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ON SOME LIMIT DISTRIBUTIONS FOR GEOMETRIC RANDOM SUMS

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Abstract

We define and give the various characterizations of a new subclass of geometrically infinitely divisible random variables. This subclass, called geometrically semistable, is given as the set of all these random variables which are the limits in distribution of geometric, weighted and shifted random sums. Introduced class is the extension of, considered until now, classes of geometrically stable [5] and geometrically strictly semistable random variables [10]. All the results can be straightforward transfered to the case of random vectors in $\mathbb{R}^d$.

Keywords: random sum, infinite divisibility, semistability, geometric infinite divisibility, geometric stability, geometric semistability, characteristic function, limit distribution, Lévy process.

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1. Introduction

In this paper we characterize a new class of limit distributions. The starting point of our considerations is the concept of random sum, i.e., a sum of random variables, where the number of summands is also a random variable. The random summation scheme arises in many areas of mathematical modeling of real phenomena. Let us mention several theories which successfully use the idea of random sum. These are reliability, renewal, queueing theories, physics, financial mathematics and insurance mathematics. For example, in reliability theory, consider a system with an operating unit which is subjected to shocks that arrise in random moments. If N denotes the random number of shocks that the unit surrvives until its death, and X_k is the time between kth and $(k-1)$th shock, then $S = X_1 + X_2 + \cdots + X_N$ (with a convention that $S = 0$ if $N = 0$) is the lifetime of the system. In actuarial setting, consider a portfolio of insurence policies. If $\{N_t, t \geq 0\}$ denotes the number of claims occuring in the time interval $(0, t]$ and X_k is the cost of the kth claim, then $S_t = X_1 + X_2 + \cdots + X_{N_t}$, $t \geq 0$ (with $S_t = 0$ if $N_t = 0$) represents the aggregate claims process. Obviously, for every fixed t, S_t is a random sum.

In the paper we will consider the geometric random sums of a random variable (r.v.) X, that is the r.v. of the form $\sum_{k=1}^{T(p)} X_k$, where $X, X_1, X_2, \ldots$ are i.i.d. r.v.'s, and r.v. $T(p)$ independent of the summands has geometric distribution with the parameter $p \in (0,1)$, i.e., $\mathbb{P}(T(p) = n) = p(1-p)^{n-1}$, $n = 1, 2, \ldots$ These assumptions about geometric summation we will assume throughout the paper. Such random sum represents, for example, a lifetime of a system with rapid repair in reliability. In such system if an operating unit with a random lifetime X fails, it is immediately replaced by the identical unit available with a probability $q = 1 - p$ close to one. The monograph [3] deals with geometric random sums and illustrates their properties very wide as well as their possible applications in risk analysis, reliability and queueing. The geometric random sums appear also in the existing in the literature definitions of *geometrically infinitely divisible* (GID) [4, 14], *geometrically strictly stable* (GSSt) [5, 6, 7, 14], *geometrically stable* (GSt) [5, 6, 7] and *geometrically strictly semistable* (GSSe) [10, 13] r.v.'s. Let us recall briefly some fundamental facts about mentioned classes as they will be important to us.

A r.v. X is

- GID iff

$$\forall p \in (0,1) \ \exists \text{ r.v. } X_p : \ \sum_{k=1}^{T(p)} X_{p,k} \stackrel{d}{=} X, \tag{1}$$

- GSSt iff

$$(2) \qquad \forall p \in (0,1) \ \exists a_p > 0 \ : \ a_p \sum_{k=1}^{T(p)} X_k \stackrel{d}{=} X,$$

- GSt iff

$$(3) \qquad \exists a_p > 0 \ \exists b_p \in \mathbb{R} \ \exists \text{ r.v. } Y \ : \ a_p \sum_{k=1}^{T(p)} (Y_k + b_p) \stackrel{d}{\longrightarrow} X, \ \text{as } p \to 0,$$

- GSSe iff

$$(4) \qquad \exists p \in (0,1) \ \exists a > 0 \ : \ a \sum_{k=1}^{T(p)} X_k \stackrel{d}{=} X,$$

where $\stackrel{d}{=}$, $\stackrel{d}{\longrightarrow}$ denote the equality in distribution and the convergence in distribution, respectively. Whenever r.v. X has the property GID (GSSt, GSt, GSSe) we will say that also its distribution and characteristic function (ch.f.) have this property.

The idea of GID r.v.'s originate from V.M. Zolotarev, who asked about such r.v.'s, X for which the following condition is satisfied:

$$(5) \qquad \forall p \in (0,1) \ \exists \text{ r.v. } X_p \ : \ X \stackrel{d}{=} \epsilon_p X + X_p,$$

where ϵ_p, X, X_p are independent r.v.'s, and ϵ_p has distribution: $\mathbb{P}(\epsilon_p = 0) = p$, $\mathbb{P}(\epsilon_p = 1) = 1 - p$. In [4] it is shown that the set of r.v.'s which satisfy (5) coincides with the set of these r.v.'s for which (1) holds. The authors of [4] proved the one to one correspondence between GID and *infinitely divisible* (ID) ch.f.'s, namely for ch.f. ϕ, they obtained that ϕ is GID ch.f. if and only if $\exp\{1 - 1/\phi\}$ is ID ch.f. (for ID distributions see e.g. [16]). They noticed that every GID r.v. is ID and gave also the first characterization of GSSt ch.f. (see Theorem 3 in [4]). Similar, to the mentioned above, one to one correspondence between GSt and St ch.f. one can find in [6, 7], and between GSSe and SSe in [13]. The authors of [14] cite a collection of examples which justify the applicable character of GID and GSSt distributions in problems from reliability, renewal theories and financial mathematics.

