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BAYESIAN AND GENERALIZED CONFIDENCE INTERVALS ON VARIANCE RATIO AND ON THE VARIANCE COMPONENT IN MIXED LINEAR MODELS

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Abstract

The paper deals with construction of exact confidence intervals for the variance component σ_1^2 and ratio θ of variance components σ_1^2 and σ^2 in mixed linear models for the family of normal distributions $\mathcal{N}_t(0, \ \sigma_1^2 W + \sigma^2 I_t)$. This problem essentially depends on algebraic structure of the covariance matrix W (see Gnot and Michalski, 1994, Michalski and Zmyślony, 1996). In the paper we give two classes of bayesian interval estimators depending on a prior distribution on $(\sigma_1^2\ , \sigma^2)$ for:

1) the variance components ratio θ - built by using test statistics obtained from the decomposition of a quadratic form $y'Ay$ for the Bayes locally best estimator of σ_1^2, Michalski and Zmyślony (1996),

2) the variance component σ_1^2 - constructed using Bayes point estimators from BIQUE class (Best Invariant Quadratic Unbiased Estimators, see Gnot and Kleffe, 1983, and Michalski, 2003).

In the paper an idea of construction of confidence intervals using generalized p-values is also presented (Tsui and Weerahandi, 1989, Zhou and Mathew, 1994). Theoretical results for Bayes interval estimators and for some generalized confidence intervals by simulations studies for some experimental layouts are illustrated and compared (cf Arendacká, 2005).

Keywords: mixed linear models; variance components; hypothesis testing; confidence intervals; generalized p-values.

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1. Introduction

We consider a mixed linear model

$$y = X\beta + X_1\beta_1 + e, \tag{1}$$

where y is an (n x 1) observed vector, X is a known (n x q)-matrix of rank $s, s \leq q$, X_1 is a known (n x q_1)-matrix of rank $s_1, s_1 \leq q_1$, β is a q-vector of parameters corresponding to fixed effects, while an unobservable random vector β_1 and a vector of random errors e are stochastically independent and normally distributed with zero mean and the covariance matrix $\sigma_1^2 I_{q_1}$ and $\sigma^2 I_n$, respectively. Under these assumptions we obtain

$$E(y) = X\beta \ , \quad Var(y) = \sigma_1^2 V_1 + \sigma^2 I_n \ , \quad V_1 = X_1 X_1'. \tag{2}$$

In this paper we restrict our attention to quadratic estimators $y'Ay$ which are invariant under the group of translations $g(y) = y + X\beta$, i.e., for which $AX = 0$. It can be checked that if B is $(n-s)$ x n-matrix such that $BB' = I_{n-s}, B'B = I - XX^+$ then $t = By$ is a maximal invariant statistic under this group of translations. The model for t is as folows

$$E(t) = 0 \ , \quad Cov(t) = E(tt') = \sigma_1^2 W + \sigma^2 I_{n-s} \ , \quad W = BV_1B'. \tag{3}$$

Denote by $\alpha_1 > \alpha_2 > ... > \alpha_{h-1} > \alpha_h = 0$ the ordered sequence of different eigenvalues of W. Let $W = \sum_{j=1}^{h} \alpha_j E_j E_j'$ be the spectral decomposition of W.

Following Olsen *et al.* (1976) let us consider the random vector $Z = (Z_1, ..., Z_{h-1}, Z_h)$ where $Z_i = t'E_it/\nu_i$, for $i = 1, ..., h-1, h$ and $\nu_1, ..., \nu_{h-1}, \nu_h$ are the multiplicities of α's. Under normality of y the random variables Z_i's are stochastically independent, and $\nu_i Z_i/(\alpha_i\sigma_1^2 + \sigma^2)$ are central chi-squared distributed with ν_i degrees of freedom, $i = 1, ..., h$. The model for $Z = (Z_1, ..., Z_{h-1}, Z_h)$ is as follows

$$(4) \qquad \begin{cases} E(Z) = L(\sigma_1^2, \sigma^2)' \quad ; \quad L' = \begin{bmatrix} \alpha_1 & ... & \alpha_h \\ 1 & ... & 1 \end{bmatrix} \\ \\ Var(Z) = 2\text{diag}\left\{(\alpha_i\sigma_1^2 + \sigma^2)^2/\nu_i\right\} \quad . \end{cases}$$

In Section 2 we use the results of point estimation for any function $f'\sigma = f_1\sigma_1^2 + f_2\sigma^2$ to construct possibly shortest exact Bayes confidence intervals on variance component σ_1^2. It follows from results of Seely (1970) that in the model (3) each function $f'\sigma$ is invariantly estimable iff matrices W and I are linearly independent or equivalently iff the number h of different eigen values of W satisfies: $h \geq 2$.

In Subsection 2.2 and Section 3 we show that the problem of interval estimation for variance componnet σ_1^2 or for variance ratio $\theta = \sigma_1^2/\sigma^2$ is also connected with testing the hypothesis about σ_1^2

$$(5) \qquad H_\sigma : \sigma_1^2 \leq \sigma_0^2 \quad vs \quad K_\sigma : \sigma_1^2 > \sigma_0^2$$

or the hypothesis about θ

$$(6) \qquad H_\theta : \theta \leq \theta_0 \quad vs \quad K_\theta : \theta > \theta_0.$$

It is known fact, that in general, tests which have good statistical properties (most powerful tests and locally best tests) lead to good confidence intervals at fixed confidence level. In Section 3 we give a class of the Bayes confidence intervals on the variance ratio θ constructed by using test statistics $F_{A_+^-}(\sigma_*)$ obtained from the decomposition of a quadratic form $t'At$ for the Bayes locally best estimator of σ_1^2 at point $\sigma_* = (\sigma_1^*, \sigma^*)'$ (see Michalski and Zmyślony, 1996). Unfortunately, we can not directly construct the confidence interval for variance component σ_1^2 due to the presence of nuisance parameter σ^2. Therefore, a useful approach is to construct confidence interval on σ_1^2 using generalized p-values and generalized test variables. In Section 4 we show the idea of generalized p-values and generalized test variables which was introduced by Tsui and Weerahandi (1989) and further developed by Weerahandi (1991, 1993).

2. Bayesian interval confidence

In this section we consider Bayes approach to construct possibly shortest confidence intervals on the variance component or the variance ratio using results for estimable functions $f'\sigma$ and test statistics derived from locally best invariant quadratic and unbiased estimators for variance component σ_1^2.

