1]$.

A modification of this approach is proposed in [3, 8, 10]. Given values for the probabilities x_i, the partial derivatives of the multilinear bound indicate changes of the x_i which would decrease the value of the bound. Depending on the partial derivatives, x_i is reset to 0 or 1. To allow for some further flexibility in [3], a parameter $b \geq 0$ is used in order to decide which values to modify in which way.

Here we apply the approach in [3] for bipartite graphs. For a bipartite graph $G = (V, E)$ with vertex set $V = S \cup T = \{v_1, v_2, \ldots, v_n\}$, we derive upper bounds on the domination number γ of G in terms of the minimum degrees, δ_1 and δ_2, of the vertices in the colour classes S and T, respectively, $\rho = \frac{|S|}{|V|}$, and n.

The following Theorem 1 is the main result of that paper and is applicable if a result $\gamma \leq \min_{(x_1,\ldots,x_n)\in[0,1]^n} f(x_1, \ldots, x_n)$ for a multilinear function $f : R^n \to R$ associated to the graph G is known (e.g., such results can be found in [1, 3, 8, 9, 10]) and the function f has a certain property P_b, where $b \geq 0$ is the mentioned parameter used in [3]. The rest of the paper is organized as follows. As an example how to apply Theorem 1, in Lemma 2 a special function f having property P_1 is considered. The resulting upper bounds on γ by using the function f of Lemma 2 are contained in the following corollaries. Finally, we give some numerical bounds on $\frac{\gamma}{|V|}$ and compare them with bounds in [1, 2, 3, 5, 6, 8, 9, 10, 14].

Given a multilinear function $f(x_1, \ldots, x_n)$, $S \subseteq \{v_1, \ldots, v_n\}$, some $x, y \in [0, 1]$ and some $b \geq 0$, consider the following algorithm $A_b(x, y)$.

Algorithm. $A_b(x, y)$

1. For i from 1 to n do: if $v_i \in S$ then $x_i := x$ else $x_i := y$.
2. For i from 1 to n do: if $f_{x_i}(x_1, \ldots, x_n) > -b$ then $x_i := 0$ else $x_i := 1$.
3. For i from 1 to n do: if $f_{x_i}(x_1, \ldots, x_n) \leq -b$ then $x_i := 1$.
4. Output $(x_1, \ldots, x_n)$.

Theorem 1. *Let* $G = (V, E)$ *be a bipartite graph with vertex set* $V = S \cup T = \{v_1, v_2, \ldots, v_n\}$, $|S| = s, |T| = t$ *and minimum degree* δ. *Let*

$f(x_1, \ldots, x_n)$ be a multilinear function such that

$$\gamma \leq \min_{(x_1,\ldots,x_n)\in[0,1]^n} f(x_1, \ldots, x_n). \tag{1}$$

Furthermore, for some $b \geq 0$ and every $x, y \in [0,1]$, let the Algorithm $A_b(x,y)$ produce a vector $(x_1, x_2, \ldots, x_n)$, where the property that $x_k = 0$ for all $1 \leq k \leq n$ with $v_k \in N_G[v_i] \cup N_G[v_j]$ for some $1 \leq i < j \leq n$ implies $\mathrm{dist}_G(v_i, v_j) \geq 3$. Given $x, y \in [0,1]$, then let $z_i = x$ if $v_i \in S$ else $z_i = y$ for $i = 1, \ldots, n$. Then

$$\gamma \leq \min_{x,y\in[0,1]} \left(\frac{\delta}{\delta(1+b)+b} f(z_1, \ldots, z_n) + \frac{b(\delta x + 1)}{\delta(1+b)+b} s + \frac{b(\delta y + 1)}{\delta(1+b)+b} t \right).$$

Before we proceed to the proof of Theorem 1, we introduce some terminology. Given the situation described in Theorem 1, we will call a vertex $v_i \in V$ *critical* if $x_k = 0$ for all $1 \leq k \leq n$ with $v_k \in N_G[v_i]$. The property described in Theorem 1 means that Algorithm $A_b(x,y)$ produces a vector $(x_1, x_2, \ldots, x_n)$ for which the critical vertices have pairwise distance at least three. If the function f — associated to the graph G — has this property, then we say that f *has property $\mathcal{P}_b$*.

