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STOCHASTIC VORTICES IN PERIODICALLY RECLASSIFIED POPULATIONS

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Abstract

Our paper considers open populations with arrivals and departures whose elements are subject to periodic reclassifications. These populations will be divided into a finite number of sub-populations.

Assuming that:

a) entries, reclassifications and departures occur at the beginning of the time units;

b) elements are reallocated at equally spaced times;

c) numbers of new elements entering at the beginning of the time units are realizations of independent Poisson distributed random variables;

we use Markov chains to obtain limit results for the relative sizes of the sub-populations corresponding to the states of the chain. Namely we will obtain conditions for stability of the relative sizes for transient and recurrent states as well as for all states. The existence of such stability corresponds to the existence of a stochastic structure based either on the transient or on the recurrent states or even on all states. We call these structures stochastic vortices because the structure is maintained despite entrances, departures and reallocations.

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1. Introduction

Let us consider an open population divided into sub-populations. These populations play a relevant part in many problems. For instance, we may consider the drivers which are clients of an insurance company. According to their records they are placed in one of the *Bonus-Malus* classes. A similar example is given by the clients of a bank each of which is placed in a *Credit-Rating* level. The list of examples can, very easily, be extended.

It is easily seen that to manage these populations it is very important to have information about the relative sizes of the sub-populations.

We obtain limit results for these relative sizes assuming that:

a) entries, reallocations and departures occur at equally spaced times;

b) probabilities of reallocation of the population elements are stable;

c) numbers of entries are given by independent Poisson distributed random variables, for populations with a finite number of sub-populations.

Our treatment will be based on finite, discrete parameter, homogeneous Markov chains. We consider the possibility of more than one recurrent class.

The stability of relative sizes of sub-populations, despite entrances, departures and reallocations, shows the existence of a structure. We call these structures stochastic vortices.

An interesting problem occurs, when for the one step transition matrix of a recurrent class, we have more than one module 1 eigenvalues. Then, from the Frobenius theorem, it may be shown that there is a limit cycle for the transition probabilities between states in that class. Nevertheless, under general conditions the relative sizes of the corresponding sub-populations will be stable.

2. Population structure

Our study will consider populations:

- divided into k sub-populations, corresponding to k Markov chain states, grouped into w communication classes;

- k_d^+ transient states grouped into d transient classes, each one with $k_j, j = 1, \ldots, d$, states;
- $k - k_d^+$ recurrent states grouped into r recurrent classes, with k_{d+j} , $j = 1, \ldots, r$, states. Note that $d + r = w$;
- we order the classes in such a way that each one can only be accessed from classes with lower or equal indexes;

According to this, the one step transition matrix of the Markov chain underlying the population will be

$$\text{(1)} \qquad \boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{1,1} & \cdots & \boldsymbol{P}_{1,d} & \boldsymbol{P}_{1,d+1} & \cdots & \boldsymbol{P}_{1,w} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{P}_{d,1} & \cdots & \boldsymbol{P}_{d,d} & \boldsymbol{P}_{d,d+1} & \cdots & \boldsymbol{P}_{d,w} \\ \boldsymbol{P}_{d+1,1} & \cdots & \boldsymbol{P}_{d+1,d} & \boldsymbol{P}_{d+1,d+1} & \cdots & \boldsymbol{P}_{d+1,w} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{P}_{w,1} & \cdots & \boldsymbol{P}_{w,d} & \boldsymbol{P}_{w,d+1} & \cdots & \boldsymbol{P}_{w,w} \end{bmatrix}$$

with $\boldsymbol{P}_{i,j}$ the $k_i \times k_j$ sub-matrix of the transition probabilities between the states of the transient classes with index i , $i = 1, \ldots, d$ [recurrent class with index $i - d$, $i = d + 1, \ldots, w$] and the states of the transient classes with index j , $j = 1, \ldots, d$ [recurrent class with index $j - d$, $j = d + 1, \ldots, w$].

It is easily seen that

$$\text{(2)} \qquad \begin{cases} \boldsymbol{P}_{l,h} = 0 \quad , \ l > h, \\ \boldsymbol{P}_{l,h} = 0 \quad , \ h \neq l \ , \ l = d + 1, \ldots, w, \end{cases}$$

so the one step transition matrix can be written as

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{1,1} & \dots & \boldsymbol{P}_{1,d} & \boldsymbol{P}_{1,d+1} & \dots & \boldsymbol{P}_{1,w} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \dots & \boldsymbol{P}_{d,d} & \boldsymbol{P}_{d,d+1} & \dots & \boldsymbol{P}_{d,w} \\ \boldsymbol{O} & \dots & \boldsymbol{O} & \boldsymbol{P}_{d+1,d+1} & \dots & \boldsymbol{O} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \dots & \boldsymbol{O} & \boldsymbol{O} & \dots & \boldsymbol{P}_{w,w} \end{bmatrix}. \tag{3}$$

