

ON THE MATRIX FORM OF KRONECKER LEMMA

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Abstract

A matrix generalization of Kronecker's lemma is presented with assumptions that make it possible not only the unboundedness of the condition number considered by Anderson and Moore (1976) but also other sequences of real matrices, not necessarily monotone increasing, symmetric and nonnegative definite. A useful matrix decomposition and a well-known equivalent result about convergent series are used in this generalization.

Keywords: matrix Kronecker lemma, matrix analysis, convergence.

2000 Mathematics Subject Classification: 15A23 (15A09), 40A05.

1. INTRODUCTION

A result due to Kronecker which is a *sine qua non* for probability theory (see [2], page 114) states the following:

Kronecker lemma. *If $\{a_k\}$ and $\{q_k\}$ are sequences of real numbers for which $\sum q_k^{-1} a_k$ is convergent and q_k is monotone, increasing and positive such that $q_k \rightarrow \infty$ as $k \rightarrow \infty$ then*

$$\lim_{n \rightarrow \infty} q_n^{-1} \sum_{k=1}^n a_k = 0.$$

In 1976, Anderson and Moore consider conditions on sequences of matrices \mathbf{Q}_k and vectors \mathbf{a}_k that permitted the following matrix generalization of the Kronecker lemma:

$$(1) \quad \lim_{n \rightarrow \infty} \mathbf{Q}_n^{-1} \sum_{k=1}^n \mathbf{a}_k = \mathbf{0}.$$

One of the condition to establish (1) is that the ratio of the largest eigenvalue to the smallest eigenvalue of \mathbf{Q}_n must be bounded for all n (termed *the condition number*). Nevertheless, if \mathbf{Q}_k is a monotone increasing sequence of $p \times p$ nonnegative definite real symmetric matrices (i.e., $\mathbf{Q}_k - \mathbf{Q}_{k-1}$ is nonnegative definite for all k) and $\lambda_{\max}(\mathbf{Q}_n)/\lambda_{\min}(\mathbf{Q}_n)$ is bounded then

$$\lambda_1(\mathbf{Q}_n) \cdots \lambda_p(\mathbf{Q}_n) \asymp [\mathbf{Q}_n]_{11} \cdots [\mathbf{Q}_n]_{pp}, \quad n \rightarrow \infty$$

provided Hadamard's inequality (see [3], page 477), that is, all principal entries of \mathbf{Q}_n , unless a constant, are asymptotically equivalent*. In this way, the Anderson & Moore's hypothesis restrict, in some sense, the choices of the sequences of matrices \mathbf{Q}_k in the problem of the generalization of the classical Kronecker lemma.

In this work, starting from a fundamental assumption for the unidimensional version of Kronecker lemma, the monotony[†] of the real sequence q_k (see [4], pages 37 and 181), we will provide sufficient conditions to get (1)

* $\alpha_n \asymp \beta_n$ means that $\alpha_n = O(\beta_n)$ and $\beta_n = O(\alpha_n)$.

[†]A real sequence is *monotone increasing* (resp. *monotone decreasing*) if $\alpha_n \leq \alpha_{n+1}$, $\forall n \in \mathbb{N}$ (resp. $\alpha_n \geq \alpha_{n+1}$, $\forall n \in \mathbb{N}$).

in a different set of hypothesis for the sequence of matrices \mathbf{Q}_k from those considered by Anderson and Moore. This assumptions will allow to consider not only cases where the ratio of the largest eigenvalue to the smallest eigenvalue of \mathbf{Q}_k is unbounded but also cases where the sequence \mathbf{Q}_k is not necessarily symmetric, monotone increasing and nonnegative definite. The technique used in our approach consists in a useful matrix decomposition of \mathbf{Q}_k^{-1} with the purpose to get

$$\mathbf{Q}_k^{-1} = \mathbf{U}_k \bar{\mathbf{Q}}_k^{-1}$$

with \mathbf{U}_k upper triangular such that $\lim_{k \rightarrow \infty} \mathbf{U}_k = \mathbf{U}_\infty$ exists, is finite and nonsingular, and $\bar{\mathbf{Q}}_k^{-1}$ is convergent to the null matrix. In this process, the generalization is obtained with the aid of the classical Kronecker lemma and also with the following result about convergent series: given the real sequences $\{x_k\}$ and $\{y_k\}$ then the two properties above are equivalent,

- (a) $\sum |x_k - x_{k+1}| < \infty$;
- (b) if the series $\sum y_k$ converges then so does the series $\sum x_k y_k$.