It is easy to see that every GSSt r.v. is GID. Also GSt and GSSe r.v.'s are the extensions of GSSt ones. From among mentioned r.v.'s, the GSt r.v.'s were the most intensively studied. For the excellent survey of GSt laws we refer to [7] and references therein. The theory of GSt distributions is very developed at present. Very useful, of practical point of view, distributions are GSt. Let us mention the Linnik, Mittag-Leffler, Laplace, asymetric Laplace, exponential distributions. The results concernig GSSe distributions one can find in [8, 10, 13], althought in [8] and [13] they are not called the GSSe. Let us emphasize that the set of GSt distributions does not contain all the GSSe laws, and there are GSt laws which are not GSSe. In this paper we want to characterize a new subset of GID r.v.'s which contains GSt r.v.'s as well as GSSe. These new r.v.'s we shall call *geometrically semistable* (GSe).

2. Connections between GSe and Se distributions

Let us recall some facts concerning *stable* (St) and *semistable* (Se) distributions as we shall often use them in our proofs. An ID r.v. X, its ch.f. ϕ and its distribution are (see [16], p. 69)

- St iff

$$\forall r>0 \ \exists a>0 \ \exists b\in\mathbb{R} \ \forall t\in\mathbb{R}: \ \phi(at)e^{itb}=\phi(t)^r, \tag{6}$$

- Se iff

$$\exists r>0 \ \exists a>0, a\neq 1 \ \exists b\in\mathbb{R} \ \forall t\in\mathbb{R}: \ \phi(at)e^{itb}=\phi(t)^r. \tag{7}$$

If (6), (7) holds with $b=0$ then X is called *strictly stable* (SSt), *strictly semistable* (SSe), respectively. Notice that for a nontrivial case of Se ch.f. (by trivial Se ch.f. we mean $\phi(t)\equiv 1$) it is enough to consider $r\in(0,1)$ and $a\in(0,1)$ in (7) (or equivalently $r>1$ and $a>1$). There are alternative characterizations of St and Se r.v.'s (see e.g. [12, 15, 16]). We cite a one of them, for Se.

Lemma 1 ([12], Theorem 2.1)**.** *A r.v. X is Se if and only if there exist the sequences $\{a_n\}\subset\mathbb{R}^+=(0,\infty)$, $a_n\to 0$, $\{b_n\}\subset\mathbb{R}$ and the i.i.d. r.v.'s $Y_1,Y_2,\dots$ such that*

$$a_n \sum_{j=1}^{k_n} Y_j + b_n \xrightarrow{d} X, \quad as\ n \to \infty \tag{8}$$

with some increasing sequence $\{k_n\} \subset \mathbb{N}$, $k_n/k_{n+1} \overset{n\to\infty}{\longrightarrow} r \in (0,1)$.

From the Theorem 2.3 of [12] one can infer the following fact.

Remark 1. If for r.v. X the convergence (8) holds with i.i.d. r.v.'s $Y_1, Y_2, \ldots$ and $a_n > 0$, $b_n \in \mathbb{R}$, $k_n \in \mathbb{N}$ such that

$$k_n/k_{n+1} \overset{n\to\infty}{\longrightarrow} 1,$$

then X is St.

In the sequel we shall indicate the connections between, proposed by us, GSe laws and Se distributions. The new class - GSe distributions - we will define as follows.

Definition 1. *A r.v. X (its ch.f. and its distribution) is GSe if there exist the sequences $\{a_n\} \subset \mathbb{R}^+$, $\{b_n\} \subset \mathbb{R}$, a constant $p_0 \in (0,1)$ and a r.v. Y such that*

$$a_n \sum_{k=1}^{T(p_0^n)} (Y_k + b_n) \xrightarrow{d} X, \ \ as\ n \to \infty. \tag{9}$$

Such a way of defining GSe distributions guarantees the intended location of GSe in the class of GID distributions. Indeed, comparing this definition with the definitions of GSt and GSSe r.v.'s, which are given in Introduction, one can see that every GSt or GSSe r.v. is GSe. The fact that GSe r.v.'s are GID easily follows from the Theorem 1 in [10] which states that r.v. X is GID if and only if there are $p_n \in (0,1)$, $p_n \overset{n\to\infty}{\longrightarrow} 0$ and r.v.'s Y_n such that $\sum_{k=1}^{T(p_n)} Y_{n,k} \xrightarrow{d} X$, as $n \to \infty$.

2.1. Correspondence between GSe and Se distributions

Now we formulate the main result of the section 2 which reflects a connection between the new class of GSe distribution and the well known class of Se laws.