Definition 2.1. An estimator $y'Ay$ is Bayesian invariant quadratic and unbiased (BIQU) of $f'\sigma$ with respect to $U = (u_{ij})_{i,j=1,2}$ (or with respect to prior distribution τ, such that $\mathbf{E}_\tau \sigma\sigma' = U$), if A minimizes the Bayesian risk $\mathbf{Var}_\tau(y'Ay)$ in the class of symmetric and positive definite matrices that satisfied conditions: $AX = 0, \quad \mathbf{E}(y'Ay) = f'\sigma$.

Let $\mathcal{U}$ be a class of symmetric and positive definite with nonnegative elements matrices U. It is known (see e.g., Gnot and Kleffe, 1983, or Gnot, 1991) that a class $\mathcal{U}$ can be with accuracy to multiplication by constant characterized using two nonnegative parameters u, v as follows

$$\mathcal{U} = \left\{ U = U_{u,v} = \begin{bmatrix} u^2+v & u \\ u & 1 \end{bmatrix}, \quad u, v \geq 0 \right\} \cup U_0$$

$$= U_{u,\infty} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

2.1. Bayesian interval estimators on σ_1^2

We consider a class of admissible invariant quadratic and unbiased estimates for a given function $f'\sigma = f_1\sigma_1^2 + f_2\sigma^2$, which are Bayesian with respect to prior distribution τ with $\mathbf{E}_\tau \sigma\sigma' = U$. It follows from Gnot and Kleffe (1983) that for a given function $f_1\sigma_1^2 + f_2\sigma^2$ the class $\mathcal{A}_{IU}$ of admissible invariant quadratic unbiased estimators in the model given by (3) coincides with the linear combinations of minimal sufficient statistics Z_i as follows

$$\mathcal{A}_{IU} = \left\{ \hat{\gamma}(u,v) = \sum_{i=1}^{h} (\lambda_1\alpha_i + \lambda_2)\nu_i w_i(u,v) Z_i, \quad u,v \geq 0 \right\} \ \cup \ \mathcal{A}_0,$$

where $w_i(u,v) = (1+2u\alpha_i + (u^2+v)\alpha_i^2)^{-1}$, and class $\mathcal{A}_0$ consists of limiting estimates $\hat{\gamma}(\infty)$ obtained as v tends to infinity which are Bayesian with respect to τ_0 with U_0, i.e.,

$$\mathcal{A}_0 = \left\{ \hat{\gamma}(\infty) = \sum_{i=1}^{h-1} \frac{\lambda_1\nu_i}{\alpha_i} Z_i + \lambda_2\nu_h Z_h \right\}.$$

Here λ_1 and λ_2 are chosen such that $\hat{\gamma}(u,v)$ or $\hat{\gamma}(\infty)$ is unbiased for $f_1\sigma_1^2 + f_2\sigma^2$, i.e.,

$$\begin{cases} \sum_{i=1}^{h} (\lambda_1\alpha_i + \lambda_2) w_i(u,v)\alpha_i\nu_i = f_1 \\ \sum_{i=1}^{h} (\lambda_1\alpha_i + \lambda_2) w_i(u,v)\nu_i = f_2 \end{cases}$$

for $\hat{\gamma}(u,v)$, while for $\hat{\gamma}(\infty)$ we have

$$\lambda_1 \mathrm{rank}(W) = f_1 \quad \text{and} \quad \lambda_1 \sum_{i=1}^{h-1} \frac{\nu_i}{\alpha_i} + \lambda_2\nu_h = f_2.$$

Now, we give the construction of an exact $1-p$ confidence interval for σ_1^2 based on BIQUE of σ_1^2 according to the following algorithm **A1-5**:

A1. Choose the BIQUE $\hat{\sigma}_1^2 = \sum_{i=1}^{h} a_i Z_i$ of σ_1^2 with respect to the prior distribution τ on (σ_1^2, σ^2) for given parameters $u, v \geq 0$:

$$\mathrm{E}\ (\hat{\sigma}_1^2) = \sigma_1^2 \ \ ; \sum_{i=1}^{h} a_i \alpha_i = 1 \ \ ; \sum_{i=1}^{h} a_i = 0.$$

A2. Calculate the variance of $\hat{\sigma}_1^2$:

$$\mathrm{Var}\ (\hat{\sigma}_1^2) = 2\sigma_1^4 \sum_{i=1}^{h} \frac{a_i^2}{\nu_i} (1/\theta + \alpha_i)^2.$$

A3. Determine the exact probability distribution of $\hat{\sigma}_1^2$:

$$\hat{\sigma}_1^2 \sim \sigma_1^2 \sum_{i=1}^{h} \frac{a_i}{\nu_i} (1/\theta + \alpha_i) \chi^2_{\nu_i},$$

so that for fixed σ_1^2 and for each $\theta \in (0, \infty)$ we have

$$\frac{\hat{\sigma}_1^2}{\sigma_1^2} \sim \sum_{i=1}^{h} \frac{a_i}{\nu_i} (1/\theta + \alpha_i) \chi^2_{\nu_i} = \sum_{i=1}^{h} (\sqrt{2}/2) \mathrm{SE}_\theta (a_i Z_i) \chi^2_{\nu_i},$$

where $\mathrm{SE}(\cdot)$ determines the standard error of an estimator.

A4. Find the quantiles C_{p_1} and C_{p_2} from the distribution of quadratic form $Q(\theta) = \sum_{i=1}^{h} b_i(\theta) \chi^2_{\nu_i}$ where $b_i(\theta) = (a_i/\nu_i)(1/\theta + \alpha_i)$, such that

$$Pr \left\{ C_{p_1}(\theta) \leq \frac{\hat{\sigma}_1^2}{\sigma_1^2} \leq C_{p_2}(\theta) \right\} = 1 - p_1 - p_2 = 1 - p \quad \text{or}$$

$$Pr \Big\{ \hat{\sigma}_1^2 / C_{p_2}(\theta) \leq \sigma_1^2 \leq \hat{\sigma}_1^2 / C_{p_1}(\theta) \Big\} = 1 - p_1 - p_2 = 1 - p.$$

A5. Optimal choice of $(C_{p_1}(\theta), C_{p_2}(\theta))$

(**A**5.1) choose the optimal pair $(C^*_{p_1}(\theta), C^*_{p_2}(\theta))$ for fixed θ:

$$(C^*_{p_1}(\theta), C^*_{p_2}(\theta)) = Arg\left\{\min_{\substack{C_{p_1}, C_{p_2}\\ p_1+p_2=p}} (1/C_{p_1}(\theta) - 1/C_{p_2}(\theta))\right\},$$

(**A**5.2) find $\theta^\star$ for $\theta \in (0, \infty)$:

$$\theta^* = Arg\left\{\max_{\theta\in(0,\infty)} (1/C^*_{p_1}(\theta) - 1/C^*_{p_2}(\theta))\right\}.$$

The *maxmin* $(1-p)$ confidence interval on variance component σ_1^2 constructed using the above algorithm as $[\hat{\sigma}_1^2/C_{p_2}(\theta^*)\,,\ \hat{\sigma}_1^2/C_{p_1}(\theta^*)\ ]$ guarantees good surroundings of the point estimator of σ_1^2 and gives a sort of protection against the worst possible scenario.