To lighten the writing, from now on we will put

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{U} \\ \boldsymbol{O} & \boldsymbol{V} \end{bmatrix} \tag{4}$$

with

- $\boldsymbol{K}$ - the $k_d^+ \times k_d^+$ transition matrix between transient states;
- $\boldsymbol{U}$ - the $k_d^+ \times (k - k_d^+)$ transition matrix between the transient and the recurrent states;
- $\boldsymbol{O}$ - the $(k - k_d^+) \times k_d^+$ null matrix;
- $\boldsymbol{V}$ - the $(k - k_d^+) \times (k - k_d^+)$ transition matrix between the recurrent states.

Lemma 1. *The n-th step transition matrix will be*

$$\boldsymbol{P}^n = \begin{bmatrix} \boldsymbol{K}^n & \boldsymbol{U}_n \\ \boldsymbol{O} & \boldsymbol{V}^n \end{bmatrix} \tag{5}$$

with

$$\boldsymbol{U}_n = \boldsymbol{U}_{n-1} \cdot \boldsymbol{V} + \boldsymbol{U} \cdot \boldsymbol{K}^{n-1}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{6}$$

Proof. Since the Markov chain is homogeneous we will have $P(n) = P^n$ and the thesis is easily established through mathematical induction. ■

3. Limits and limit cycles

Let us assume that $\boldsymbol{P}$ is a $k \times k$ diagonilizable matrix, i.e., there exists a basis of eigenvectors of the matrix $\boldsymbol{P}$.

With $S(\boldsymbol{P})$ the sprectrum of the matrix $\boldsymbol{P}$, we know that

$$S(\boldsymbol{P}) = \bigcup_{l=1}^{w} S(\boldsymbol{P}_{l,l}), \tag{7}$$

so the eigenvalues of $\boldsymbol{P}_{l,l}$; $l = 1, \ldots, w$, will also be eigenvalues of $\boldsymbol{P}$.
Under very general conditions, see Schott [6], we will have:

$$\boldsymbol{P} = \sum_{j=1}^{k} \eta_j \, \boldsymbol{\alpha}_j \, \boldsymbol{\beta}_j^T, \tag{8}$$

where $\eta_j \; \left[\boldsymbol{\alpha}_j \, ; \, \boldsymbol{\beta}_j^T\right]$; $j = 1, \ldots, k$, are the eigenvalues [left and right eigenvectors] of $\boldsymbol{P}$.

Now, the n-step transition matrix $\boldsymbol{P}(n)$ will be, see also Schott [6], the n-th power of $\boldsymbol{P}$ and so,

$$\boldsymbol{P}(n) = \boldsymbol{P}^n = \sum_{j=1}^{k} \eta_j^n \, \boldsymbol{\alpha}_j \, \boldsymbol{\beta}_j^T. \tag{9}$$

For the sub-matrices $\boldsymbol{P}_{l,h}$, $l = 1, \ldots, w$, $h = 1, \ldots, w$, of the transition matrix $\boldsymbol{P}$, considering $k_i^+ = \sum_{j=1}^{i} k_j$, we will have

$$\boldsymbol{P}_{l,h} = \sum_{j=k_{h-1}^+ +1}^{k_h^+} \eta_j \, \boldsymbol{\alpha}_{j,l} \, \boldsymbol{\beta}_{j,h}^T \quad , \; l = 1, \ldots, w \, , \; h = 1, \ldots, w, \tag{10}$$

as well as

$$(11) \qquad \boldsymbol{P}_{l,h}(n) = \sum_{j=k_{h-1}^{+}+1}^{k_h^{+}} \eta_j^n \,\boldsymbol{\alpha}_{j,l}\, \boldsymbol{\beta}_{j,h}^T \quad , \; l \le h \;,\; h = 1, \dots, d.$$

From Parzen [4], we know that the transition probabilities between the transient states tend to zero, as $n \to +\infty$, so

$$(12) \qquad \lim_{n\to+\infty} p_{l,h}(n) = 0 \;,\; l = 1, \dots, d \;,\; h = 1, \dots, d,$$

and so, from (11) and (12), we can conclude that for the transient states we have $|\eta_j| < 1 \;,\; j = 1, \dots, k_d^{+}$.