Theorem 1. *If a ch.f.* ϕ *is GSe then* $\exp\{1 - 1/\phi\}$ *is Se ch.f. Conversely, if* ψ *is Se ch.f. then* $(1 - \log\psi)^{-1}$ *is GSe ch.f.*

Proof. Let X denote a r.v. with the ch.f. ϕ. From the Definition 1 there are: r.v. Y, $a_n \in \mathbb{R}^+$, $b_n \in \mathbb{R}$ and $p_0 \in (0,1)$ for which (9) holds. Denote by φ the ch.f. of Y and let $S_n = a_n \sum_{k=1}^{T(p_0^n)} (Y_k + b_n)$. Since for $t \in \mathbb{R}$

$$\begin{aligned} \mathbb{E}\exp\{itS_n\} &= \mathbb{E}\mathbb{E}(\exp\{itS_n\} | T(p_0^n)) \\ &= \sum_{m=1}^{\infty} p_0^n (1 - p_0^n)^{m-1} \varphi(a_n t)^m e^{itma_n b_n} \\ &= \frac{p_0^n \varphi(a_n t) e^{ita_n b_n}}{1 - (1 - p_0^n)\varphi(a_n t) e^{ita_n b_n}}, \end{aligned}$$

then (9) can be replaced by equivalent convergence of ch.f.'s

$$\frac{p_0^n \varphi(a_n t) e^{ita_n b_n}}{1 - (1 - p_0^n)\varphi(a_n t) e^{ita_n b_n}} \overset{n\to\infty}{\longrightarrow} \phi(t) \quad \text{for every } t \in \mathbb{R}.$$

Denote $f_n(t) = \varphi(a_n t) e^{ita_n b_n}$. Since ϕ is GSe ch.f., then it is ID and it has no zeros. Hence we have

$$\frac{f_n(t) - 1}{p_0^n f_n(t)} \overset{n\to\infty}{\longrightarrow} 1 - \frac{1}{\phi(t)}.$$

Therefore $(f_n(t) - 1)/f_n(t) \overset{n\to\infty}{\longrightarrow} 0$ and this implies that $f_n(t) \overset{n\to\infty}{\longrightarrow} 1$. Thus $p_0^{-n}(f_n(t) - 1) \overset{n\to\infty}{\longrightarrow} 1 - 1/\phi(t)$ and also

$$[p_0^{-n}](f_n(t) - 1) \overset{n\to\infty}{\longrightarrow} 1 - 1/\phi(t), \tag{10}$$

where $[x]$ denotes the greatest integer not greater than x. This notation for the integer part of real x we will use throughout the paper.

Since

$$[p_0^{-n}]\log f_n(t) = [p_0^{-n}](f_n(t)-1) + o(f_n(t)-1)[p_0^{-n}]$$

and (10) holds then

$$\varphi(a_n t)^{[p_0^{-n}]} e^{ita_n b_n [p_0^{-n}]} \overset{n\to\infty}{\longrightarrow} \exp\{1 - 1/\phi(t)\}\,.$$

The function $\exp\{1-1/\phi(t)\}$ is a limit of the ch.f.'s sequence and it is continuous at $t = 0$, then from the *Lévy-Cramér Continuity Theorem* it represents a ch.f. of some distribution. Observing that $[p_0^{-n}]/[p_0^{-n-1}] \overset{n\to\infty}{\longrightarrow} p_0 \in (0,1)$ and applying Lemma 1 we infer that $\exp\{1-1/\phi(t)\}$ is ch.f. which corresponds to some Se distribution.

For the proof of the second part of the theorem assume that ψ is Se ch.f. Then there exist the constants $a, r \in (0,1)$, $b \in \mathbb{R}$ such that $\psi(at)e^{itb} = \psi(t)^r$ for every $t \in \mathbb{R}$. Hence

$$\psi(t) = (\psi(at)\exp\{itb\})^{r^{-1}} = \left(\psi(a^2t)\exp\{itb(a+r)\}\right)^{r^{-2}}$$

$$= \left(\psi(a^n t)\exp\left\{itb(a^{n-1} + a^{n-2}r + \cdots + ar^{n-2} + r^{n-1})\right\}\right)^{r^{-n}}.$$

Finally, for every $n \in \mathbb{N}$

$$\psi(t) = \begin{cases} \left(\psi(a^n t)\exp\left\{itb\frac{a^n - r^n}{a-r}\right\}\right)^{r^{-n}}, & \text{if} \quad a \neq r, \\ \left(\psi(r^n t)\exp\left\{itbnr^{n-1}\right\}\right)^{r^{-n}}, & \text{if} \quad a = r. \end{cases} \tag{11}$$

In the case $a \neq r$ we can write

$$r^{-n}\log\Big(\psi(a^n t)\exp\Big\{itb\frac{a^n - r^n}{a-r}\Big\}\Big) \overset{n\to\infty}{\longrightarrow} \log\psi(t) \quad \text{for every } t \in \mathbb{R}. \tag{12}$$

Denote $f_n(t) = \psi(a^n t)\exp\{itb(a^n - r^n)/(a-r)\}$. Then from (12) we infer that $\log f_n(t) \overset{n\to\infty}{\longrightarrow} 0$ and farther $f_n(t) \overset{n\to\infty}{\longrightarrow} 1$. The Taylor's expansion formula applied to the function $\log f_n(t)$ leads us to

$$r^{-n}(f_n(t)-1)\Big(1+\frac{o(f_n(t)-1)}{f_n(t)-1}\Big) \overset{n\to\infty}{\longrightarrow} \log\psi(t).$$