***Proof of Theorem* 1.** Let G, b and f be as in the statement of Theorem 1. Since f is multilinear, we have for all $x_1, \ldots, x_n, y \in \mathbb{R}$

$$\begin{aligned} & f(x_1, \ldots, x_{i-1}, x_i + y, x_{i+1}, \ldots, x_n) \\ & = f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \\ & + \frac{\partial}{\partial x_i} f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \cdot y. \end{aligned} \tag{2}$$

For some $x, y \in [0,1]$, let $(x_1, \ldots, x_n)$ denote the output of Algorithm $A_b(x,y)$. Let

$$M = \{v_i \in V | x_i = 1\}.$$

Note that a vertex v_i is critical exactly if $N_G[v_i] \cap M = \emptyset$.

Claim 1. $\gamma \leq f(z_1, \ldots, z_n) - b|M| + bxs + byt$.

***Proof of Claim* 1.** By (1), $\gamma \leq f(z_1, \ldots, z_n)$. We consider the Algorithm $A_b(x,y)$. After Step 1, $(x_1, \ldots, x_n) = (z_1, \ldots, z_n)$. If during Step 2 some

$x_i = x$ is replaced by 1, then, by (2), the value of $f(x_1, \dots, x_n)$ decreases at least by $b(1-x)$. Similarly, if during Step 2 some $x_i = x$ is replaced by 0, then, by (2), the value of $f(x_1, \dots, x_n)$ increases at most by bx. Furthermore, if during Step 3 some $x_i = 0$ is replaced by 1, then $x_i = x$ was replaced by 0 in Step 2 and summing the effect of the changes in x_i made by Step 2 and Step 3, $f(x_1, \dots, x_n)$ decreases at least by $b(1-x)$ in total. Altogether,

$$\begin{aligned} f(x_1, \dots, x_n) &\le f(z_1, \dots, z_n) - b(1-x)|M \cap S| \\ &+ bx(s - |M \cap S|) - b(1-y)|M \cap T| + by(t - |M \cap T|) \\ &= f(z_1, \dots, z_n) - b|M| + bxs + byt \end{aligned}$$

which completes the proof of the claim. □

Let k be the number of critical vertices and let D be obtained by adding all critical vertices to M. Clearly, D is a dominating set of G, $\gamma \le |D| = |M| + k$, and, by Claim 1,

$$\begin{aligned} \gamma &= \left(\frac{1}{1+b} + \frac{b}{1+b}\right)\gamma \\ &\le \frac{1}{1+b}(f(z_1, \dots, z_n) - b|M| + bxs + byt) + \frac{b}{1+b}|D| \\ &= \frac{1}{1+b}(f(z_1, \dots, z_n) - b(|D| - k) + bxs + byt) + \frac{b}{1+b}|D| \\ &= \frac{1}{1+b}f(z_1, \dots, z_n) + \frac{b}{1+b}(k + xs + yt). \end{aligned} \tag{3}$$

Since f has property $\mathcal{P}_b$,

$$\gamma \le n - \delta k. \tag{4}$$

Since $\frac{\delta(1+b)}{\delta(1+b)+b} + \frac{b}{\delta(1+b)+b} = 1$, a convex combination of (3) and (4) yields

$$\begin{aligned} \gamma \le \frac{\delta(1+b)}{\delta(1+b)+b} &\left(\frac{1}{1+b}f(z_1, \dots, z_n) + \frac{b}{1+b}(k + xs + yt)\right) \\ &+ \frac{b}{\delta(1+b)+b}(n - \delta k) \end{aligned}$$

$$= \frac{\delta}{\delta(1+b)+b} f(z_1,\ldots,z_n) + \frac{b(\delta x+1)}{\delta(1+b)+b} s + \frac{b(\delta y+1)}{\delta(1+b)+b} t.$$