Besides this, in the recurrent classes we can have $u_{d+1}, \dots, u_w$ eigenvalues with modulus 1, which according to (7), will also be eigenvalues of $\boldsymbol{P}$. According to the Frobenius theorem (see Gantmacher [1]), these module 1 eigenvalues will be the roots of 1 with indexes $u_{d+1}, \dots, u_w$. The remaining eigenvalues of the $\boldsymbol{P}_{d+1,d+1}, \dots, \boldsymbol{P}_{w,w}$ will have modules lesser than 1.

Let us give to eigenvalues with module 1 the indexes $k_{h-1}^{+} + 1, \dots, k_{h-1}^{+} + u_h$, for $h = d+1, \dots, w$.

Then, due to (9) we will have

$$(13) \qquad \lim_{n\to+\infty} \boldsymbol{P}^n - \sum_{h=d+1}^{w} \sum_{j=k_{h-1}^{+}+1}^{k_{h-1}^{+}+u_h} \eta_j^n \,\boldsymbol{\alpha}_j\, \boldsymbol{\beta}_j^T = 0,$$

so we can concentrate our attention on the limits of $\boldsymbol{P}_*^n$, with

$$(14) \qquad \boldsymbol{P}_* = \sum_{h=d+1}^{w} \sum_{j=k_{h-1}^{+}+1}^{k_{h-1}^{+}+u_h} \eta_j \,\boldsymbol{\alpha}_j\, \boldsymbol{\beta}_j^T .$$

Let c be the least common multiple of the $u_{d+1}, \ldots, u_w$. Then

$$\boldsymbol{P}_*^c = \sum_{h=d+1}^{w} \sum_{j=k_{h-1}^{+}+1}^{k_{h-1}^{+}+u_h} \boldsymbol{\alpha}_j \, \boldsymbol{\beta}_j^T \tag{15}$$

and with $s = 0, \ldots, c-1$,

$$\boldsymbol{P}_*^{cn+s} = \boldsymbol{P}_*^s \ , \ n \in \mathbb{N}_0. \tag{16}$$

Since $\boldsymbol{P}_*^s$, $s = 0, \ldots, c-1$ are distinct matrices we will have a limit cycle with period c, which, due to (13), implies that $\boldsymbol{P}^n$ has also a limit cycle with period c.

If $u_{d+1} = \ldots = u_w = 1$, then $c = 1$ and the cycle degenerates on the "constant" matrix $\boldsymbol{P}_*$, and so,

$$\lim_{n \to +\infty} \boldsymbol{P}_*^n = \boldsymbol{P}_* \tag{17}$$

which, due to (13), implies that

$$\lim_{n \to +\infty} \boldsymbol{P}^n = \boldsymbol{P}_* \, . \tag{18}$$

4. Stochastic vortices

4.1. Model Presentation

In this section we will make several assumptions:

- The elements entering the population in each period of time will be independent and Poisson distributed with mean λ_i , $i \in \mathbb{N}_0$, so being N_i , $i \in \mathbb{N}_0$ the number of elements entering the population in the i-th year, we will have

$$N_i \sim P(\lambda_i) \ , \ i \in \mathbb{N}_0. \tag{19}$$

- Mean values λ_i , $i \in \mathbb{N}_0$, will be given by

$$\lambda_i = a + b\,\theta^i \ , \ \theta > 0 \, ; \, i \in \mathbb{N}_0. \tag{20}$$

  Note that this is quite a general assumption. For instance, we can obtain $\lambda_i = b\theta^i$ if $a = 0$, which represents a population with a geometric growth as well as $\lambda_i = a(1 - e^{-\delta\, i})$, if $b = -a$ and $\theta = e^{-\delta}$, which represents a population with an asymptotic growth.

- New elements entering the population in the i-th period will be allocated to the different sub-populations according to the components of $\boldsymbol{p}_i$, $i \in \mathbb{N}_0$. We will also consider the sub-vector $\boldsymbol{t}_i$ $[\boldsymbol{r}_i]$ of $\boldsymbol{p}_i$ whose components are the probabilities for entering in transient [recurrent] states, thus

$$\boldsymbol{p}_i^T = \left[\begin{array}{c|c} \boldsymbol{t}_i^T & \boldsymbol{r}_i^T \end{array} \right].$$

- The one-step transition matrix is given by (4) so the n-step transition matrix is given by (5).

4.2. Vortices based on the transient states

For vortices based on the transient states, we will only consider the $\boldsymbol{K}$ sub-matrix of (4) given by

$$\boldsymbol{K} = \sum_{j=1}^{k_d^+} \eta_j \, \boldsymbol{\alpha}_j^\circ \, \boldsymbol{\beta}_j^{\circ T}, \tag{21}$$

where the $\boldsymbol{\alpha}_j^\circ$ $\left[\boldsymbol{\beta}_j^{\circ T}\right]$, $j = 1, \ldots, k_d^+$, are the left [right] eigenvectors of $\boldsymbol{K}$. We recall that the eigenvalues of $\boldsymbol{K}$ will be eigenvalues of $\boldsymbol{P}$.