2.2. Bayesian interval estimators on $\theta = \sigma_1^2/\sigma^2$

Let us consider locally best unbiased quadratic invariant (LBUQI) estimator of σ_1^2 at point $\sigma_* = (\sigma_1^*, \sigma^*)'$. To present a convenient form of this estimator let us define a random variable K as follows

$$Pr\{K = \alpha_j\} = \nu_j/c(\sigma_1^*\alpha_j + \sigma^*)^2, \quad j = 1, 2, ..., h$$

and

$$c = \sum_{j=1}^{h} \nu_j/(\sigma_1^*\alpha_j + \sigma^*)^2.$$

From Lemma 4.1 (Michalski and Zmyślony, 1996) we have that a LBUQI estimator of σ_1^2 at point $\sigma_* = (\sigma_1^*, \sigma^*)'$ is $t'A^*t$, where

$$A^* = [c\mathbf{Var}(K)]^{-1}\sum_{j=1}^{h}\frac{\alpha_j - \mathbf{E}(K)}{(\sigma_1^*\alpha_j + \sigma^*)^2}E_j, \tag{7}$$

while $\mathbf{E}(K)$ and $\mathbf{Var}(K)$ denote the expectation and variance of K, respectively. Hence we have the following decomposition of $A^* = A_+^* - A_-^*$, where

$$A_+^* = [c\mathbf{Var}(K)]^{-1} \sum_{\alpha_j > \mathbf{E}(K)} \frac{\alpha_j - \mathbf{E}(K)}{(\sigma_1^* \alpha_j + \sigma^*)^2} E_j \,,$$

$$A_-^* = [c\mathbf{Var}(K)]^{-1} \sum_{\alpha_j < \mathbf{E}(K)} \frac{\mathbf{E}(K) - \alpha_j}{(\sigma_1^* \alpha_j + \sigma^*)^2} E_j \,.$$

Now, let us consider the test statistics in the following form

$$F_{\mathbf{A}_-^+} = \frac{t' A_+^* t}{t' A_-^* t} = \frac{\displaystyle\sum_{\alpha_j > \mathbf{E}(K)} \frac{\alpha_j - \mathbf{E}(K)}{(\sigma_1^* \alpha_j + \sigma^*)^2} t' E_j t}{\displaystyle\sum_{\alpha_j < \mathbf{E}(K)} \frac{\mathbf{E}(K) - \alpha_j}{(\sigma_1^* \alpha_j + \sigma^*)^2} t' E_j t} \tag{8}$$

The probability distribution of the ratio statistics for each fixed $\sigma = (\sigma_1^2, \sigma^2)'$ is as follows

$$F_{\mathbf{A}_-^+} = \frac{t' A_+^* t}{t' A_-^* t} \sim \frac{\displaystyle\sum_{\alpha_j > \mathbf{E}(K)} \frac{\alpha_j - \mathbf{E}(K)}{(\sigma_1^* \alpha_j + \sigma^*)^2} (\sigma_1^2 \alpha_j + \sigma^2) \chi^2_{\nu_j}}{\displaystyle\sum_{\alpha_j < \mathbf{E}(K)} \frac{\mathbf{E}(K) - \alpha_j}{(\sigma_1^* \alpha_j + \sigma^*)^2} (\sigma_1^2 \alpha_j + \sigma^2) \chi^2_{\nu_j}} \,. \tag{9}$$

or for each θ

$$F_{\mathbf{A}_-^+} = \frac{t' A_+^* t}{t' A_-^* t} \sim \frac{\displaystyle\sum_{\alpha_j > \mathbf{E}(K)} \frac{\alpha_j - \mathbf{E}(K)}{(\sigma_1^* \alpha_j + \sigma^*)^2} (\theta \alpha_j + 1) \chi^2_{\nu_j}}{\displaystyle\sum_{\alpha_j < \mathbf{E}(K)} \frac{\mathbf{E}(K) - \alpha_j}{(\sigma_1^* \alpha_j + \sigma^*)^2} (\theta \alpha_j + 1) \chi^2_{\nu_j}} \,. \tag{10}$$

The test statisic given by (8) can be used for testing hypothesis given by (6) and for a construction of the confidence interval on variance ratio θ. To build an exact confidence interval on θ we consider test statistics given by (8) as a function of the parameter θ, namely

$$F_{\mathbf{A}^+_-}(\theta) = \frac{\displaystyle\sum_{\alpha_j > \mathbf{E}(K)} \frac{\alpha_j - \mathbf{E}(K)}{(\sigma_1^*\alpha_j + \sigma^*)^2(\theta\alpha_j + 1)} t' E_j t}{\displaystyle\sum_{\alpha_j < \mathbf{E}(K)} \frac{\mathbf{E}(K) - \alpha_j}{(\sigma_1^*\alpha_j + \sigma^*)^2(\theta\alpha_j + 1)} t' E_j t}. \tag{11}$$

It is not difficult to show that $F_{\mathbf{A}^+_-}(\theta)$ is a strictly convex and decreasing function of $\theta \in (-1/\alpha_1, \infty)$. Next, for each fixed θ the probability distribution of $F_{\mathbf{A}^+_-}(\theta)$ is the same as the distrubution of the ratio of positive linear combinations of independent central chi-squared random variables, i.e.,

$$F_{\mathbf{A}^+_-}(\theta) \sim \frac{\displaystyle\sum_{j=1}^{\kappa} b_j \chi^2_{\nu_j}}{\displaystyle\sum_{j=\kappa+1}^{h} b_j^* \chi^2_{\nu_j}}, \tag{12}$$

where

$$b_j = \frac{\alpha_j - \mathbf{E}(K)}{(\sigma_1^*\alpha_j + \sigma^*)^2} \quad \text{for } \; j = 1, ..., \kappa : \alpha_j > \mathbf{E}(K)$$

and

$$b_j^* = \frac{\mathbf{E}(K) - \alpha_j}{(\sigma_1^*\alpha_j + \sigma^*)^2} \quad \text{for } \; j = \kappa + 1, ..., h : \alpha_j < \mathbf{E}(K).$$

