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CERTAIN NEW M-MATRICES AND THEIR PROPERTIES WITH APPLICATIONS

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Abstract

The M_n-matrix was defined by Mohan [21] who has shown a method of constructing $(1, -1)$-matrices and studied some of their properties. The $(1, -1)$-matrices were constructed and studied by Cohn [6], Ehrlich [9], Ehrlich and Zeller [10], and Wang [34]. But in this paper, while giving some resemblances of this matrix with a Hadamard matrix, and by naming it as an M-matrix, we show how to construct partially balanced incomplete block designs and some regular graphs by it. Two types of these M-matrices have been considered. Also we will make a mention of certain applications of these M-matrices in signal and communication processing, and network systems and end with some open problems.

Keywords: M-matrices, non-orthogonality, orthogonal number, Hadamard matrix, partially balanced incomplete block (PBIB) design, regular graph.

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1. Introduction

The $(1,-1)$-matrices have been investigated by many authors who also studied their properties. For an $n \times n$ $(1,-1)$-matrix having the largest possible determinant (Hadamard's maximum determinant problem) called as a Hadamard matrix. Ehrlich [9], and Ehrlich and Zeller [10] studied these binary matrices, and Seifer [28] and Wang [34] investigated their properties, and Kahn, Kolmos and Szemeredi [18] studied the probability of these matrices to be singular. Again for the Hadamard matrix, refer to Geramita and Seberry [12], Seberry and Yamada [27]. These matrices have wide applications in the construction of codes, designs and graphs [12, 27], and of sequences that were used in signal processing (Fan and Darnell [11]). As the matrices discussed here are non-orthogonal, this property of non-orthogonality is well used in image reconstruction in image analysis, for details refer to Teague [32]. The concepts of orthogonal matrices, quasi-orthogonal matrices by Jafarkhani [15], and non-orthogonal matrices are used in the construction of space-time block codes by Chang, Hua, Xia and Sudler [5]. A 4×4 quasi-orthogonal code matrix is that the columns are grouped such that, a pair of columns of different groups are orthogonal but the columns in the same group are not orthogonal, as per the definition of Jafarkhani [15]. The M_n-matrices were defined by Mohan [21] as the matrix constructed from the formula $M_n = (d_i \odot d_h d_j) \bmod n$ by suitably defining d_i, d_h, d_j and $\odot$. By using this M_n-matrix pattern Vasic and Milenkovic [33, pp. 1165] gave a method of construction of low-density parity check codes. Kageyama and Mohan [16, 17] used these M_n-matrices for the construction of μ-resolvable and affine μ-resolvable balanced incomplete block (BIB) and partially BIB (PBIB) designs. And in [21] two types of $(1,-1)$-matrices were constructed and some of their properties were studied. Now we consider those two types of matrices, i.e., when (i) n is a prime, and (ii) $n+1$ is a prime, and study them further.

The Hadamard matrix (H-matrix) of order n is a square $(1,-1)$-matrix, which satisfies $HH' = nI_n$, where I_n is the identity matrix of order n.

Now while making the comparative study of the present matrix and the H-matrix we will show how best we can make use of them, to construct some symmetric PBIB (SPBIB) designs and some regular bipartite graphs.

In the literature, many people worked on PBIB designs, for example, refer to Bose [3], Goethals and Seidel [13], Liu [19], Mohan [20], Raghavarao [24], and especially Ramanujacharyulu [26], Shrikhande [29] and Sprott [31] studied the designs with three and more associate classes. For many other properties of BIB, SBIB and PBIB designs, Raghavarao [25] is a good reference.

Definition 1.1. A BIB design is an arrangement of v symbols in b blocks of size k each such that every symbol occurs in r blocks and each pair of symbols occurs in λ blocks.

The parameters of a BIB design satisfy $vr = bk, \lambda(v-1) = r(k-1)$, $b \geq v$. A BIB design is said to be symmetric if $v = b$, and consequently $r = k$.

Bose and Nair [4] defined a PBIB design in which some pairs of symbols do not occur in a constant number of times as in a BIB design. For defining a PBIB design we require the concept of an association scheme with m associate classes.