The elements which entered the population in the i-th time period will, after n periods, have the parameters

$$\boldsymbol{p}_i^T \, \boldsymbol{P}^n = \left[\ \boldsymbol{t}_i^T \, \boldsymbol{K}^n \ \ | \ \ \boldsymbol{t}_i^T \, \boldsymbol{U}_n + \boldsymbol{r}_i^T \, \boldsymbol{V}_n \ \right] \tag{22}$$

for the Poisson distribution.

We can now establish

Proposition 1. *After n reclassifications, the expected dimension of the different sub-populations is given by the components of the vector*

$$(23)\qquad \boldsymbol{\lambda}_n^{++T} = \left[\sum_{i=0}^{n} \lambda_i \, \boldsymbol{t}_i^T \, \boldsymbol{K}^{n-i} \;\middle|\; \sum_{i=0}^{n} \lambda_i \, \boldsymbol{t}_i^T \, \boldsymbol{U}_{n-i} + \sum_{i=0}^{n} \lambda_i \boldsymbol{r}_i^T \, \boldsymbol{V}_{n-i} \right].$$

Proof. Since the expected number of new elements entering the population in the i-th period is λ_i , $i \in \mathbb{N}_0$, we will have:

- At the beginning of "period 0", the expected number of elements in the different sub-populations will be the components of

$$\boldsymbol{\lambda}_0^{++T} = \lambda_0 \, \boldsymbol{p}_0^T.$$

- At the beginning of the next period we will expect λ_1 entrances, which will be allocated to the different states according to the components of $\lambda_1 \, \boldsymbol{p}_1$. The elements entered in the first period will be reallocated according to the components of $\lambda_0 \, \boldsymbol{p}_0 \, P$. So, at the beginning of "period 1", the vector of sub-populations parameters will be

$$\boldsymbol{\lambda}_1^{++T} = \lambda_1 \, \boldsymbol{p}_1^T + \lambda_0 \, \boldsymbol{p}_0^T \, P$$

$$= \sum_{i=0}^{1} \lambda_i \, \boldsymbol{p}_i^T \, \boldsymbol{P}^{n-i},$$

where $\boldsymbol{P}^0$ is the identity matrix.

- Reasoning as before, at the beginning of "period 2", we will have

$$\boldsymbol{\lambda}_2^{++T} = \lambda_2 \, \boldsymbol{p}_2^T + \lambda_1 \, \boldsymbol{p}_1^T \, \boldsymbol{P} + \lambda_0 \, \boldsymbol{p}_0^T \, \boldsymbol{P}^2$$

$$= \sum_{i=0}^{2} \lambda_i \, \boldsymbol{p}_i^T \, \boldsymbol{P}^{n-i}.$$

- Likewise, after n periods of time, the expected dimensions of the sub-populations states will be the components of

$$\boldsymbol{\lambda}_n^{++T} = \sum_{i=0}^{n} \lambda_i \, \boldsymbol{p}_i^T \, \boldsymbol{P}^{n-i}.$$

Remembering that

$$\boldsymbol{p}_i^T = \left[\begin{array}{c|c} \boldsymbol{t}_i^T & \boldsymbol{r}_i^T \end{array} \right],$$

we will get

$$\begin{aligned}
\boldsymbol{\lambda}_n^{++T} &= \sum_{i=0}^{n} \lambda_i \left[\begin{array}{c|c} \boldsymbol{t}_i^T & \boldsymbol{r}_i^T \end{array} \right] \left[\begin{array}{cc} \boldsymbol{K}^{n-i} & \boldsymbol{U}_{n-i} \\ \boldsymbol{O} & \boldsymbol{V}_{n-i} \end{array} \right] \\
&= \sum_{i=0}^{n} \lambda_i \left[\begin{array}{c|c} \boldsymbol{t}_i^T \, \boldsymbol{K}^{n-i} & \boldsymbol{t}_i^T \, \boldsymbol{U}_{n-i} + \boldsymbol{r}_i^T \, \boldsymbol{V}_{n-i} \end{array} \right] \\
&= \left[\begin{array}{c|c} \displaystyle\sum_{i=0}^{n} \lambda_i \, \boldsymbol{t}_i^T \, \boldsymbol{K}^{n-i} & \displaystyle\sum_{i=0}^{n} \lambda_i \, \boldsymbol{t}_i^T \, \boldsymbol{U}_{n-i} + \sum_{i=0}^{n} \lambda_i \, \boldsymbol{r}_i^T \, \boldsymbol{V}_{n-i} \end{array} \right].
\end{aligned}$$