Now, in order to construct the $1 - p$ confidence intervals ($p_1 + p_2 = p$) we must find the critical values C_{p_1} and C_{p_2} such that

$$Pr\left\{ C_{p_1} \leq \frac{\displaystyle\sum_{j=1}^{\kappa} b_j \chi^2_{\nu_j}}{\displaystyle\sum_{j=\kappa+1}^{h} b_j^* \chi^2_{\nu_j}} \leq C_{p_2} \right\} = 1 - p_1 - p_2 \tag{13}$$

or equivalently

$$
(14) \qquad \begin{cases} Pr\left\{\sum_{j=1}^{\kappa} b_j\chi^2_{\nu_j} - \sum_{j=\kappa+1}^{h} b_j^* C_{p_1}\chi^2_{\nu_j} > 0\right\} & = \quad 1-p_1 \\ \\ Pr\left\{\sum_{j=\kappa+1}^{h} b_j^* C_{p_2}\chi^2_{\nu_j} - \sum_{j=1}^{\kappa} b_j\chi^2_{\nu_j} > 0\right\} & = \quad p_2. \end{cases}
$$

Thus the solutions C_{p_1} and C_{p_2} of (13 or 14) are based on the distribution of quadratic forms $Q = \sum_{i=1}^{m} \lambda_i \chi^2_{\nu_i}$ (for details see e.g., Imhof, 1961, Davies, 1980). Next, we must solve following nonlinear equations

$$
(15) \qquad F_{\mathbf{A}_{-}^{+}}(\bar{\theta}) = C_{p_1} \quad \text{and} \quad F_{\mathbf{A}_{-}^{+}}(\underline{\theta}) = C_{p_2},
$$

where $F_{\mathbf{A}_{-}^{+}}(\theta)$ is determined by (11) and $0 < C_{p_1}, C_{p_2} \leq F_{\mathbf{A}_{-}^{+}}(0)$. The solution is ensured by the properties of $F_{\mathbf{A}_{-}^{+}}(\theta)$. From equations given by (15) we obtain the exact 1-p confidence interval on variance ratio θ as follows

$$
[\underline{\theta} \ , \ \bar{\theta}] = \left[F^{-1}_{\mathbf{A}_{-}^{+}}(C_{p_2}) \ , \ F^{-1}_{\mathbf{A}_{-}^{+}}(C_{p_1})\right].
$$

Finally, to obtain the shortest $1-(p_1+p_2)$ confidence interval we have to allocate properly the mass probability in the left tail (p_1) and in the right tail of the distribution (p_2). Thus for a choice of the pair (p_1^0, p_2^0) we obtain the confidence interval on θ at fixed confidence level $1-(p_1+p_2)$ whose its length $l(p_1, p_2)$ satisfies

$$
l(p_1^0, p_2^0) = \min_{\substack{p_1, p_2 \\ p_1+p_2=p}} \left[F^{-1}_{\mathbf{A}_{-}^{+}}(C_{p_1}) \ - \ F^{-1}_{\mathbf{A}_{-}^{+}}(C_{p_2})\right].
$$

Remark 2.1. The test statistic given by (8) was used by Michalski and Zmyślony (1996) for testing hypothesis $H_\sigma : \sigma_1^2 = 0$ versus $K_\sigma : \sigma_1^2 > 0$. Using the Bayes locally best unbiased quadratic invariant estimator of σ_1^2 at the point $(\sigma_1^*, \sigma^*) = (0,1)$ which is MINQUE for σ_1^2, the test statistic can be expressed as follows

$$F_{\mathbf{A}_-^+} = \frac{t' A_+^* t}{t' A_-^* t} = \frac{\sum\limits_{\alpha_j^* > 0} \alpha_j * \nu_j Z_j}{\sum\limits_{\alpha_j^* < 0} \alpha_j * \nu_j Z_j} \quad ,$$

where $\alpha_i^* = \alpha_i - \mathrm{trace}(W)/\mathrm{rank}(W)$.

3. Generalized confidence interval on σ_1^2

Following Tsui and Weerahandi (1989) we present in this section the concept of generalized p-value and generalized test variables which was further developed by Weerahandi (1991, 1993). The concept has been proposed to some testing problems where nuisance parameters are present and it is difficult or impossible to obtain a rational test at a fixed significance level.

3.1. The idea of constructing a generalized confidence interval

Let X be a random vector with a cumulative distribution function $F(x, \vartheta)$, where a vector $\vartheta = (\xi, \delta)$ represents unknown parameters: ξ is a scalar parameter of interest, δ is a nuisance parameter (scalar or vector). We are interested in testing

$$H_\xi : \xi \le \xi_0 \quad \textit{against} \quad K_\xi : \xi > \xi_0.$$

Suppose it is difficult or impossible to find a test statistic $T(X)$ whose distribution at ξ_0 is independent of the nuisance parameter δ and thereby we can not determine an appropriate critical region for a given significance level. We then consider a random variable $T(X; x, \vartheta)$, which also depends on the observed value and the parameters.

Definition 3.1. If a function $T(X; x, \vartheta)$, where $\vartheta = (\xi, \delta)$ satisfies the following conditions:

(i) for fixed ξ, the distribution of $T(X;x,\vartheta)$ does not depend on δ for each x,

(ii) the observed value of $T(X;x,\vartheta)$ (i.e., $t_{obs} = T(x;x,\vartheta)$) does not depend on unknown parameters,

(iii) for fixed x and δ, $T(X;x,\vartheta)$ is stochastically increasing in ξ i.e., $Pr\{T(X;x,\vartheta) \geq t\}$is nondecreasing in ξ it is called a generalized test variable.

Condition (iii) implies, that a generalized test variable orders the sample space and thus can be effectively used for finding a critical region in testing $H_\xi : \xi \leq \xi_0$ vs $K_\xi : \xi > \xi_0$ in the following form

$$C(x,\vartheta) = \{X : T(X;x,\vartheta) \geq T(x;x,\vartheta)\}$$

and the generalized p-value for testing H_ξ against K_ξ is

$$p(x) = \sup_{\xi\leq\xi_0} Pr\{X \in C(x,\vartheta)|\xi\} = \sup_{\xi\leq\xi_0} Pr\{T(X;x,(\xi,\delta)) \geq t_{obs}|\xi\}$$

$$= Pr\{T(X;x,(\xi_0,\delta)) \geq t_{obs}|\xi\}.$$

Condition (iii) guarantees that the expressions above for critical region are equal and thanks (i) and (ii) the generalized p-value is computable. Condition (iii) also implies that $Pr\{T(X;x,\vartheta) \geq t_{obs}|\xi\}$ becomes large as $(\xi - \xi_0)$ increases. Besides, larger p-values favour the null hypothesis , the smaller p-values favour the alternative, and so tests based on generalized p-values reject the null hypothesis for small values $p(x)$, similarly to classical p-values. Next, having determined a generalized critical region for testing $H_\xi : \xi \leq \xi_0$ versus $K_\xi : \xi > \xi_0$ we define a power function of the test based on data.