Therefore

$$r^{-n}(f_n(t)-1)/f_n(t) \overset{n\to\infty}{\longrightarrow} \log\psi(t),$$

and finally we have

$$\frac{r^n\psi(a^n t)e^{itb(a^n-r^n)/(a-r)}}{1-(1-r^n)\psi(a^n t)e^{itb(a^n-r^n)/(a-r)}} \overset{n\to\infty}{\longrightarrow} (1-\log\psi(t))^{-1}. \tag{13}$$

The function $(1-\log\psi(t))^{-1}$ is ch.f. because it is limit of the ch.f.'s sequence and it is continuous at $t=0$. Denote by X, Y the r.v.'s with ch.f.'s $(1-\log\psi)^{-1}$ and ψ, respectively. Since (13) is equivalent with

$$a^n\sum_{k=1}^{T(r^n)}\left(Y_k+\frac{b(1-(r/a)^n)}{a-r}\right) \overset{d}{\longrightarrow} X, \text{ as } n\to\infty,$$

then, accordingly to the Definition 1, r.v. X is GSe.

In the case $a = r$ we have

$$\psi(r^n t)^{r^{-n}}\exp\{itbn/r\} \overset{n\to\infty}{\longrightarrow} \psi(t) \quad \text{for every } t\in\mathbb{R}$$

and going similarly as in case $a \neq r$ we obtain

$$\frac{r^n\psi(r^n t)e^{itbnr^{n-1}}}{1-(1-r^n)\psi(r^n t)e^{itbnr^{n-1}}} \overset{n\to\infty}{\longrightarrow} (1-\log\psi(t))^{-1}.$$

Therefore we have, under notation that Y is a r.v. with ch.f. ψ and X a r.v. with ch.f. $(1-\log\psi)^{-1}$, the following convergence

$$r^n \sum_{k=1}^{T(r^n)} (Y_k + bn/r) \xrightarrow{d} X, \text{ as } n \to \infty.$$

Thus X is GSe. ■

One can see that the differences between the conditions defining GSe and GSt r.v.'s are delicate, but the class of GSe r.v.'s is essentially wider than that which is consisted of GSt r.v.'s. We give an example of a ch.f. which is GSe and does not belong to the other mentioned subclasses of GID.

Example 1. Consider a measure ν of the form

$$\nu = \sum_{n=-\infty}^{+\infty} a^{-n\alpha}\delta_{a^n}, \text{ where } a > 1, \ \alpha \in (0,2).$$

It can be verified that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(|x|^2 \wedge 1)\nu(dx) < \infty$. Hence ν is a Lévy measure of an ID distribution. Therefore, using the Lévy-Khintchine formula ([16], p. 37), we get that for every $b \in \mathbb{R}$ the function

$$\psi(t) = \exp\left\{ itb + \sum_{n=-\infty}^{+\infty} a^{-n\alpha}\Big(e^{ita^n} - 1 - ita^n \mathbf{1}_{(0,1]}(a^n)\Big)\right\}, \quad t \in \mathbb{R},$$

is ch.f. of some ID distribution. It can be checked that for such ch.f. ψ the following equation is true

$$\psi(at)e^{it(a+b(a^\alpha - a))} = \psi(t)^{a^\alpha} \text{ for every } t \in \mathbb{R},$$

which means that ψ is Se ch.f. For $b \neq a(a-a^\alpha)^{-1}$ the ch.f. ψ is not SSe. Notice that ψ is not St, because in a stable (non-gaussian) case its Lévy measure ν should be absolutely continuous and expressed as follows ([16], p. 80)

$$\nu(dx) = \begin{cases} c_1 x^{-1-\alpha}dx, & \text{if } \ x \in (0,\infty), \\ c_2|x|^{-1-\alpha}dx, & \text{if } \ x \in (-\infty, 0), \end{cases}$$

where $\alpha \in (0,2)$, $c_1 \geq 0$, $c_2 \geq 0$, $c_1+c_2 > 0$. Therefore, applying Theorem 1, we get that for $a > 1$, $\alpha \in (0,2)$, $b \neq a(a-a^\alpha)^{-1}$ the function

$$\phi(t) = \Big(1 - itb - \sum_{n=-\infty}^{+\infty} a^{-n\alpha}(e^{ita^n} - 1 - ita^n \mathbf{1}_{(0,1]}(a^n))\Big)^{-1}$$

is the ch.f. which is GSe and neither GSt nor GSSe.

2.2. Subordination and GSe distributions

Another connection between Se and GSe distributions goes through the concept of subordination of Lévy processes. By a *Lévy process* (see [16], p. 3) we mean an $\mathbb{R}$-valued stochastic process $\{X_t, t \geq 0\}$ which has independent and stationary increments, $X_0 = 0$ with probability one ($\mathbb{P}$.1), its trajectories are càdlàg (i.e., right-continuous for $t \geq 0$ and have left limits for $t > 0$) with $\mathbb{P}$.1. There is a correspondence between Lévy processes and ID distributions (for the details we refer to Theorem 7.10 in [16]). For this reason a Lévy process $\{X_t\}$ is called St, Se if the ch.f. of X_1 is St, Se, respectively. Since GSe distributions are ID, then by the analogy we can define a *GSe Lévy process* as a Lévy process $\{X_t, t \geq 0\}$ with X_1 being GSe r.v.