(see page 39 of [4]; the proof can be found in pages 186 and 187 of the same reference).

2. MATRIX KRONECKER LEMMA

We start with an important auxiliary result which will be used in the proof of the main one. Given a sequence of matrices \mathbf{A}_n , let us denote $[\mathbf{A}_n]_{ij}$ the i - j th element of \mathbf{A}_n ; $M_{ij}(\mathbf{A}_n)$ the $(p-1) \times (p-1)$ minor of \mathbf{A}_n , obtained removing the i -th row and the j -th column of \mathbf{A}_n ; for $1 \leq j \leq p$ the principal submatrix of \mathbf{A}_n will be represented by $\mathbf{A}_n(\{1, \dots, j\})$.

Proposition 1. *Let \mathbf{a}_k , $k = 1, 2, \dots$ be a sequence of real p -vectors, \mathbf{U}_k , $k = 1, 2, \dots$ a sequence of nonsingular $p \times p$ upper triangular matrices with monotone entries and positive principal diagonal elements such that $\lim_{k \rightarrow \infty} \mathbf{U}_k = \mathbf{U}_\infty$ exists, is finite and nonsingular. If $\sum \mathbf{U}_k^{-1} \mathbf{a}_k$ exists and is finite then $\lim_{n \rightarrow \infty} \mathbf{U}_n^{-1} \sum_{k=1}^n \mathbf{a}_k$ exists and is finite.*

Proof. Using back substitution it's easy to see that

$$\mathbf{U}_k^{-1} = \begin{bmatrix} \frac{1}{[\mathbf{U}_k]_{11}} - \frac{[\mathbf{U}_k]_{12}}{[\mathbf{U}_k]_{22}} \cdot [\mathbf{U}_k^{-1}]_{11} - \frac{1}{[\mathbf{U}_k]_{33}} \cdot \sum_{i=1}^2 [\mathbf{U}_k]_{i3} [\mathbf{U}_k^{-1}]_{1i} & \cdots & -\frac{1}{[\mathbf{U}_k]_{pp}} \cdot \sum_{i=1}^{p-1} [\mathbf{U}_k]_{ip} [\mathbf{U}_k^{-1}]_{1i} \\ 0 & \frac{1}{[\mathbf{U}_k]_{22}} & -\frac{[\mathbf{U}_k]_{23}}{[\mathbf{U}_k]_{33}} \cdot [\mathbf{U}_k^{-1}]_{22} & \cdots & -\frac{1}{[\mathbf{U}_k]_{pp}} \cdot \sum_{i=2}^{p-1} [\mathbf{U}_k]_{ip} [\mathbf{U}_k^{-1}]_{2i} \\ 0 & 0 & \frac{1}{[\mathbf{U}_k]_{33}} & \cdots & -\frac{1}{[\mathbf{U}_k]_{pp}} \cdot \sum_{i=3}^{p-1} [\mathbf{U}_k]_{ip} [\mathbf{U}_k^{-1}]_{3i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{[\mathbf{U}_k]_{(p-1)p}}{[\mathbf{U}_k]_{pp}} \cdot [\mathbf{U}_k^{-1}]_{(p-1)(p-1)} \\ 0 & 0 & 0 & \cdots & \frac{1}{[\mathbf{U}_k]_{pp}} \end{bmatrix}.$$