Definition 3.2. Function $\pi(x,\xi) = \Pr\{X \in C(x,(\xi,\delta))|\xi\}$ is called a data-based power function if holds:

(a) $\pi(x,\xi_0) = p(x)$,

(b) for each fixed x, $\pi(x,\xi) \sim$ R(0,1) (an uniform random variable on (0,1), for an arbitrary ξ),

(c) for each fixed x, $\pi(x,\xi)$ is a monotonic function of ξ.

Properties (a) and (b) guarantee the power function $\pi(x,\xi)$ can be used for the construction of a confidence interval on the parameter ξ. For $\pi_1, \pi_2 \in (0,1)$ such that $\pi_2 - \pi_1 = 1-p$ and a given observed x it holds:

$$\Pr\{\pi_1 \leq \pi(x,\xi) \leq \pi_2\} = 1-p.$$

Finally, by inversion we get a $(1-p)$ generalized confidence interval on the parameter ξ.

3.2. Generalized test variables

Let us consider the model given by (3) and problem of testing the hypothesis $H_\sigma : \sigma_1^2 \leq \sigma_0^2$ *vs* $K_\sigma : \sigma_1^2 > \sigma_0^2$ with an arbitrary nonnegative σ_0^2. The minimal sufficient statistics for the family of normal distributions of maximal invariant $t = By$, i.e., for $\mathcal{N}_t(0, \sigma_1^2 W + \sigma^2 I_t)$ (see sec. 1) are $U_1 = t'E_1 t$, ..., $U_{h-1} = t'E_{h-1}t$, $U_h = t'E_h t$, such that, $S_1 = U_1/(\alpha_1\sigma_1^2 + \sigma^2) \sim \chi^2_{\nu_1}, \ldots, S_{h-1} = U_{h-1}/(\alpha_{h-1}\sigma_1^2 + \sigma^2) \sim \chi^2_{\nu_{h-1}}$ and $S_h = U_h/\sigma^2 \sim \chi^2_{\nu_h}$. First, we consider the case $h = 2$, i.e., when a matrix W has only two different eigen values: $\alpha_1 > 0$ and $\alpha_2 = 0$. Then the function $T(X, x, \xi, \delta) = T((U_1, U_2), (u_1, u_2), \sigma_1^2, \sigma^2) = S_1(1/S_2 + \alpha_1\sigma_1^2/u_2)$ satisfies conditions (i)-(iii) as a generalized test variable. Its observed value $t_{obs} = u_1/u_2$ does not depend on the unknown parameters, the distribution of T is independent of the nuisance parameter σ^2 and T is stochastically increasing in σ_1^2. It can be shown that the generalized test of (5) is unique, i.e., all generalized test constructed on the base of statistics U_1, U_2 can be based on the test variable T (see Weerahandi, 1995, Theorem 5.2).

In the general case when $h > 2$, as does not exist the uniformly most accurate $1-p$ confidence interval on θ (see Michalski 1995, 2003), the same, a choice of the generalized test of (5) is not unique. Zhou and Mathew (1994) showed this using a test variable as follows

$$T_1 = T_1^{c_1,\ldots,c_{h-1}} = \sum_{i=1}^{h-1} c_i S_i \left(\frac{1}{S_h} + \frac{\alpha_i\sigma_1^2}{u_h}\right), \tag{16}$$

which is a generalized test variable for testing (5) for arbitrary positive real numbers c_i, $i = 1, ..., h-1$. A problem arises how to choose these constants.

For testing (5) with $\sigma_0^2 = 0$ the test variable T_1 coincides under H_σ, for a special sets of constants c_i, with following statistics:

– Wald's statistic $F_W = \frac{\sum_{i=1}^{h-1} U_i}{U_h}$ for $c_i = 1$ (Seely and El-Bassiouni, 1983)

– modified Wald's statistic $F_W^* = \frac{\sum_{i=1}^{h-1} \alpha_i U_i}{U_h}$ for $c_i = \alpha_i$ or more generally,

$$F(u,v) = \frac{\sum_{i=1}^{h-1} \alpha_i w_i(u,v) U_i}{U_h}, \; u,v \geq 0 \;\; \text{for} \;\; c_i = \alpha_i w_i(u,v)$$

and

$$w_i(u,v) = \frac{1}{(1+u\alpha_i)^2 + v\alpha_i^2}$$

(Gnot and Michalski, 1991, 1994). It is clear that $F_W^* = F(0,0)$.

Now, in accordance with Section 2.1, consider Bayes estimators $\hat{\sigma}_1^2(u,v)$ and $\hat{\sigma}_1^2(\infty)$ from a class $\mathcal{A}_{IU}$. The estimators can be written in the form $\sum_{i=1}^h c_i^\lambda S_i(\sigma^2 + \alpha_i \sigma_1^2)$, where constants c_i^λ are such that the unbiasedness condition holds and we have additionally to calculate $\lambda = (\lambda_1, \lambda_2)$ (see Subsection 2.1). For the limiting estimator $\hat{\sigma}_1^2(\infty)$, we obtain $c_i^\lambda = 1/(\alpha_i \text{rank}(W))$, where $\text{rank}(W) = \sum_{i=1}^{h-1} \nu_i$, and $c_h^\lambda = -\sum_{i=1}^{h-1} \frac{\nu_i}{\alpha_i}/(\nu_h \text{rank}(W))$. To obtain independence in the above expression on σ^2 we multiply this parameter by the observed value u_h and next we divide by the random variable U_h. In case all c_i^λ, $i = 1, ..., h-1$ are positive, it is easy to check that we get a generalized test variable:

$$T_\lambda = \sum_{i=1}^{h-1} c_i^\lambda S_i \left(\frac{u_h}{S_h} + \alpha_i \sigma_1^2 \right) + c_h^\lambda u_h \,, \tag{17}$$

which is stochastically increasing in σ_1^2 and whose observed value is $t_{obs}^\lambda = \sum_{i=1}^h c_i^\lambda u_i$. Let $\pi_T(u_1, ..., u_h, \sigma_1^2)$ denote the data-based power function for testing (5) based on test variable T (see Definition 3.2). The data-based power function corressponding to the test variable T_λ is of the form:

$$\pi_{T_\lambda}\left(u_1, ..., u_h, \sigma_1^2\right) = Pr\Big\{T_\lambda \geq t_{obs}^\lambda | \sigma_1^2\Big\}$$

$$(18) \qquad = Pr\left\{\sum_{i=1}^{h-1} c_i^\lambda S_i \left(\frac{u_h}{S_h} + \alpha_i \sigma_1^2\right) \geq \sum_{i=1}^{h-1} c_i^\lambda u_i | \sigma_1^2\right\}$$

$$= Pr\left\{\sum_{i=1}^{h-1} c_i^\lambda S_i \left(\frac{1}{S_h} + \frac{\alpha_i \sigma_1^2}{u_h}\right) \geq \sum_{i=1}^{h-1} \frac{c_i^\lambda u_i}{u_h} \quad | \, \sigma_1^2\right\}.$$