Since x and y were arbitrary in $[0,1]$, the theorem follows. ■

We remark that for fixed x and y the upper bound $T(b) = \frac{\delta}{\delta(1+b)+b} f(z_1,\ldots, z_n) + \frac{b(\delta x+1)}{\delta(1+b)+b} s + \frac{b(\delta y+1)}{\delta(1+b)+b} t$ on γ equals the upper bound $f(z_1,\ldots,z_n)$ if $b=0$, and that $T(b)$ is strictly decreasing in b if $f(z_1,\ldots,z_n) > \frac{\delta xs+\delta yt+n}{\delta+1}$. Hence, if $f(z_1,\ldots,z_n)$ is large then $T(b_0)$ is a reasonable upper bound on γ, where b_0 (if it exists) is the largest b such that f has property P_b.

Our next lemma is proven in [3] and gives an upper bound on the domination number in terms of a multilinear function as required for Theorem 1 (similar bounds are contained in [8]). Additionally, we have to verify property $\mathcal{P}_b$ for some b. For the sake of completeness, we give a proof of Lemma 2 here as well.

Lemma 2. *If $G=(V,E)$ is a graph with vertex set $V=\{v_1,\ldots,v_n\}$, then*

$$\gamma = \min_{(x_1,\ldots,x_n)\in[0,1]^n} f(x_1,\ldots,x_n) \tag{5}$$

where

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n}\left(x_i + \prod_{v_j\in N_G[v_i]}(1-x_j) - \frac{1}{1+d_G(v_i)}\prod_{v_j\in N_G[v_i]} x_j\right). \tag{6}$$

Furthermore, the function f in (6) *has property $\mathcal{P}_1$.*

***Proof of Lemma* 2.** Let $(x_1,\ldots,x_n)\in[0,1]^n$ and let $X\subseteq V$ be a set of vertices containing every vertex v_i independently at random with probability x_i. Let

$$X' = \{v_i\in V \mid N_G[v_i]\subseteq X\}$$

and let I be a maximum independent set in the subgraph $G[X']$ induced by X'. If

$$Y=\{v\in V| N_G[v]\cap X=\emptyset\},$$

then $(X\setminus I)\cup Y$ is a dominating set of G, and hence $\gamma \le \mathbf{E}[|X|]+\mathbf{E}[|Y|]-\mathbf{E}[|I|]$. Clearly, $\mathbf{E}[|X|]=\sum_{i=1}^n x_i$ and $\mathbf{E}[|Y|]=\sum_{i=1}^n\prod_{v_j\in N_G[v_i]}(1-x_j)$.

By the Caro-Wei inequality [4, 15],

$$\begin{aligned}\mathbf{E}[|I|] &\geq \sum_{v\in X'} \frac{1}{1+d_{G[X']}(v)} \geq \sum_{v\in V} \frac{1}{1+d_G(v)}\mathbf{P}[v\in X'] \\ &= \sum_{i=1}^{n} \frac{1}{1+d_G(v_i)} \prod_{v_j\in N_G[v_i]} x_j.\end{aligned}$$

This implies that γ is at most the expression given on the right hand side of (6). For the converse, let D be a minimum dominating set. Note that for every vertex $v_i \in V$, we have $N_G[v_i] \cap D \neq \emptyset$, since D is dominating and $N_G[v_i] \cap D \neq N_G[v_i]$, because D is minimum. Therefore, setting $x_i^* = 1$ for all $v_i \in D$ and $x_i^* = 0$ for all $v_i \in V \setminus D$ yields

$$\begin{aligned}\gamma &= \sum_{i=1}^{n}\left(x_i^* + \prod_{v_j\in N_G[v_i]}(1-x_j^*) - \frac{1}{1+d_G(v_i)}\prod_{v_j\in N_G[v_i]} x_j^*\right) \\ &= \sum_{i=1}^{n}(x_i^*+0+0) = |D| = \gamma.\end{aligned}$$

The proof of (5) is thus complete.