Putting $\mathbf{a}_k = \left[[a_k]_1 \dots [a_k]_p \right]^T$ we get

$$\mathbf{U}_k^{-1} \mathbf{a}_k = \begin{bmatrix} \frac{1}{[\mathbf{U}_k]_{11}} \cdot [a_k]_1 - \frac{[\mathbf{U}_k]_{12}}{[\mathbf{U}_k]_{22}} [\mathbf{U}_k^{-1}]_{11} \cdot [a_k]_2 - \dots - \frac{1}{[\mathbf{U}_k]_{pp}} \sum_{i=1}^{p-1} [\mathbf{U}_k]_{ip} [\mathbf{U}_k^{-1}]_{1i} \cdot [a_k]_p \\ \frac{1}{[\mathbf{U}_k]_{22}} \cdot [a_k]_2 - \frac{[\mathbf{U}_k]_{23}}{[\mathbf{U}_k]_{33}} [\mathbf{U}_k^{-1}]_{22} \cdot [a_k]_3 - \dots - \frac{1}{[\mathbf{U}_k]_{pp}} \sum_{i=2}^{p-1} [\mathbf{U}_k]_{ip} [\mathbf{U}_k^{-1}]_{2i} \cdot [a_k]_p \\ \frac{1}{[\mathbf{U}_k]_{33}} \cdot [a_k]_3 - \dots - \frac{1}{[\mathbf{U}_k]_{pp}} \sum_{i=3}^{p-1} [\mathbf{U}_k]_{ip} [\mathbf{U}_k^{-1}]_{3i} \cdot [a_k]_p \\ \vdots \\ \frac{1}{[\mathbf{U}_k]_{(p-1)(p-1)}} \cdot [a_k]_{p-1} - \frac{[\mathbf{U}_k]_{(p-1)p}}{[\mathbf{U}_k]_{pp}} [\mathbf{U}_k^{-1}]_{(p-1)(p-1)} \cdot [a_k]_p \\ \frac{1}{[\mathbf{U}_k]_{pp}} \cdot [a_k]_p \end{bmatrix}.$$

From the assumptions we conclude first that $\sum [a_k]_p$ is convergent since $[\mathbf{U}_k]_{pp}$ converge monotonically; on the other hand we can conclude also that

$$\sum \frac{[\mathbf{U}_k]_{(p-1)p}}{[\mathbf{U}_k]_{pp}} [\mathbf{U}_k^{-1}]_{(p-1)(p-1)} \cdot [a_k]_p$$

is convergent provided that

$$\sum \frac{[\mathbf{U}_k]_{(p-1)p}}{[\mathbf{U}_k]_{pp}} \cdot [a_k]$$

($[\mathbf{U}_k]_{(p-1)p}$ is monotonically convergent) and thus

$$\sum \frac{[\mathbf{U}_k]_{(p-1)p}}{[\mathbf{U}_k]_{pp}} [\mathbf{U}_k^{-1}]_{(p-1)(p-1)} \cdot [a_k]_p$$

is convergent (since $[\mathbf{U}_k^{-1}]_{(p-1)(p-1)} = \frac{1}{[\mathbf{U}_k]_{(p-1)(p-1)}}$ is monotonically convergent). Hence

$$\sum \frac{1}{[\mathbf{U}_k]_{(p-1)(p-1)}} \cdot [a_k]_{p-1}$$

is convergent and $\sum [a_k]_{p-1}$ is also convergent (since $[\mathbf{U}_k]_{(p-1)(p-1)}$ is monotonically convergent). The proof is established using the same procedure to prove the convergence of the remaining components of $\sum \mathbf{a}_k$. ■

Remark 1. The same conclusion follows replacing the sequence of matrices \mathbf{U}_k , $k = 1, 2, \dots$ by sequences \mathbf{L}_k , $k = 1, 2, \dots$ of nonsingular $p \times p$ lower triangular matrices with monotone entries and positive principal diagonal elements such that $\lim_{n \rightarrow \infty} \mathbf{L}_n = \mathbf{L}_\infty$ is finite and nonsingular.