The last equality implies that the test based on the variable T_λ with constants $c_i = c_i^\lambda$ for $i = 1, ..., h-1$ coincides with the test based on T_1 and is stochastically increasing in σ_1^2, and its observed value is $t_{obs}^\lambda = \sum_{i=1}^{h-1} c_i^\lambda u_i / u_h$. Below, we put together formerly described the following generalized test variables :

$$(19) \qquad T_1^1 = T_1^{1,...,1} = \sum_{i=1}^{h-1} S_i \left(\frac{1}{S_h} + \frac{\alpha_i \sigma_1^2}{u_h}\right) \quad \text{with} \quad t_{obs}^1 = \sum_{i=1}^{h-1} \frac{u_i}{u_h},$$

$$(20) \qquad T_1^\alpha = T_1^{\alpha_1,...,\alpha_{h-1}} = \sum_{i=1}^{h-1} \alpha_i S_i \left(\frac{1}{S_h} + \frac{\alpha_i \sigma_1^2}{u_h}\right) \quad \text{with} \quad t_{obs}^\alpha = \sum_{i=1}^{h-1} \frac{\alpha_i u_i}{u_h},$$

$$(21) \qquad T_1^\infty = T_1^{1/\alpha_1,...,1/\alpha_{h-1}} = \sum_{i=1}^{h-1} \frac{1}{\alpha_i} S_i \left(\frac{1}{S_h} + \frac{\alpha_i \sigma_1^2}{u_h}\right) \quad \text{with} \quad t_{obs}^\infty = \sum_{i=1}^{h-1} \frac{u_i}{\alpha_i u_h}.$$

And general in connection with a BIQUE $\hat{\sigma}_1^2(u,v)$ we obtain

(22)
$$T_\lambda = T_1^{c_1^\lambda,\dots,c_{h-1}^\lambda} = \sum_{i=1}^{h-1} c_i^\lambda S_i \left(\frac{1}{S_h} + \frac{\alpha_i \sigma_1^2}{u_h}\right) \qquad \text{with}$$

$$t_{obs}^\lambda = \sum_{i=1}^{h-1} \frac{c_i^\lambda \alpha_i u_i}{u_h},$$

where $c_i^\lambda = (\lambda_1 \alpha_i + \lambda_2) w_i$,

$$\lambda_1 = \frac{\sum_i^h w_i \nu_i}{\sum_i^h w_i \alpha_i^2 \nu_i \sum_i^h w_i \nu_i - \left(\sum_i^h w_i \alpha_i \nu_i\right)^2},$$

$$\lambda_2 = \frac{-\sum_i^h w_i \alpha_i \nu_i}{\sum_i^h w_i \alpha_i^2 \nu_i \sum_i^h w_i \nu_i - \left(\sum_i^h w_i \alpha_i \nu_i\right)^2}$$

and

$$w_i = \frac{1}{(1+u\alpha_i)^2 + v\alpha_i^2}, \quad \text{for} \quad u, v \geq 0.$$

The values of data-based power function based on T_1 are calculated from the following equality:

$$\pi_{T_1}(u_1,...,u_h,\sigma_1^2) = Pr\left\{\sum_{i=1}^{h-1} c_i S_i \left(\frac{1}{S_h} + \frac{\alpha_i \sigma_1^2}{u_h}\right) \geq \sum_{i=1}^{h-1} \frac{c_i u_i}{u_h} \quad | \, \sigma_1^2\right\}$$

$$= \int_0^\infty \left(1 - F_v\left(\sum_{i=1}^{h-1} \frac{c_i u_i}{u_h}\right)\right) f_{V_h}(v) dv,$$

where F_v denotes the distribution function of a linear combination of independent central χ^2 random variables, i.e.,

$$\sum_{i=1}^{h-1} c_i \left(\frac{1}{v} + \frac{\alpha_i \sigma_1^2}{u_h}\right) S_i \sim \sum_{i=1}^{h-1} c_i \left(\frac{1}{v} + \frac{\alpha_i \sigma_1^2}{u_h}\right) \chi^2_{\nu_i}$$

and f_{V_h} is density function of a $\chi^2_{\nu_h}$ distribution. Using Imhof's (1961) algorithm or the one given by Davies (1980) we may calculate probabilities for any quadratic form $Q = \sum_{j=1}^{k} b_j \chi^2_{\nu_j}$.

It is worth stressing that in the general case (here for $h > 2$) on account of nonuniqueness in testing (5) we have a great set of addmissible test variables not only over a choice of constants c_i in T_1. Weerahandi (1995) proposed in a mixed one-way classification unbalanced model the generalized test variable T_2 as follows

$$T_2 = \sum_{i=1}^{h-1} S_i - \sum_{i=1}^{h-1} \frac{u_i S_h}{u_h + \alpha_i \sigma_1^2 S_h} \tag{23}$$

(see also Arendacká, 2005). It is easy to check that T_2 is stochastically increasing, its observed value is 0, and its the data-power function has a simplified form, and so more convenient in computing, namely

$$\begin{aligned}\pi_{T_2}(u_1, ..., u_h, \sigma_1^2) &= Pr\left\{\sum_{i=1}^{h-1} S_i - \sum_{i=1}^{h-1} \frac{u_i S_h}{u_h + \alpha_i \sigma_1^2 S_h} \geq 0 \ \ | \, \sigma_1^2\right\} \\ &= Pr\left\{\sum_{i=1}^{h-1} S_i \geq \sum_{i=1}^{h-1} \frac{u_i S_h}{u_h + \alpha_i \sigma_1^2 S_h} \ \ | \, \sigma_1^2\right\},\end{aligned}$$

where $\quad \sum_{i=1}^{h-1} S_i \sim \chi^2_{\nu_1 + ... + \nu_{h-1}}.$

4. Comparison - numerical calculations

In this section we compare effects of the test variables which were presented previously and used for constructing confidence intervals on the variance component σ_1^2 in two examples of model (5). A special case of model (1) is a mixed two-way classification model