Definition 1.2. Given v symbols 1, 2, ..., v, a relation satisfying the following conditions is said to form an association scheme with m associate classes:

1. Any two symbols are either 1st, 2nd, ..., or mth associate. The relation of association is symmetric, that is, if the symbol α is the ith associate of the symbol β, then β is the ith associate of α.
2. Each symbol α has n_i ith associates, the number n_i being independent of α.
3. If any two symbols α and β are the ith associates, then the number of symbols that are the jth associate of α and the ℓth associate of β is $p^i_{j\ell}$ and is independent of the pair of α and β.

The numbers $v, n_i, p^i_{j\ell}, i, j, \ell = 1, 2, ..., m$, are called parameters of the association scheme.

Now a PBIB design is defined as follows.

Definition 1.3. Given an association scheme with m associate classes for the v symbols. Then an m-associate PBIB design with parameters $v, b, r, k, \lambda_i, i = 1, 2, ..., m$, is such that

1. the v symbols are arranged in b blocks of size $k(< v)$ each,
2. every symbol occurs at most once in a block,
3. every symbol occurs exactly in r blocks,
4. if any two symbols α and β are the ith associates, then they occur together in λ_i blocks, the number λ_i being independent of the particular pair of ith associates α and β.

The numbers $v, b, r, k, \lambda_i, i = 1, 2, ..., m$, are called parameters of the design. An m-associate PBIB design satisfies the following conditions:

$$vr = bk, \ \sum_{i=1}^{m} n_i = v - 1, \ \sum_{i=1}^{m} n_i\lambda_i = r(k-1), \ \sum_{\ell=1}^{m} p^i_{j\ell} = n_j - \delta_{ij},$$

where δ_{ij} is the Kronecker delta, taking the value 1 when $i = j$, otherwise 0. When $b = v$, the PBIB design is said to be symmetric.

Definition 1.4. Let $G = (V, E)$, where V is a set of vertices and E is a set of edges, joining any two vertices, be called a graph. The number of edges passing through a vertex is called its valence. In a graph if the valence of each vertex has the same constant then it is said to be regular.

Definition 1.5. If the vertex set V has two complementary sets V_1, V_2 ($V = V_1 \cup V_2$) such that each edge of the graph has one end in V_1 and the other end in V_2, then the graph is said to be bipartite.

Mohan [21], defined an M_n-matrix (a_{ij}) with $a_{ij} = d_i \odot d_h d_j$ mod n, by suitably defining d_i, d_h, d_j and $\odot$, which will be given below.

Definition 1.6. When n is a prime, an M_n-matrix (a_{ij}) is given as a matrix obtained from $a_{ij} = 1 + (i-1)(j-1)$ mod n for $i, j = 1, 2, ..., n$.

This is an $n \times n$ symmetric matrix and has been used in the construction of graphs (refer to [21]) called M_n-graphs, which are also defined as follows.

Definition 1.7. Given an M_n-matrix and C_k be its columns, which are numbered as $1, 2, ..., n$ and a_{ij} be the elements of it, and let $V_1 = \{C_k\}, V_2 = \{a_{ij}\}$ be the two sets of the vertices. Now there is an edge a_{ijk} if a_{ij} is in C_k. Then the graph $(V_1, V_2, \{a_{ijk}\})$, $i, j, k = 1, 2, ..., n$, is called an M_n-graph.

As an extension of these concepts, we define an M-matrix as follows.

Definition 1.8. When n is a prime, consider the M_n-matrix of order n by $M_n = (a_{ij})$ with $a_{ij} = 1 + (i-1)(j-1) \bmod n$ for $i, j = 1, 2, ..., n$. In the resulting matrix retain 1 as it is and substitute -1's for odd numbers and $+1$'s for even numbers. (We can substitute 1 for odd numbers and -1 for even numbers, in that case the change of sign occurs in its determinant.) Let the resulting matrix M be called an M-matrix of Type I. This is a symmetric matrix of order n.

Definition 1.9. When $n+1$ is a prime, the M-matrix of Type II is essentially obtained through a matrix (a_{ij}) with $a_{ij} = ij \bmod n+1$ for $i, j = 1, 2, ..., n$. In this matrix since each row or column has n elements where n is even, 1 to n elements do come in all the columns and rows, and each element comes once in each row and each column. In the resulting matrix substitute 1 for even numbers and -1 for odd numbers and also for 1, (or 1 for odd numbers keeping the 1 in the matrix as 1 itself and -1 for even numbers). Then this resulting matrix M is called an M-matrix of Type II. Each row (column) consists of an even number of $+1$'s and -1's numbering to $n/2$. This is a symmetric matrix of order n.