Now we proceed to the proof that f has property $\mathcal{P}_1$. Therefore, let $x, y \in [0,1]$, let $(x_1,\dots,x_n)$ be the output of Algorithm $A_1(x,y)$ and let v_i and v_j be two critical vertices. For contradiction, we assume that $N_G[v_i] \cap N_G[v_j] \neq \emptyset$. Note that after the execution of Step 2, the values x_l for all $v_l \in N_G[v_i] \cup N_G[v_j]$ are 0 and remain 0 throughout the execution of Step 3. For $1 \leq k \leq n$ we have

$$\begin{aligned}&\frac{\partial}{\partial x_k} f(x_1,\dots,x_n) \\ &= 1 - \sum_{v_l\in N_G[v_k]}\left(\prod_{v_m\in N_G[v_l]\setminus\{v_k\}}(1-x_m) + \frac{1}{1+d_G(v_l)}\prod_{v_m\in N_G[v_l]\setminus\{v_k\}} x_m\right).\end{aligned}$$

If $v_j \in N_G[v_i]$, then during the execution of Step 3

$$\frac{\partial}{\partial x_i} f(x_1,\dots,x_n) \leq 1 - \prod_{v_m\in N_G[v_i]\setminus\{v_i\}}(1-x_m) - \prod_{v_m\in N_G[v_j]\setminus\{v_i\}}(1-x_m) = -1,$$

and if $v_k \in N_G(v_i) \cap N_G(v_j)$, then during the execution of Step 3

$$\frac{\partial}{\partial x_k} f(x_1, \ldots, x_n) \leq 1 - \prod_{v_m \in N_G[v_i] \setminus \{v_k\}} (1 - x_m) - \prod_{v_m \in N_G[v_j] \setminus \{v_k\}} (1 - x_m) = -1.$$

In both cases, we obtain the contradiction that either x_i or x_k would be set to 1 by Step 3 and the proof is complete. ■

Theorem 1 and Lemma 2 immediately imply the following result for $b = 1$.

Corollary 3. *If $G = (V, E)$ is a bipartite graph with vertex set $V = S \cup T = \{v_1, v_2, \ldots, v_n\}$, $|S| = s, |T| = t$ and minimum degree δ, then*

$$\begin{aligned}\gamma \ \leq\ & \frac{1}{2\delta + 1}\Bigg((2\delta x + 1)s + (2\delta y + 1)t \\ & + \delta \sum_{v \in S} \left((1 - x)(1 - y)^{d_G(v)} - \frac{1}{1 + d_G(v)} x y^{d_G(v)}\right) \\ & + \delta \sum_{v \in T} \left((1 - y)(1 - x)^{d_G(v)} - \frac{1}{1 + d_G(v)} y x^{d_G(v)}\right)\Bigg)\end{aligned}$$

for every $x, y \in [0, 1]$.

Clearly, the following corollary holds.

Corollary 4. *Let $G = (V, E)$ be a bipartite graph with vertex set $V = S \cup T = \{v_1, v_2, \ldots, v_n\}$, δ_1 and δ_2 the minimum degrees in S and T, respectively, $\delta_1 \leq \delta_2$ and $\rho \in [0, 1]$ such that $|S| = \rho|V|$.*

Then $\gamma \leq h(x, y)|V| \leq g(x, y)|V|$ for every $x, y \in [0, 1]$, where

$$h(x, y) =$$

$$\frac{2\delta_1 x\rho + 2\delta_1 y(1 - \rho) + 1 + \delta_1 \rho(1 - x)(1 - y)^{\delta_1} + \delta_1(1 - \rho)(1 - y)(1 - x)^{\delta_2}}{2\delta_1 + 1}$$

and

$$g(x, y) = \frac{2\delta_1 x\rho + 2\delta_1 y(1 - \rho) + 1 + \delta_1 \rho(1 - y)^{\delta_1} + \delta_1(1 - \rho)(1 - x)^{\delta_2}}{2\delta_1 + 1}.$$

We also can derive the following bound.