Now we recall the definition of a subordination for Lévy processes. Let $\{X_t, t \geq 0\}$ be a Lévy process and $\{Y_t, t \geq 0\}$ a nondecreasing Lévy process (i.e., it has nondecreasing trajectories with $\mathbb{P}$.1) independent of $\{X_t\}$. The transformation giving a process $\{Z_t := X_{Y_t}, t \geq 0\}$ is called a subordination ([16], p. 197). We will say that $\{Z_t\}$ is subordinate to $\{X_t\}$ by the subordinator $\{Y_t\}$.

Proposition 1. *Let $\{X_t, t \geq 0\}$ be a Se Lévy process with ch.f. ψ of X_1, and let $\{Y_t, t \geq 0\}$ be a gamma process which is independent of $\{X_t\}$, with $\mathbb{E}Y_1 = a > 0$. Then $\{Z_t = X_{Y_t}, t \geq 0\}$ is GSe Lévy process with ch.f. $(1 - a\log\psi)^{-1}$ of Z_1.*

Proof. Suppose $\{X_t, t \geq 0\}$ is a Se Lévy process, and let $\psi(s) = \mathbb{E}\exp\{isX_1\}$, $s \in \mathbb{R}$. For a gamma process $\{Y_t, t \geq 0\}$ it is known that it is a subordinator and the Laplace transform of Y_t is as follows

$$\mathbb{E}\exp\{-uY_t\} = (1+au)^{-t}, \;\; u \geq 0,$$

where $a = \mathbb{E}Y_1 > 0$. Then using the Theorem 30.1 of [16] we have that $\{Z_t = X_{Y_t}, t \geq 0\}$ is a Lévy process, and due to the formula (30.6) (from the same theorem) for ch.f. of $Z_t = X_{Y_t}$ we have immediately that

$$\mathbb{E}\exp\{isZ_t\} = (1 - a\log\psi(s))^{-t}.$$

Since ψ^a is Se ch.f. then the Theorem 1 implies that Z_1 is GSe r.v. ■

With a concept of the subordination we can derive a Lévy-Khintchine representation for the ch.f. of GSe distribution.

Theorem 2. *Let ϕ be a ch.f. Then it is GSe ch.f. if and only if it admits the following representation*

$$\phi(s) = \exp\left\{\int_{\mathbb{R}}(e^{isx} - 1)\nu(dx)\right\}, \tag{14}$$

with

$$\nu(B) = \int_{(0,\infty)} \mu^{*u}(B)u^{-1}e^{-u}du, \quad B \in \mathcal{B}(\mathbb{R}\setminus\{0\}), \tag{15}$$

where μ^{*u} denotes a Se distribution with ch.f. $\psi^u = \exp\{u(1-1/\phi)\}$, and $\mathcal{B}(\mathbb{R}\setminus\{0\})$ is the class of Borel subsets of $\mathbb{R}\setminus\{0\}$.

Proof. Assume that ϕ is GSe ch.f. Let $\{X_t, t \geq 0\}$ be a Se Lévy process with the ch.f. $\psi = \exp\{1-1/\phi\}$ of X_1, and let $\{Y_t, t \geq 0\}$ be a gamma process independent of $\{X_t\}$, $\mathbb{E}Y_1 = 1$. If φ denotes ch.f. of Y_1 then $\varphi(s) = (1-is)^{-1}$, $s \in \mathbb{R}$. Since φ is ID ch.f. then it has a Lévy-Khintchine representation, which is in fact

$$\varphi(s) = \exp\left\{\int_{(0,\infty)}(e^{isx} - 1)x^{-1}e^{-x}dx\right\}.$$

Thus the Lévy measure ρ for the gamma process $\{Y_t\}$, with $\mathbb{E}Y_1 = 1$, is expressed as

$$\rho(dx) = x^{-1}e^{-x}\mathbf{1}_{(0,\infty)}(x)dx.$$

Consider a GSe Lévy process $\{Z_t, t \geq 0\}$ as subordinate to $\{X_t\}$ by the subordinator $\{Y_t\}$. Then ϕ is ch.f. of Z_1. Applying the Theorem 30.1 of [16] we get

$$\phi(s) = \exp\left\{ is\gamma + \int_{\mathbb{R}} \Big(e^{isx} - 1 - isx\mathbf{1}_{[-1,1]}(x) \Big) \nu(dx) \right\}, \tag{16}$$

with

$$\gamma = \int_{(0,\infty)} \rho(du) \int_{[-1,1]} x\mu^{*u}(dx)$$

and ν such as in (15). Notice that $\int_{(0,\infty)} u^{1/2}\rho(du) < \infty$ and, in view of Lemma 30.3 of [16], there exists a constant c such that for every u the following inequality holds $\int_{[-1,1]} |x|\mu^{*u}(dx) \leq cu^{1/2}$. Therefore

$$\int_{[-1,1]} |x|\nu(dx) = \int_{(0,\infty)} \rho(du) \int_{[-1,1]} |x|\mu^{*u}(dx) < \infty$$

and (16) turns into (14). The opposite implication is obvious since $\psi = \exp\{1 - 1/\phi\}$ is Se ch.f. ∎

We observe that the given in (15) Lévy measure ν of GSe distribution is the same as the Lévy measure of the so called 1-potential measure V^1 on $\mathbb{R}$ of some Se Lévy process. This measure is defined as

$$V^1(B) = \int_0^\infty e^{-u}\mu^{*u}(B)du, \quad B \in \mathcal{B}(\mathbb{R})$$

where μ^{*u} denotes the distribution of r.v. X_u from the Se Lévy process $\{X_t, t \geq 0\}$. For the details concerning q-potential measures ($q \geq 0$) of the Lévy processes see [16], page 203. Therefore we infer the following assertion.