■

So, after n periods, the parameters vector for the transient states will, with $c_{i,j} = \boldsymbol{t}_i^T \cdot \boldsymbol{\alpha}_j$, $i \in \mathbb{N}_0,$, $j = 1, \ldots, k_d^+$, be given by

$$\begin{aligned}
\boldsymbol{\lambda}_n^{+T} &= \sum_{i=0}^{n} \lambda_i \, \boldsymbol{t}_i^T \, \boldsymbol{K}^{n-i} = \sum_{i=0}^{n} \lambda_i \, \boldsymbol{t}_i^T \, \sum_{j=1}^{k_d^+} \eta_j^{n-i} \boldsymbol{\alpha}_j \, \boldsymbol{\beta}_j \\
&= \sum_{j=1}^{k_d^+} \left(\sum_{i=0}^{n} \lambda_i \, c_{i,j} \, \eta_j^{n-i} \right) \boldsymbol{\beta}_j^T.
\end{aligned} \tag{24}$$

Let us now establish

Lemma 2. *If* $\lim_{i\to+\infty} c_{i,j} = c_j \;\;,\;\; j = 1,\ldots,k_d^+$, *then* $\lim_{i\to+\infty} d_{i,j} = c_{i,j} - c_j = 0$, *and we will have*

$$\lim_{n\to+\infty} \frac{\sum_{i=0}^{n} \lambda_i \, d_{i,j} \, \eta_j^{n-i}}{\sum_{i=0}^{n} \lambda_i \, \eta_j^{n-i}} = 0 \;\;,\;\; j = 1,\ldots,k_d^+ . \tag{25}$$

Proof. With fixed $\overline{n}$, we have

$$\lim_{n\to+\infty} \sum_{i=0}^{\overline{n}} \lambda_i \, d_{i,j} \, \eta_j^{n-i} = \lim_{n\to+\infty} \eta_j^{n-\overline{n}} \sum_{i=0}^{\overline{n}} \lambda_i \, d_{i,j} \, \eta_j^{\overline{n}-i} = 0$$

since that $|\eta_j| < 1 \;,\; j = 1,\ldots,k_d^+$.

To conclude the proof, we point out that, $\forall \varepsilon > 0$, there exists $\overline{n}$ such that, for $i > \overline{n}$, $|d_{i,j}| < \varepsilon$, since $\lim_{i\to+\infty} d_{i,j} = 0$. ■

Corollary 1. *If* $\lim_{i\to+\infty} d_{i,j} = 0$, *then*

$$\lim_{n\to+\infty} \frac{\sum_{i=0}^{n} \lambda_i \, c_{i,j} \, \eta_j^{n-i}}{c_j \sum_{i=0}^{n} \lambda_i \, \eta_j^{n-i}} = 1.$$

So, if $\lim_{i\to+\infty} d_{i,j} = 0, j = 1,\ldots,k_d^+$ we can replace $c_{i,j}$ by $c_j, j = 1,\ldots,k_d^+$, while obtaining the limits for the parameter and quotients.

Considering

$$u_{n,j} = \sum_{i=0}^{n} \lambda_i \, c_j \, \eta_j^{n-i},$$

we then get

$$
(26)\quad \begin{aligned}
u_{n,j} &= \sum_{i=0}^{n} \lambda_i \, c_j \, \eta_j^{n-i} = \sum_{i=0}^{n} (a + b\theta^i) \, c_j \, \eta_j^{n-i} \\
&= c_j \, a \sum_{i=0}^{n} \eta_j^{n-i} + c_j \, b \, \eta_j^n \sum_{i=0}^{n} \left(\frac{\theta}{\eta_j} \right)^i \\
&= c_j \, a \frac{1 - \eta_j^{n+1}}{1 - \eta_j} + c_j \, b \frac{1 - \left(\frac{\theta}{\eta_j} \right)^{n+1}}{1 - \frac{\theta}{\eta_j}} \\
&= c_j \, a \frac{1 - \eta_j^{n+1}}{1 - \eta_j} + c_j \, b \, \frac{\theta^{n+1} - \eta_j^{n+1}}{\theta - \eta_j} \, , \; j = 1, \ldots, k_d^+ ,
\end{aligned}
$$

so, if

- $0 < \theta < 1 \, , \quad \lim_{n \to +\infty} u_{n,j} = \dfrac{c_j \, a}{1 - \eta_j} \, , \; j = 1, \ldots, k_d^+ ;$
- $\theta = 1 \, , \quad \lim_{n \to +\infty} u_{n,j} = \dfrac{c_j \, (a + b)}{1 - \eta_j} \, , \; j = 1, \ldots, k_d^+ ;$
- $\theta > 1 \, , \quad \lim_{n \to +\infty} \dfrac{u_{n,j}}{\theta^{n+1}} = \dfrac{1}{\theta - \eta_j} \, , \; j = 1, \ldots, k_d^+ .$