Corollary 5. *Let $G = (V, E)$ be a bipartite graph with vertex set $V = S \cup T = \{v_1, v_2, \ldots, v_n\}$, δ_1 and δ_2 the minimum degrees in S and T, respectively, $\delta_1 \le \delta_2$ and $\rho \in [0, 1]$ such that $|S| = \rho|V|$.*

Then $\gamma \le \phi(x, y)|V|$ for every $x, y \in [0, \frac{1}{2}]$, where

$$\phi(x,y) =$$

$$\frac{1}{2\delta_1+1}\left(2\delta_1 x\rho + 2\delta_1 y(1-\rho) + 1 + \delta_1\rho\left((1-x)(1-y)^{\delta_1} - \frac{1}{1+\delta_1}xy^{\delta_1}\right)\right.$$
$$\left. + \delta_1(1-\rho)\left((1-y)(1-x)^{\delta_2} - \frac{1}{1+\delta_2}yx^{\delta_2}\right)\right).$$

***Proof of Corollary* 5.**

Claim 2. If $0 \le p, q \le \frac{1}{2}$ (p and q real numbers) and $m \ge n$ (m and n positive integers), then

$$(1-p)(1-q)^m - \frac{1}{m+1}pq^m \le (1-p)(1-q)^n - \frac{1}{n+1}pq^n.$$

***Proof of Claim* 2.** In case $p = 0$ or $q = 0$ nothing is to prove.
Let $p, q > 0$. We prove that
$(1-p)(1-q)^{k+1} - \frac{1}{k+2}pq^{k+1} \le (1-p)(1-q)^k - \frac{1}{k+1}pq^k$ if $k \ge 1$.

Because of $(1-p)(1-q)^{k+1} = (1-p)(1-q)^k - (1-p)q(1-q)^k$, this inequality is equivalent to $\frac{1}{q(k+1)} \le (\frac{1-p}{p})(\frac{1-q}{q})^k + \frac{1}{k+2}$. From $p \le \frac{1}{2}$, it follows $\frac{1-p}{p} \ge 1$. Hence, it suffices to show that $\frac{1}{q(k+1)} \le (\frac{1-q}{q})^k = (\frac{1}{q} - 1)^k$ is true because $\frac{1}{q} \ge 2$, and that the function $(k+1)(z-1)^k - z$ is increasing in z if $z \ge 2$ and $k \ge 1$. □

Let $0 \le x, y \le \frac{1}{2}$. Using Claim 2, Corollary 3 implies

$$\gamma \le \frac{1}{2\delta+1}\left((2\delta x+1)s + (2\delta y+1)t + \delta s\left((1-x)(1-y)^{\delta_1} - \frac{1}{1+\delta_1}xy^{\delta_1}\right)\right.$$
$$\left. + \delta t\left((1-y)(1-x)^{\delta_2} - \frac{1}{1+\delta_2}yx^{\delta_2}\right)\right),$$

and because $s = \rho|V|$, $t = (1-\rho)|V|$ and $\delta = \delta_1$, Corollary 5 is proven. ■

It is easy to calculate $\min(g) = \min\{g(x,y) \mid 0 \le x,y \le 1\}$ by analytical methods (e.g., see [9]). It follows $\min(g) = g(x^*,y^*)$, where $x^* = \max\{0, 1-\big(\frac{2(1-\rho)}{\delta_1\rho}\big)^{\frac{1}{\delta_1-1}}\}$ and $y^* = \max\{0, 1-\big(\frac{2\rho}{\delta_2(1-\rho)}\big)^{\frac{1}{\delta_2-1}}\}$. If $\delta_1 \ge 1$ and $\frac{\delta_1}{2^{\delta_1}} \le \frac{1-\rho}{\rho} \le \frac{2^{\delta_2}}{\delta_2}$, then $x^*, y^* \le \frac{1}{2}$. Hence, we obtain compact expressions as bounds on $\frac{\gamma(G)}{|V|}$ as follows.