Remark 2. A set of GSe distributions coincides with the set of 1-potential measures of Se Lévy processes.

The analogous statements can be formulated for GSSt, GSt, GSSe and GID distributions.

3. Other characterizations for GSe random variable

In this section we present the other theorems which characterize GSe r.v. as a limit (in the sense of convergence in distributions) of a sequence of the geometric random sums.

Proposition 2. *A r.v. X is GSe if and only if there exist the sequences $\{a_n\} \subset \mathbb{R}^+$, $\{b_n\} \subset \mathbb{R}$, and a r.v. Y such that*

$$a_n \sum_{k=1}^{T(p_n)} (Y_k + b_n) \xrightarrow{d} X, \text{ as } n \to \infty, \tag{17}$$

where $\{p_n\} \subset (0,1)$ and $p_n \to 0$, $p_{n+1}/p_n \to p_0 \in (0,1]$.

Proof. The first implication is obvious. For the proof of the second let us denote by ϕ, ψ the ch.f.'s of r.v.'s X, Y, respectively. From the assumptions we have

$$\frac{p_n \psi(a_n t) e^{ita_n b_n}}{1-(1-p_n)\psi(a_n t)e^{ita_n b_n}} \overset{n\to\infty}{\longrightarrow} \phi(t) \quad \text{for every } t \in \mathbb{R}.$$

Going similarly as in the proof of the Theorem 1 we obtain

$$\psi(a_n t)^{k_n} e^{ita_n b_n k_n} \overset{n\to\infty}{\longrightarrow} \exp\{1 - 1/\phi(t)\},$$

where $k_n = [p_n^{-1}]$. The assumption about sequence $\{p_n\}$ yields $k_n/k_{n+1} \overset{n\to\infty}{\longrightarrow} p_0 \in (0,1]$. If $p_0 \in (0,1)$ then, according to the Lemma 1, we can state that the limit function $\exp\{1-1/\phi\}$, which is continuous at zero, is ch.f. of some Se distribution. Hence, by the Theorem 1, ϕ is GSe ch.f. If $p_0 = 1$ then, according to the Remark 1, $\exp\{1-1/\phi\}$ is St ch.f. Since every St ch.f. is Se, then the proof is complete. ■

In the studies of GSe r.v.'s it has appeared a question: whether can we omit the condition $p_{n+1}/p_n \to p_0 \in (0,1]$ in Proposition 2 and still preserve geometric semistability of the limit X in (17)? The following example shows that the answer is negative.

Example 2. Following the Khintchine's method ([2] Chapter 7, §36) choose a sequence $\{k_n\} \subset \mathbb{N}$ which increases so rapidly that

$$k_n \sum_{j=n+1}^{\infty} k_j^{-1} \overset{n\to\infty}{\longrightarrow} 0$$

and satisfies

$$k_n^{-1/2} \lambda_n^{-3/2} \sum_{j=1}^{n-1} k_j^{1/2} \lambda_j^{3/2} \overset{n\to\infty}{\longrightarrow} 0,$$

where $\lambda_1 = 1$, $\lambda_j = j \sum_{k=1}^{j-1} \lambda_k$ for $j \geq 2$.

Denote

$$a_n = k_n^{1/2} \lambda_n^{1/2}, \quad b_n = \sum_{j=1}^{n-1} k_j^{-1} a_j,$$

and let Y be an ID r.v. with ch.f.

$$\varphi(t) = \exp\left\{ \sum_{j=1}^{\infty} k_j^{-1} (e^{ita_j} - 1) \right\}, \quad t \in \mathbb{R}.$$

It can be shown that

$$k_n \log\left(\varphi(a_n^{-1} t) e^{-ita_n^{-1} b_n} \right) \overset{n\to\infty}{\longrightarrow} \log \psi(t),$$

where $\psi(t) = \exp\{e^{it} - 1\}$ is ch.f. of the Poisson distribution. So we have the expression similar to (12), therefore going similarly as in the proof of the Theorem 1 we obtain

$$a_n^{-1} \sum_{k=1}^{T(k_n^{-1})} (Y_k - b_n) \xrightarrow{d} X, \text{ as } n \to \infty,$$

where r.v. X has ch.f. equal to $(1 - \log \psi(t))^{-1} = (2 - e^{it})^{-1}$. This means that X has the following geometric distribution: $\mathbb{P}(X = n) = 2^{-n-1}$, $n = 0, 1, \ldots$ R.v. X is ID and GID, but is not GSe since ψ is not Se ch.f. (ψ does not satisfy (7)).

It turns out that a skipping of the condition $p_{n+1}/p_n \to p_0 \in (0,1]$ in (17) leads to the geometric infinite divisibility of the limit, only. However, a substitution of this condition for the one similar on sequence $\{a_n\}$ assures that the limit stays in the class GSe.