$$y_{ijk} = \beta_j + \eta_i + \epsilon_{ijk}, \quad i = 1, ..., s; \quad j = 1, ..., b; \quad k = 1, ..., n_{ij},$$

corresponding to block design $BD(s, b, n, N)$, in which n experimental units are arranged in b blocks (with fixed effects β_j) and treated by s treatment (with random effects η_i) according to the incidence matrix $N = \Delta D'$. The matrix form of the above model can be presented as follows

$$y = D'\beta + \Delta'\eta + \epsilon\,.$$

In the numerical calculations we use the $(s$ x $b)$ matrix C associated with the block design and given by $C = diag\{r_1, ...r_s\} - N diag\{1/k_1, ..., 1/k_b\}N'$, where $\mathbf{\Delta 1}_n = N\mathbf{1}_b = (r_1, ..., r_s)$ is the vector of treatment replications and $D\mathbf{1}_n = N'\mathbf{1}_s = (k_1, ..., k_b)$ is the vector of block sizes, $n = \sum_{ij} n_{ij}$ is the total number of observations. Here $\mathbf{1}_a$ denotes a vector of ones. Further, we apply the theorem, that the eigenvalues of the variance-covariance matrix W in model (3) are the same as the positive eigenvalues of C which has a very convenient form. For comparison two examples of block designs were chosen having different properties. The first example refers to a binary block design which is very close to the so called partially balanced incomplete block design (PBIBD) with a small number of observations and is connected with empirical data. The second block design is a complete unbalanced and additionally orthogonal block design with a large number of observations and refers to simulated data. Differentiated numbers of observations in the examples imply essentially different mulipicities of zero eigenvalue of matrix W. The simulation studies were executed, similarly as in the paper of Arendacká (2005), for 5 pairs of the values of parameters $(\sigma_1^2, \sigma^2) \in \{(0.1, 10), (0.5, 2), (1, 1), (2, 0.5), (5, 0.2)\}$, such that their product equals 1.

The simulated probabilities of coverage are in both examples based on 2000 simulations, while the average length of confidence intervals on 20 simulations. The values of the data-based power function $\pi(y, \sigma_1^2)$ were computed by numerical integration following the expressions in the Subsection 3.2. The bounds of the 95% generalized confidence intervals $[\underline{\sigma_1^2}, \overline{\sigma_1^2}]$ corresponding to generalized test variables T_1^1, T_1^α (based on $\hat{\gamma}(0,0)$), $T_1^{1/\alpha}$ (based on $\hat{\gamma}(\infty)$) and T_2 were obtained by solving $\pi(y, \underline{\sigma_1^2}) = 0.025$ and $\pi(y, \overline{\sigma_1^2}) = 0.975$. Next, Using algorithm **A1-5** given in Subsection 2.1 we present 95% Bayes confidence intervals $[\hat{\sigma}_1^2/C_{p_2}(\theta^*),\ \hat{\sigma}_1^2/C_{p_1}(\theta^*)\ ]$ for specific pairs (u,v): (0,0), (0,1),(1,0), (1,1) and (0, ∞) (see Table 4.).

The computations were realized using a modified procedure of Imhof (1961) and also using package MATHEMATICA ver. 5.1, including in the to comparison some numerical results from the paper of Arendacká (2005).

Example 1. Let us consider an empirical example of two-way classification model (see Kala *et al.*, 1992, cf. also Michalski, 1995) corresponding to block design BD(v,b,n,N) with the following incidence matrix N

$$N = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

with parameters $s = 6$, $b = 6$, $n = 18$, so that $(r_1, ..., r_6) = (k_1, ..., k_6) = 3\,\mathbf{1}_6$. According to this incidence matrix an experiment was done, in which the influence of various types of lupine on total crop was investigated. Assuming that types used in the experiment were chosen at random out of an infinite population of types we can apply the mixed model with two variance components given by (1). The vector y of observations ordered according to treatments is as follows

$$y = (62.8, 54.9, 59.1, 73.8, 69.0, 62.9, 68.4, 67.3, 70.8,$$
$$73.0, 72.3, 73.0, 72.1, 73.2, 71.6, 71.6, 77.9, 78.8)'.$$

Table 1. The eigenvalues α_i and their the multiplicities ν_i of matrix W.

i	1	2	3
α_i	$2\frac{2}{3}$	2	0
ν_i	3	2	n-rank(C)-b=7

Table 2. The probabilities of coverage of a true value σ_1^2 by the generalized confidence intervals constructed on base of the different test variables.

(σ_1^2, σ^2)	(0.1,10)	(0.5,2)	(1,1)	(2,0.5)	(5,0.2)
T_1^1	0.9540	0.9485	0.9490	0.9505	0.9515
T_1^α	0.9491	0.9503	0.9495	0.9525	0.9513
$T_1^{1/\alpha}$	0.9515	0.9516	0.9560	0.9465	0.9535
T_2	0.9520	0.9495	0.9560	0.9480	0.9520

Table 3. The average lenghts of the constructed 95% generalized confidence intervals.

(σ_1^2, σ^2)	(0.1,10)	(0.5,2)	(1,1)	(2,0.5)	(5,0.2)
T_1^1	17.88	16.49	18.08	21.12	33.41
T_1^α	19.47	21.93	20.88	34.35	42.60
$T_1^{1/\alpha}$	14.92	11.47	13.67	19.14	23.29
T_2	13.44	12.09	13.34	17.41	25.26

Table 4. The Bayes generalized confidence intervals at significance level $1-(p_1+p_2)=0.95$ and their lengths $l(p_1,p_2)$ for choosen pairs (u,v).

(u,v)	$\underline{\sigma_1^2}$	$\overline{\sigma_1^2}$	p_1	p_2	$l(p_1,p_2)$
(0,0)	0.0	10.299	0.038	0.012	10.299
	0.110	14.262	0.025	0.025	14.152
(0,1)	0.0	12.178	0.037	0.013	12.178
	0.119	16.889	0.025	0.025	16.770
(1,0)	0.0	18.323	0.038	0.012	18.323
	0.133	21.180	0.025	0.025	21.047
(1,1)	0.0	20.640	0.035	0.015	20.640
	0.139	24.444	0.025	0.025	24.305
(u,∞)	0.0	9.456	0.038	0.012	9.456
	0.121	14.128	0.025	0.025	14.007

Example 2. Consider an example of the two-way classification model (see Gnot and Michalski, 1994, Michalski and Zmyślony, 1995) corresponding to the orthogonal block design BD(v,b,n,N) with the following incidence matrix N

$$N=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \\ 12 & 12 & 24 \end{bmatrix};\quad r=(4,4,4,8,48)\,;\qquad n=68.$$

In this model we obtain

Table 5. The eigenvalues α_i and theirs the multiplicities ν_i of matrix W.

i	1	2	3	4
α_i	18.2984	6.1722	4	0
ν_i	1	1	2	61

Table 6. The probabilities of coverage of a true value σ_1^2 by the generalized confidence intervals constructed based on the different test variables.