Corollary 6. *$\frac{\gamma}{|V|} \le h(x^*,y^*)$. If $\frac{\delta_1}{2^{\delta_1}} \le \frac{1-\rho}{\rho} \le \frac{2^{\delta_2}}{\delta_2}$, then $\frac{\gamma}{|V|} \le \phi(x^*,y^*)$.*

Since both S and T are dominating, it follows $\frac{\gamma}{|V|} \le \min\{\rho, 1-\rho\}$. If $\frac{\delta_1}{2^{\delta_1}} > \frac{1-\rho}{\rho}$ or $\frac{1-\rho}{\rho} > \frac{2^{\delta_2}}{\delta_2} \ge \frac{2^{\delta_1}}{\delta_1}$ (see Corollary 6), then $\min\{\rho, 1-\rho\} < \frac{\delta_1}{\delta_1+2^{\delta_1}}$, and if δ_1 is large, then $\min\{\rho, 1-\rho\}$ is an attractive bound on $\frac{\gamma}{|V|}$ in this case.

Numerical evaluations show that quite often the trivial upper bound $\min\{\rho, 1-\rho\}$ is smaller then $\min(h) = \min\{h(x,y) \mid 0 \le x,y \le 1\}$ or $\min(\phi) = \min\{\phi(x,y) \mid 0 \le x,y \le \frac{1}{2}\}$. Thus, we will consider the bound $B = \min\{\min(h), \min(\phi), \rho, 1-\rho\}$.

We list the following upper bounds C, D, E and F on $\frac{\gamma(G)}{|V|}$ which are in terms of δ and hold for arbitrary graphs. $C = \frac{\ln(\delta+1)+1}{\delta+1}$ (see [1]), $D = \frac{1}{\delta+1}\sum_{i=1}^{\delta+1}\frac{1}{i}$ (see [2, 14]), $E = 1-\big(\frac{1}{\delta+1}\big)^{\frac{1}{\delta}}\frac{\delta}{\delta+1}$ (see [5, 6]),
$F = \frac{1}{2\delta+1}\big((2\delta x_0+1)+\delta\big((1-x_0)^{\delta+1}-\frac{1}{1+\delta}x_0^{\delta+1}\big)\big)$, where x_0 is the unique solution of $(\delta+1)(1-x)^\delta + x^\delta = 2$ in $\big[0, \frac{1}{2}\big]$ (see [3]).

An upper bound on $\frac{\gamma}{|V|}$ for an arbitrary graph G in terms of δ and the maximum degree Δ is given in [8]. If Δ is not limited for a class of graphs in question (and this is the case in the class of bipartite graphs being considered here), this bound tends to E if Δ tends to infinity.

The following upper bound H on $\frac{\gamma}{|V|}$ for a bipartite graph G in terms of δ and ρ was established in [11].
If $\frac{e\delta}{\delta^2-1+e(\delta+1)} \le \rho \le \frac{1}{2}$ then $\frac{\gamma}{|V|} \le H = \frac{1}{\delta+1} + \frac{\rho}{\delta^2-1}\big(\ln\big(\frac{\delta(1-\rho)-\rho}{(\delta^2-1)\rho}\big) - \delta\ln\big(\frac{\delta\rho-(1-\rho)}{(\delta^2-1)(1-\rho)}\big)\big) + \frac{(1-\rho)}{\delta^2-1}\big(\ln\big(\frac{\delta\rho-(1-\rho)}{(\delta^2-1)(1-\rho)}\big) - \delta\ln\big(\frac{\delta(1-\rho)-\rho}{(\delta^2-1)\rho}\big)\big)$.

To our knowledge, upper bounds on $\frac{\gamma}{|V|}$ for a bipartite graph G in terms of δ_1, δ_2 and ρ are rare in the literature. Here we present such a bound I which was proven in [9].
$\frac{\gamma}{|V|} \le I = \min\{\rho x + (1-\rho)y + \rho(1-x)(1-y)^{\delta_1} + (1-\rho)(1-y)(1-x)^{\delta_2} \mid 0 \le x,y \le 1\}$.