Proposition 3. *A r.v. X is GSe if and only if there exist the sequences $\{p_n\} \subset (0,1)$, $p_n \to 0$, $\{b_n\} \subset \mathbb{R}$, and a r.v. Y such that*

$$a_n \sum_{k=1}^{T(p_n)} (Y_k + b_n) \xrightarrow{d} X, \ \ \textit{as } n \to \infty,$$

where $\{a_n\} \subset \mathbb{R}^+$, $a_{n+1}/a_n \to a \in (0,1]$.

Proof. Assume that X is GSe r.v., and denote by ϕ its ch.f. Then, according to the Theorem 1, $\psi = \exp\{1 - 1/\phi\}$ is Se ch.f. and therefore there are some constants $a, r \in (0,1)$, $b \in \mathbb{R}$ such that $\psi(at)e^{itb} = \psi(t)^r$ for every $t \in \mathbb{R}$. Similarly as in the course of the proof of Theorem 1 one obtains (11). Consider the case $a \neq r$. If $a = r$, the proof is analogous. Thus we have

$$(\psi(a^n t) \exp\{itb(a^n - b^n)/(a-r)\})^{k_n} \overset{n\to\infty}{\longrightarrow} \psi(t)$$

for every $t \in \mathbb{R}$, with $k_n = [r^{-n}]$. So denoting by Y any r.v. with ch.f. equal to ψ we can write

$$a^n \sum_{k=1}^{T(k_n^{-1})} \Big(Y_k + b(a-r)^{-1}(1 - (r/a)^n)\Big) \xrightarrow{d} X, \ \text{as } n \to \infty.$$

For the opposite implication, going similarly as in the proof of the first part of Theorem 1, one gets

$$(18) \qquad \Big(\psi(a_n t)e^{ita_n b_n}\Big)^{[p_n^{-1}]} \overset{n\to\infty}{\longrightarrow} \exp\{1 - 1/\phi(t)\} \quad \text{for every } t \in \mathbb{R},$$

where ϕ, ψ are the ch.f.'s of r.v.'s X, Y, respectively. Therefore

$$\left(\left|\psi\Big(\frac{a_{n+1}}{a_n}\cdot a_n t\Big)\right|^{[p_n^{-1}]}\right)^{[p_{n+1}^{-1}]/[p_n^{-1}]} \stackrel{n\to\infty}{\longrightarrow} |\exp\{1-1/\phi(t)\}|\,. \tag{19}$$

But on the other hand we have

$$\left|\psi\Big(\frac{a_{n+1}}{a_n}\cdot a_n t\Big)\right|^{[p_n^{-1}]} \stackrel{n\to\infty}{\longrightarrow} |\exp\{1-1/\phi(at)\}|\,,$$

so the sequence $[p_{n+1}^{-1}]/[p_n^{-1}]$ in (19) has to be convergent. Denote its limit by p and let $\varphi(t)=\exp\{1-1/\phi(t)\}$. If $p<1$ then the equality $|\varphi(t)|=|\varphi(at)|^p$ yields

$$|\varphi(t)|=|\varphi(a^n t)|^{p^n} \stackrel{n\to\infty}{\longrightarrow} 1 \quad \text{for every } t\in\mathbb{R}.$$

Hence $\varphi(t)=e^{itx}$ for some $x\in\mathbb{R}$ and $\phi(t)=(1-itx)^{-1}$ is (except $x=0$) ch.f. of an exponential distribution with a mean equal to x. For any $x\in\mathbb{R}$ ch.f. $(1-itx)^{-1}$ is GSe.

If $p\geq 1$ then (by Lemma 1 and Remark 1) the convergence (18) means that $\exp\{1-1/\phi\}$ is Se ch.f. and, as a consequence of the Theorem 1, ϕ is then GSe ch.f. ■

Remark 3.

(i) If r.v.'s X, Y and the sequences $\{a_n\}\subset\mathbb{R}^+$, $\{b_n\}\subset\mathbb{R}$, $\{p_n\}\subset(0,1)$, $p_n\to 0$ are such that (17) holds with $p_{n+1}/p_n\to 1$ or $a_{n+1}/a_n\to 1$, then X is GSt r.v.

(ii) If X is GSt r.v. then there are sequences $\{a_n\}\subset\mathbb{R}^+$, $\{b_n\}\subset\mathbb{R}$, $\{p_n\}\subset(0,1)$, $p_n\to 0$, r.v. Y such that (17) holds with $p_{n+1}/p_n\to 1$ and $a_{n+1}/a_n\to 1$.

Proof. The assertion (i) easily follows from Proposition 2 and Proposition 3. For (ii), if X is GSt then (3) is satisfied, i.e., for some positive constants a_p, real b_p and r.v. Y convergence $a_p\sum_{k=1}^{T(p)}(Y_k+b_p)\stackrel{d}{\longrightarrow}X$ takes place, where $p\in(0,1)$, $p\to 0$. In particular, this convergence holds when one consider it with p_n instead of p, where $\{p_n\}\subset(0,1)$ and $p_n\stackrel{n\to\infty}{\longrightarrow}0$.