(σ_1^2, σ^2)	(0.1,10)	(0.5,2)	(1,1)	(2,0.5)	(5,0.2)
T_1^1	0.9640	0.9465	0.9460	0.9500	0.9525
T_1^α	0.9480	0.9420	0.9445	0.9525	0.9475
$T_1^{1/\alpha}$	0.9520	0.9520	0.9570	0.9465	0.9540
T_2	0.9520	0.9485	0.9570	0.9470	0.9540

Table 7. The average lenghts of the constructed 95% generalized confidence intervals.

(σ_1^2, σ^2)	(0.1,10)	(0.5,2)	(1,1)	(2,0.5)	(5,0.2)
T_1^1	13.11	6.49	16.08	21.29	36.41
T_1^α	17.47	9.93	29.88	36.35	70.60
$T_1^{1/\alpha}$	14.82	6.47	13.20	19.44	28.29
T_2	14.48	6.44	13.17	19.41	28.26

Table 8. The Bayes generalized confidence intervals at significance level $1 - (p_1 + p_2) = 0.95$ and their lengths $l(p_1, p_2)$ for choosen pairs (u, v).

(u,v)	$\underline{\sigma_1^2}$	$\overline{\sigma_1^2}$	p_1	p_2	$l(p_1,p_2)$
(0,0)	0.0	12.135	0.037	0.013	12.135
	0.120	15.202	0.025	0.025	15.082
(0,1)	0.0	14.175	0.036	0.014	14.175
	0.113	17.934	0.025	0.025	17.823
(1,0)	0.0	24.123	0.038	0.012	24.123
	0.135	27.180	0.025	0.025	27.045
(1,1)	0.0	15.641	0.034	0.016	15.641
	0.159	18.443	0.025	0.025	18.284
(u,∞)	0.0	12.988	0.037	0.013	12.988
	0.106	15.401	0.025	0.025	15.295

DISCUSSION

The inspection of the tables leads to the following general remarks:

(i) The simulated probabilities of coverage don't indicate that the factual probabilities of coverage of a true parameter σ_1^2 by the constructed generalized confidence intervals are lower than the postulated confidence level $1 - p = 0.95$. It seems that all used generalized test variables perform equally well.

(ii) Comparing the average lengths of the constructed confidence intervals, seems that some precaution is needed when using the test variable T_1^α (see Tables 3 and 7). Probably, it is caused by the low stability of confidence intervals lengths for different pairs (σ_1^2, σ^2).

(iii) The results in Tables 3 and 7 indicate that the test variable $T_1^{1/\alpha}$ and T_2 give shorter intervals as the ratio σ_1^2/σ^2 increases while T_1^1 behaves in the opposite way, its performance improves as the ratio decreases.

(iv) The results for the Bayesian confidence intervals on the variance component σ_1^2 indicate unambiguously an adventage for the Bayes approach for constructing the confidence intervals (see Tables 4 and 8). The confidence intervals are shorter and the method gives more stabile confidence intervals than using the generalized test variables. Besides, the lengths of the $(1-p)$ confidence intervals essentially depend on the allocation of the mass probability $p = p_1 + p_2$ for the two tails of the probability distribution of the statistics used to their construction. It is confirmed that the symmetrical determination of the lower and the upper quantiles, C_{p_1} and C_{p_2}, respectively in this problem is not recommendable.

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References

[1] B. Arendacká, *Generalized confidence intervals on the variance component in mixed linear models with two variance components*, Statistics **39** (4) (2005), 275–286.

[2] R.K. Burdick and F.A. Graybill, *Confidence intervals on variance components*, Marcel Dekker, Inc., New York, Basel, Hong Kong 1992.

[3] R.B. Davies, *The distribution of a linear combination of χ^2 random variables*, Applied Statistics **29** (1980), 323–333.

[4] S. Gnot, *Bayes estimation in linear models: A cordinate - free approach*, J. Mulivariate Anal. **13** (1983), 40–51.

[5] S. Gnot and J. Kleffe, *Quadratic estimation in mixed linear models with two variance components*, J. Statist. Plann. Inference **8** (1983), 267–279.

[6] S. Gnot, *Estimation of variance components in linear models. Theory and applications*, WNT, Warszawa 1991 (in Polish).

[7] S. Gnot and A. Michalski, *Tests based on admissible estimators in two variance components models*, Statistics **25** (1994), 213–223.

[8] J.P. Imhof, *Computing the distribution of quadratic forms in normal variables*, Biometrika **48** (1961), 419–426.

[9] A.I. Khuri, T. Mathew and B.K. Sinha, *Statistical Tests for Mixed Linear Models*, Wiley & Sons, New York, Toronto 1998.

[10] A. Michalski, *Confidence intervals on the variance ratio in two variance components models*, Discussiones Mathematicae - Algebra and Stochastic Methods **15** (1995), 179–188.

[11] A. Michalski and R. Zmyślony, *Testing hypotheses for variance components in mixed linear models*, Statistics **27** (1996), 297–310.

[12] A. Olsen, J. Seely and D. Birkes, *Invariant quadratic unbiased estimation for two variance components*, Ann. Statist. **4** (1976), 878–890.

[13] J. Seely, *Minimal sufficient statistics and completeness for multivariate normal families*, Sankhyā A **39** (1977), 170–185.

[14] J. Seely and Y. El-Bassiouni, *Applying Wald's variance components test*, Ann. Statist. **11** (1983), 197–201.

[15] K.W. Tsui and S. Weerahandi, *Generalized P-values in significance testing of hypotheses in the presence of nuisance parameters*, J. Amer. Statist. Assoc. **84** (1989), 602–607.

[16] S. Weerahandi, *Testing variance components in mixed linear models with generalized P-values*, J. Amer. Statist. Assoc. **86** (1991), 151–153.

[17] S. Weerahandi, *Generalized confidence intervals*, J. Amer. Statist. Assoc. **88** (1993), 899–905.

[18] S. Weerahandi, *Exact Statistical Methods for Data Analysis*, Springer-Verlag, New York 1995.

[19] L. Zhou and T. Mathew, *Some tests for variance components using generalized p-values*, Technometrics **36** (1994), 394–402.

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