It is easy to see that $C = \min\{x+e^{-x(\delta+1)} \mid 0 \le x \le 1\}$ and $E = \min\{x+(1-x)^{\delta+1} \mid 0 \le x \le 1\}$. Because $1-x \le e^{-x}$, it follows $E \le C$. Again, because $1-x \le e^{-x}$, it follows that $I \le \min\{\psi(x,y) = \rho x + (1-\rho)y + \rho e^{-x-\delta_1 y} + (1-\rho)e^{-y-\delta_2 x} \mid 0 \le x, y \le 1\}$. In [11], it is shown that $H = \psi(\hat{x}, \hat{y})$ for special values $\hat{x}, \hat{y} \in [0,1]$, and hence, $I \le H$.

We conclude this paper by presenting some numerical results for B with some special values of ρ, δ_1 and δ_2 (see Table 1) and comparing them with the corresponding values of D, E, F and I in Table 2. Note that D, E and F do not depend on the choice of ρ and δ_2, and that these bounds are valid for arbitrary graphs. The outcome of this comparison is the large difference between this general bounds and B.

Table 1

ρ	δ_2	$\delta_1 = 3$	$\delta_1 = 5$	$\delta_1 = 10$	$\delta_1 = 20$	$\delta_1 = 40$
0.1	3	0.1	-	-	-	-
	30	0.1	0.1	0.1	0.0831	-
	60	0.1	0.1	0.1	0.0788	0.0606
	100	0.1	0.1	0.0989	0.0769	0.0576
0.3	3	0.3	-	-	-	-
	30	0.2927	0.2498	0.1961	0.1443	-
	60	0.2837	0.2403	0.1826	0.1286	0.0896
	100	0.2796	0.2360	0.1760	0.1213	0.0818
0.5	3	0.4890	-	-	-	-
	30	0.3761	0.3012	0.2164	0.1564	-
	60	0.3609	0.2835	0.1964	0.1349	0.0949
	100	0.3535	0.2746	0.1862	0.1240	0.0835
0.7	3	0.3	-	-	-	-
	30	0.3	0.2721	0.1932	0.1411	-
	60	0.3	0.2549	0.1728	0.1191	0.0859
	100	0.3	0.2455	0.1621	0.1075	0.0739
0.9	3	0.1	-	-	-	-
	30	0.1	0.1	0.1	0.0857	-
	60	0.1	0.1	0.1	0.0777	0.0574
	100	0.1	0.1	0.1	0.0714	0.0503

Table 2

ρ	δ_1	δ_2	B	I	D	E	F
0.1	3	30	0.1	0.1	0.521	0.528	0.490
0.1	3	60	0.1	0.1			
0.1	3	100	0.1	0.1			
0.1	10	30	0.1	0.1	0.275	0.285	0.270
0.1	10	60	0.1	0.1			
0.1	10	100	0.099	0.1			
0.1	20	30	0.083	0.092	0.174	0.182	0.174
0.1	20	60	0.079	0.087			
0.1	20	100	0.077	0.085			
0.1	40	60	0.061	0.065	0.105	0.111	0.107
0.1	40	100	0.058	0.062			
0.5	3	30	0.376	0.360			
0.5	3	60	0.361	0.339			
0.5	3	100	0.353	0.329			
0.5	10	30	0.216	0.214			
0.5	10	60	0.196	0.189			
0.5	10	100	0.186	0.177			
0.5	20	30	0.156	0.160			
0.5	20	60	0.135	0.133			
0.5	20	100	0.124	0.121			
0.5	40	60	0.095	0.097			
0.5	40	100	0.084	0.084			
0.9	3	30	0.1	0.1			
0.9	3	60	0.1	0.1			
0.9	3	100	0.1	0.1			
0.9	10	30	0.1	0.095			
0.9	10	60	0.1	0.081			
0.9	10	100	0.1	0.071			
0.9	20	30	0.086	0.085			
0.9	20	60	0.077	0.067			
0.9	20	100	0.071	0.056			
0.9	40	60	0.057	0.057			
0.9	40	100	0.050	0.046			

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