One can choose $p_n = 1/n$, $n = 2, 3, \ldots$ Consider (3) and replace p with $p_n = 1/n$ and a_p, b_p with corresponding to $p_n = 1/n$ constants a_n, b_n. The constants a_p in (3), see the Theorem 3.1 of [5], have a form (in general) $a_p = Cp^{1/\alpha}(1 + o(1))$, where $\alpha \in (0, 2]$ and $C > 0$. Thus $a_n = Cn^{-1/\alpha}(1 + o(1))$. This ends the proof. ∎

From the above statements one can write some new conditions which can define GSt r.v.

4. LIMIT TYPE RESULT FOR GSE CHARACTERISTIC FUNCTION

In this part some limit representation for GSe ch.f. is established.

Theorem 3. *The ch.f. ϕ is GSe if and only if*

$$\phi(t) = \lim_{n\to\infty} (1 + r_n(1 - \psi(a_n t)) + itb_n)^{-1} \quad \textit{for every } t \in \mathbb{R}, \tag{20}$$

where ψ is ch.f., $\{a_n\} \subset \mathbb{R}^+$, $a_n \to 0$, $\{b_n\} \subset \mathbb{R}$, $\{r_n\} \subset \mathbb{R}^+$ is an increasing sequence such that $r_n/r_{n+1} \to r \in (0, 1]$.

Proof. *Necessity.* If ϕ is GSe ch.f. then, by the Theorem 1, $\exp\{1-1/\phi\}$ is Se ch.f. From the Lemma 1 there exist a ch.f. ψ, $a_n \in \mathbb{R}^+$, $a_n \to 0$, $b_n \in \mathbb{R}$, and $k_n \in \mathbb{N}$ with property $k_n/k_{n+1} \stackrel{n\to\infty}{\longrightarrow} r \in (0, 1)$, for which

$$\psi(a_n t)^{k_n} e^{itb_n} \stackrel{n\to\infty}{\longrightarrow} \exp\{1 - 1/\phi(t)\} \quad \text{for every } t \in \mathbb{R}.$$

Hence

$$k_n \log \psi(a_n t) + itb_n \stackrel{n\to\infty}{\longrightarrow} 1 - 1/\phi(t)$$

and using the Taylor's expansion formula to the function $\log \psi$ we obtain

$$k_n(\psi(a_n t) - 1)\left(1 + \frac{o(\psi(a_n t) - 1)}{\psi(a_n t) - 1}\right) + itb_n \stackrel{n\to\infty}{\longrightarrow} 1 - 1/\phi(t).$$

Thus

$$k_n(\psi(a_nt) - 1) + itb_n \overset{n\to\infty}{\longrightarrow} 1 - 1/\phi(t).$$

Now it is easy to see that for every $t \in \mathbb{R}$ we have

$$(1 + k_n(1 - \psi(a_nt)) + itb_n)^{-1} \overset{n\to\infty}{\longrightarrow} \phi(t).$$

Sufficiency. Denote $h(t) = 1/\phi(t) - 1$, $t \in \mathbb{R}$. From the assumptions we can write

$$h(t) = \lim_{n\to\infty}(r_n(1 - \psi(a_nt)) + itb_n).$$

If by F we denote the distribution function with ch.f. ψ, then we have

$$r_n \int_{-\infty}^{+\infty} (e^{ia_ntx} - 1)dF(x) - itb_n \overset{n\to\infty}{\longrightarrow} -h(t).$$

Since F is the distribution function of a probability measure, then $\int_{-\infty}^{+\infty}(e^{ia_ntx} - 1)dF(x)$ is the logarithm of some ID ch.f. γ at the point a_nt. Hence we can write

$$r_n \log\gamma(a_nt) - itb_n \overset{n\to\infty}{\longrightarrow} -h(t)$$

and in consequence

$$\gamma(a_nt)^{r_n}e^{-itb_n} \overset{n\to\infty}{\longrightarrow} e^{-h(t)},$$

but this implies that also

$$\gamma(a_nt)^{[r_n]}e^{-itb_n[r_n]/r_n} \overset{n\to\infty}{\longrightarrow} e^{-h(t)}.$$

The limit function $\exp\{-h(t)\}$ is continuous at $t = 0$. Thus it is ch.f. of some distribution. Since $r_n/r_{n+1} \overset{n\to\infty}{\longrightarrow} r \in (0,1]$, then $[r_n]/[r_{n+1}] \overset{n\to\infty}{\longrightarrow} r$. If $r \in (0,1)$ then the Lemma 1 implies that $\exp\{-h(t)\} = \exp\{1-1/\phi(t)\}$ is Se ch.f. Hence, due to the Theorem 1, ϕ is GSe ch.f. In the case $r = 1$, applying the Remark 1, we infer that $\exp\{1 - 1/\phi\}$ is St ch.f. and in consequence ϕ is GSt. ∎

Corollary 1. *Let ϕ be a function described by* (20) *in the Theorem* 3. *It is*

(i) *GSSe ch.f., iff* $b_n = 0$ *for every* $n \in \mathbb{N}$*;*

(ii) *GSt ch.f., iff* $r_n/r_{n+1} \stackrel{n\to\infty}{\longrightarrow} 1$*;*

(iii) *GSSt ch.f., iff* $b_n = 0$ *for every* $n \in \mathbb{N}$ *and* $r_n/r_{n+1} \stackrel{n\to\infty}{\longrightarrow} 1$.

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