We point out that, if $\theta = \eta'_j$, then

$$
\lim_{n \to +\infty} \eta_j'^n \sum_{i=0}^{n} \left(\frac{\theta}{\eta'_j} \right)^i = \lim_{n \to +\infty} n \, \eta_j'^n = 0.
$$

We thus establish

Proposition 2. *If* $\lim_{i \to +\infty} d_{i,j} = 0 \, , \; j = 1, \ldots, k_d^+$, *we have*

- $0 < \theta < 1\,,\quad \lim_{n\to+\infty} \boldsymbol{\lambda}_n^{+T} = \sum_{j=1}^{k} \frac{c_j\, a}{1-\eta_j}\, \boldsymbol{\beta}_j^T\,;$

- $\theta = 1\,,\quad \lim_{n\to+\infty} \boldsymbol{\lambda}_n^{+T} = \sum_{j=1}^{k} \frac{c_j\,(a+b)}{1-\eta_j}\, \boldsymbol{\beta}_j^T\,;$

- $\theta > 1\,,\quad \lim_{n\to+\infty} \frac{\boldsymbol{\lambda}_n^{+T}}{\theta^{n+1}} = \sum_{j=1}^{k} \frac{c_j\, b}{\theta-\eta_j}\, \boldsymbol{\beta}_j^T\,.$

So we will have finite limits for the parameters vector if $0 < \theta \leq 1$. Moreover, if $\theta > 1$, the parameters will grow to $+\infty$ "proportionally" to θ^{n+1}.

In both cases, the limits

$$\lim_{n\to+\infty} \frac{\lambda_{n,i}^{+}}{\lambda_{n,i'}^{+}}\,,\ i = 1,\dots,k_d^+\,,\ i' = 1,\dots,k_d^+\,,$$

exist and are different from zero.

The $\lambda_{n,i}^{+}$ and $\lambda_{n,i'}^{+}$ measure the dimensions of the i and i' sub-populations, so the existence of the previous limit represents the stabilization of the relative dimension of the two sub-populations. This stabilization is the main characteristic of the stochastic vortices, so we have.

Proposition 3. *If $\lambda_i = a + b\,\theta^i$, where $\theta > 0$, $i \in \mathbb{N}_0$, and if $\lim_{i\to+\infty} d_{i,j} = 0$, $j = 1,\dots,k_d^+$, there exists a vortice with support in the transient states of the Markov chain.*

4.3. Other vortices

Let us now consider all states of the Markov chain, this is, besides sub-populations associated to the transient states, we will consider the ones associated to the recurrent states.

Note that if a recurrent state is isolated in it's communication class the population associated to it acts as a "black hole".

We then have

$$\boldsymbol{P}^n = \sum_{j=1}^{n} \eta_j^n\, \boldsymbol{\alpha}_j\, \boldsymbol{\beta}_j^T \tag{27}$$

so, reasoning as before, we see that after n periods of time, the parameters vector is given by

$$\boldsymbol{\lambda}_n^{++T} = \sum_{i=0}^{n} \lambda_i \, \boldsymbol{p}_i^T \, \boldsymbol{P}^{n-i} = \sum_{j=1}^{k} \left(\sum_{i=0}^{n} \lambda_i \, m_{i,j} \, \eta_j^{n-i} \right) \boldsymbol{\beta}_j^T \tag{28}$$

with $m_{i,j} = \boldsymbol{p}_i^T \cdot \boldsymbol{\alpha}_j$, $i \in \mathbb{N}_0$, $j = 1, \ldots, k$.

If $\lim_{i \to +\infty} m_{i,j} - m_j = 0$, $j = 1, \ldots, k$, we can, as before, replace the $m_{i,j}$, $j = 1, \ldots, k$, by the m_j, lightening the writing.