Theorem 1. Let $p \geq 2$ and $\mathbf{a}_k, k = 1, 2, \dots$ be a sequence of real p -vectors and $\mathbf{Q}_k, k = 1, 2, \dots$ a sequence of nonsingular $p \times p$ matrices with monotone entries and positive principal diagonal elements such that:

(i) $\lim_{n \rightarrow \infty} \mathbf{Q}_n^{-1} = \mathbf{O}$;

(ii) for each $2 \leq i \leq p$,

$$\frac{M_{ij}(\mathbf{Q}_n(\{1, \dots, i\})) [\mathbf{Q}_n(\{1, \dots, i\})]_{ii}}{\det(\mathbf{Q}_n(\{1, \dots, i\}))}$$

converges monotonically to c_j for all $1 \leq j \leq i$ with $c_i \neq 0$;

(iii) for each $2 \leq i \leq p$,

$$\frac{[\mathbf{Q}_n]_{ij}}{[\mathbf{Q}_n]_{ii}}$$

converges monotonically for all $1 \leq j < i$.

If $\sum \mathbf{Q}_k^{-1} \mathbf{a}_k$ exists and is finite then

$$\lim_{n \rightarrow \infty} \mathbf{Q}_n^{-1} \sum_{k=1}^n \mathbf{a}_k = \mathbf{0}.$$

Proof. For $p = 2$, putting

$$\mathbf{a}_k = \begin{bmatrix} a_k \\ b_k \end{bmatrix}, \quad \mathbf{Q}_k = \begin{bmatrix} q_k & s_k \\ r_k & t_k \end{bmatrix}$$

and

$$\bar{\mathbf{Q}}_k = \begin{bmatrix} q_k & 0 \\ r_k & t_k \end{bmatrix}$$

we get

$$\begin{aligned}
\mathbf{Q}_k^{-1} &= \mathbf{Q}_k^{-1} \cdot \bar{\mathbf{Q}}_k \bar{\mathbf{Q}}_k^{-1} \\
&= \begin{bmatrix} \frac{t_k}{\det(\mathbf{Q}_k)} & -\frac{s_k}{\det(\mathbf{Q}_k)} \\ -\frac{r_k}{\det(\mathbf{Q}_k)} & \frac{q_k}{\det(\mathbf{Q}_k)} \end{bmatrix} \begin{bmatrix} q_k & 0 \\ r_k & t_k \end{bmatrix} \cdot \begin{bmatrix} q_k^{-1} & 0 \\ -\frac{r_k}{q_k t_k} & t_k^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -\frac{s_k t_k}{\det(\mathbf{Q}_k)} \\ 0 & \frac{q_k t_k}{\det(\mathbf{Q}_k)} \end{bmatrix} \cdot \begin{bmatrix} q_k^{-1} & 0 \\ -\frac{r_k}{q_k t_k} & t_k^{-1} \end{bmatrix}.
\end{aligned}$$

The last Proposition 1 ensures the convergence of

$$\sum \begin{bmatrix} q_k^{-1} & 0 \\ -\frac{r_k}{q_k t_k} & t_k^{-1} \end{bmatrix} \begin{bmatrix} a_k \\ b_k \end{bmatrix}$$

and the classical Kronecker lemma implies

$$\lim_{n \rightarrow \infty} \begin{bmatrix} q_n^{-1} & 0 \\ -\frac{r_n}{q_n t_n} & t_n^{-1} \end{bmatrix} \sum_{k=1}^n \begin{bmatrix} a_k \\ b_k \end{bmatrix} = \mathbf{0}$$

that is,

$$\lim_{n \rightarrow \infty} \mathbf{Q}_n^{-1} \sum_{k=1}^n \mathbf{a}_k = \mathbf{0}.$$

By induction, supposing that the result is valid for p and assuming

$$\boldsymbol{\alpha}_k = \begin{bmatrix} \mathbf{a}_k \\ \beta_k \end{bmatrix}, \quad \mathbf{R}_k = \begin{bmatrix} \mathbf{Q}_k & \mathbf{u}_k \\ \mathbf{v}_k^T & y_k \end{bmatrix}$$

and

$$\bar{\mathbf{R}}_k = \begin{bmatrix} \mathbf{Q}_k & \mathbf{0} \\ \mathbf{v}_k^T & y_k \end{bmatrix}$$