It will be useful to group the eigenvalues of $\boldsymbol{P}$ in three groups:

1. The first group will have the η_j in which $|\eta_j| < 1$. $\mathcal{D}_1$ will be the set of these eigenvalues indexes. So, $1, \ldots, k_d^+ \in \mathcal{D}_1$, but $\mathcal{D}_1$ can contain other eigenvalues;
2. The second group will contain the roots of 1 different from 1. $\mathcal{D}_2$ will be the corresponding set of indexes. Observe that $\mathcal{D}_2$ can be empty;
3. Lastly, $\mathcal{D}_3$ will be the set of indexes of eigenvalues equal to 1 so we have $\#(\mathcal{D}_3) = w - d$, since we have an eigenvalue equal to 1 for each recurrent communication class.

Let us consider the case $0 < \theta < 1$. We then have

$$\begin{aligned} j \in \mathcal{D}_1 \ : u_{n,j} &= \sum_{i=0}^{n} \lambda_i \, m_j \, \eta_j^{n-i} \\ &= \sum_{i=0}^{n} a \, m_j \, \eta_j^{n-i} + \sum_{i=0}^{n} b \, \theta^i \, m_j \, \eta_j^{n-i} \\ &= a \, m_j \, \frac{1 - \eta_j^{n+1}}{1 - \eta_j} + b \, m_j \, \frac{\eta_j^{n+1} - \theta^{n+1}}{\eta_j - \theta} \xrightarrow[n \to +\infty]{} \frac{a \, m_j}{1 - \eta_j} \, ; \end{aligned} \tag{29}$$

$$\begin{aligned} j \in \mathcal{D}_2 \ : u_{n,j} \ &= \ \sum_{i=0}^{n} \lambda_i \, m_j \, \eta_j^{n-i} \\ &= \ a \, m_j \, \frac{1 - \eta_j^{n+1}}{1 - \eta_j} + b \, m_j \, \frac{\eta_j^{n+1} - \theta^{n+1}}{\eta_j - \theta} \, ; \end{aligned} \tag{30}$$

$$j \in \mathcal{D}_3 \ : \frac{1}{n}\, u_{n,j} \ = \ \frac{1}{n} \sum_{i=0}^{n} \lambda_i \, m_j \, \eta_j^{n-i} \tag{31}$$

$$= \ a\, m_j \ + \frac{b}{n}\, m_j \, \frac{1-\theta^{n+1}}{1-\theta} \ \xrightarrow[n\to+\infty]{} a\, m_j .$$

If $j \in \mathcal{D}_2$, we have,

$$\lim_{n\to+\infty} \sum_{i=0}^{n} \lambda_i \, m_j \, \eta_j^{n-i} - \left(a\, m_j \, \frac{1-\eta_j^{n+1}}{1-\eta_j} + b\, m_j \, \frac{\eta_j^{n+1}}{\eta_j - \theta} \right) = 0. \tag{32}$$

Note that, with z the least common multiple of the root 1 indexes, we will have

$$a\, m_j \, \frac{1-\eta_j^{lz+h}}{1-\eta_j} + b\, m_j \, \frac{\eta_j^{lz+h}}{\eta_j - \theta} \ = \ a\, m_j \, \frac{1-\eta_j^{h}}{1-\eta_j} + b\, m_j \, \frac{\eta_j^{h}}{\eta_j - \theta}, \tag{33}$$

$$j \in \mathcal{D}_2 \ , \ h = 0, \ldots, z-1,$$

and so, $u_{n,j}$ varies ciclically with period z.

So, the only

$$u_{n,j} = \sum_{i=0}^{n} \lambda_i \, m_j \, \eta_j^{n-i},$$

which are divergent, are the ones corresponding to $j \in \mathcal{D}_3$.

We have

$$\lim_{n\to+\infty} \frac{1}{n} \, \boldsymbol{\lambda}_n^{++T} - \sum_{j\in\mathcal{D}_3} u_{n,j} \, \boldsymbol{\beta}_j^T = 0. \tag{34}$$

As the non-null components of $\boldsymbol{\beta_j}$, with $j > k_d^+$ correspond to recurrent states we see that besides the vortices based on the transient states we also have a stochastic vortice based on the recurrent states.

We thus have

$$\lim_{n\to+\infty} \frac{\lambda_{n,i}^+}{\lambda_{n,i'}^+} = 0 \ , \ i \le k_d^+ \le i' \tag{35}$$

as well as

$$\lim_{n\to+\infty} \frac{\lambda_{n,i}^+}{\lambda_{n,i'}^+} = l_{i,i'} > 0 \ , \ k_d^+ \le i \le i'. \tag{36}$$

The vortices are clearly and easily distinguished since the limit parameters of the states in the first one are finite and for those in the second ones are infinite.