we have

$$\begin{aligned} \mathbf{R}_k^{-1} &= \mathbf{R}_k^{-1} \cdot \bar{\mathbf{R}}_k \bar{\mathbf{R}}_k^{-1} \\ &= \begin{bmatrix} \mathbf{I} & - (y_k - \mathbf{v}_k^T \mathbf{Q}_k^{-1} \mathbf{u}_k)^{-1} y_k \mathbf{Q}_k^{-1} \mathbf{u}_k \\ \mathbf{0}^T & (y_k - \mathbf{v}_k^T \mathbf{Q}_k^{-1} \mathbf{u}_k)^{-1} y_k \end{bmatrix} \cdot \begin{bmatrix} \mathbf{Q}_k^{-1} & \mathbf{0} \\ -y_k^{-1} \mathbf{v}_k^T \mathbf{Q}_k^{-1} & y_k^{-1} \end{bmatrix}. \end{aligned}$$

Finally, assumption (ii) and the Proposition 1 permit us to conclude that

$$\sum \begin{bmatrix} \mathbf{Q}_k^{-1} & \mathbf{0} \\ -y_k^{-1} \mathbf{v}_k^T \mathbf{Q}_k^{-1} & y_k^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{a}_k \\ \beta_k \end{bmatrix}$$

is convergent provided that

$$\begin{aligned} & \left[- (y_k - \mathbf{v}_k^T \mathbf{Q}_k^{-1} \mathbf{u}_k)^{-1} y_k \mathbf{Q}_k^{-1} \mathbf{u}_k \right]_j \\ &= \frac{M_{pj}(\mathbf{Q}_n) [\mathbf{Q}_n]_{pp}}{\det(\mathbf{Q}_n)}, \quad j = 1, \dots, p-1 \end{aligned}$$

and

$$(y_k - \mathbf{v}_k^T \mathbf{Q}_k^{-1} \mathbf{u}_k)^{-1} y_k = \frac{M_{pp}(\mathbf{Q}_n) [\mathbf{Q}_n]_{pp}}{\det(\mathbf{Q}_n)}.$$

Since each component of $y_k^{-1} \mathbf{v}_k^T$ converges monotonically, the induction hypothesis and the classical Kronecker lemma guarantees that

$$\lim_{n \rightarrow \infty} \begin{bmatrix} \mathbf{Q}_n^{-1} & \mathbf{0} \\ -y_n^{-1} \mathbf{v}_n^T \mathbf{Q}_n^{-1} & y_n^{-1} \end{bmatrix} \sum_{k=1}^n \begin{bmatrix} \mathbf{a}_k \\ \beta_k \end{bmatrix} = \mathbf{0}$$

and the thesis is established provided that

$$\begin{bmatrix} \mathbf{I} & -(y_k - \mathbf{v}_k^T \mathbf{Q}_k^{-1} \mathbf{u}_k)^{-1} y_k \mathbf{Q}_k^{-1} \mathbf{u}_k \\ \mathbf{0}^T & (y_k - \mathbf{v}_k^T \mathbf{Q}_k^{-1} \mathbf{u}_k)^{-1} y_k \end{bmatrix}$$

converges to a (finite) non-singular matrix. ■

Remark 2. The result above, allows us to consider a huge class of sequences of matrices \mathbf{Q}_k , namely,

$$\mathbf{Q}_k = \begin{bmatrix} k^3 & k \\ k & k^2 + 1 \end{bmatrix}$$

or even cases where \mathbf{Q}_k is non-symmetric. On the other hand, let us note down that there are some class of sequences of matrices for which no "true generalization" of Kronecker lemma can be developed: e.g.

$$\mathbf{Q}_k = \begin{bmatrix} k^5 + 1 & \frac{k^3}{2} \\ \frac{k^3}{2} & k \end{bmatrix} \quad \text{and} \quad \mathbf{a}_k = \begin{bmatrix} 2k^2 \\ 1 \end{bmatrix}.$$

Situations where the sequence of matrices is as the one presented above, only can be treated imposing conditions directly on the sequence of vectors \mathbf{a}_k (which not happens in the classical version of the Kronecker lemma).

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Received 6 September 2009