For the case $\theta = 1$, we get very similar results, since that

$$\begin{aligned} j \in \mathcal{D}_1 \ : \ u_{n,j} \ &= \ a\, m_j \sum_{i=0}^{n} \eta_j^{n-i} + b\, m_j \sum_{i=0}^{n} \eta_j^{n-i} \\ &\xrightarrow[n\to+\infty]{} \ \frac{(a+b)\, m_j}{1-\eta_j}, \end{aligned} \tag{37}$$

$$j \in \mathcal{D}_2 \ : \ u_{n,j} = (a+b)\, m_j \, \frac{1-\eta_j^{n+1}}{1-\eta_j}, \tag{38}$$

$$j \in \mathcal{D}_3 \ \ : \frac{1}{n}\, u_{n,j} = (a+b) m_j. \tag{39}$$

If $j \in \mathcal{D}_2$, $u_{n,j}$ exibits a cyclic behaviour.

So we establish

Proposition 4. *When $\lambda_i = a + b\,\theta^i$, with $0 < \theta \leq 1$ and $\lim_{i\to\infty} m_{i,j} - m_j = 0$, $j = 1, \ldots, k$, two vortices are established, one in the transient states and another one in the recurrent states with parameters that grow to infinity proportionally to the number of time periods.*

When $\theta > 1$, we have

$$(40)\qquad \begin{aligned} j \in \mathcal{D}_1 \,:\, \frac{u_{n,j}}{\theta^{n+1}} &= \frac{a\,m_j}{\theta^{n+1}}\,\frac{1-\eta_j^{n+1}}{1-\eta_j} + \frac{b\,m_j}{\theta^{n+1}}\,\frac{\theta^{n+1}-\eta_j^{n+1}}{\theta-\eta_j} \\ &\xrightarrow[n\to+\infty]{} \frac{b\,m_j}{\theta-\eta_j}, \end{aligned}$$

$$(41)\qquad \begin{aligned} j \in \mathcal{D}_2 \,:\, \frac{u_{n,j}}{\theta^{n+1}} &= \frac{a\,m_j}{\theta^{n+1}}\,\frac{1-\eta_j^{n+1}}{1-\eta_j} + \frac{b\,m_j}{\theta^{n+1}}\,\frac{\theta^{n+1}-\eta_j^{n+1}}{\theta-\eta_j} \\ &\xrightarrow[n\to+\infty]{} \frac{b\,m_j}{\theta-\eta_j}, \end{aligned}$$

$$(42)\qquad \begin{aligned} j \in \mathcal{D}_3 \,:\, \frac{u_{n,j}}{\theta^{n+1}} &= \frac{n\,a\,m_j}{\theta^{n+1}}\,\frac{b\,m_j}{\theta^{n+1}}\,\frac{\theta^{n+1}-1}{\theta-1} \\ &\xrightarrow[n\to+\infty]{} \frac{b\,m_j}{\theta-1} \end{aligned}$$

so

$$(43)\qquad \lim_{n\to+\infty} \frac{\boldsymbol{\lambda}_n^{++T}}{\theta^{n+1}} = b\sum_{j=1}^{k} \frac{m_j}{\theta-\eta_j}\,\boldsymbol{\beta}_j^T$$

and then we get.

Proposition 5. *When $\lambda_i = a+b\theta^i$, with $\theta > 1$ and $\lim\limits_{i\to+\infty} m_{i,j} - m_j = 0, j = 1, \ldots, k$, there is a stochastic vortice based on all states. The state parameters grow to $+\infty$ proportionally to θ^n.*

4.4. Validation

In this point we will assume that the expression for the intensity of new elements arrival is already adjusted and, so, we have

$$\tilde{\lambda}_i = \tilde{a} + \tilde{b}\,\tilde{\theta}^i. \tag{44}$$

We can then use (24) and (28) to obtain adjusted parameters $\tilde{\boldsymbol{\lambda}}_n^{+}$ e $\tilde{\boldsymbol{\lambda}}_n^{++}$.

Let $\vec{N}_n = (N_{n,1}, \ldots, N_{n,l})$ be the vector of the number of elements in each of the vortices already considered and $\tilde{\boldsymbol{\lambda}}_n^{\circ} = \left(\tilde{\lambda}_{n,1}^{\circ}, \ldots, \tilde{\lambda}_{n,l}^{\circ}\right)$ the vector of adjusted parameters.

When the vortice is established and the entrances expression is (44), then

$$X^2 = \sum_{i=1}^{l} \frac{\left(N_{n,i} - \tilde{\lambda}_{n,i}^{\circ}\right)^2}{\tilde{\lambda}_{n,i}^{\circ}}$$

is "almost" a χ_l^2.

In this way we can validate the existence of vortices within our population structure.

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