We discuss these two types of matrices, while giving examples for their constructions and applications of these matrices in the later sections.

Take an M-matrix of Type I and the Hadamard matrix (H-matrix). Then we see some resemblances and differences between them.

1. In both the M-matrix and the H-matrix all the elements are either $+1$ or -1.
2. In both the matrices the first row and the first column could consist of $+1$ only.
3. In the H-matrix the row sums (except the first row) are all zeros and in the M-matrix each of the row sums (except the first row) is one. When in the H-matrix the row sums are some constant, it is called a regular H-matrix, but in the case of the M-matrix it is 1.

4. The H-matrix may exist for $n = 2$ or $n \equiv 0 \bmod 4$, but the M-matrix exists for any prime n.
5. The H-matrices are used in the construction of codes, graphs and designs, refer to Seberry and Yamada [27]. In this paper it will be shown that the M-matrix can be used in the construction of certain designs and graphs.
6. If H_n and H_m are two H-matrices then their Kronecker product $H_n \otimes H_m$ is also an H-matrix, but it is not so in the case of the M-matrices of Type I.
7. From an H-matrix a symmetric BIB design can be constructed and it will be shown that from an M-matrix a symmetric PBIB design can be constructed.
8. The H-matrices are orthogonal matrices and these M-matrices are non-orthogonal.

There are wide applications of matrices/BIB designs and their associated graphs in information and communication systems and in network systems. For further details refer to Aupperle and Meyer [1], Bhuyan and Agarwal [2], Colbourn [7], Colbourn, Dinitz and Stinson [8], Hawkes [14], Mohan and Kulkarni [22], Nguyen, Vo and Lee [23], and Skillicorn [30].

2. Statements

As these two types of M-matrices are non-orthogonal, the orthogonal numbers are defined for them as follows.

Definition 2.1. The orthogonal number of a given M-matrix with entries ± 1 is the sum of the products of the corresponding numbers in two given rows of the matrix (called an inner product of the rows). Consider any two rows $R_\ell = (r_1, r_2, ..., r_n)$ and $R_m = (s_1, s_2, ..., s_n)$ and then the orthogonal number denoted by g is given by $g = (R_\ell \cdot R_m) = \sum_{i=1}^{n} r_i s_i$.

The M-matrix of Type I. The elements of the principal diagonal of this matrix have a special property, i.e., if D_n is the principal diagonal of the M-matrix of order n, then we get its elements by $x^2 - 2x + 2 \bmod n$ for some positive integer x. Thus, for example, $D_3 = (1\ 2\ 2), D_5 = (1\ 2\ 5\ 5\ 2)$ and $D_7 = (1\ 2\ 5\ 3\ 3\ 5\ 2)$, leaving the first element the remaining $(n-1)/2$ elements just repeat in the reverse order. It is a modular property.

Proposition 2.1. *For a given M-matrix of Type I of order n, in each of its rows and columns, the number of $+1$'s is $(n+1)/2$ and the number of -1's is $(n-1)/2$.*

Proof. Since n is a prime and the first row and the first column have unities, in each row (column) the first element is 1 and among the remaining $n-1$ places of each row (column) there occur each of 2, 3, ..., n exactly once. And among these $n-1$ elements half of them are even numbers and the other half of them are odd numbers. Consequently as we replace even numbers by -1's and odd numbers by -1's, it holds that $(n+1)/2$ elements are $+1$'s (as we add the first element 1 also), and $(n-1)/2$ elements are -1's. ■

Proposition 2.2. *In an M-matrix of Type I of order n (prime), the orthogonal number between any two distinct rows is given by $4k+2-n$, where k is the number of unities in the selected set and $0 \leq k \leq (n-1)/2$.*

Proof. Let R_i and R_j be the two given rows in the M-matrix. Their inner product $(R_i \cdot R_j)$ has to be calculated for $i \neq j$ and $i, j \geq 2$. By elementary transformations we can make the row R_i with the first $(n+1)/2$ elements as $+1$'s and the next $(n-1)/2$ elements as -1's. The orthogonal numbers of the matrix remain invariant by such elementary transformations. Now consider the other row R_j which has n elements. The first element is $+1$ and the remaining $n-1$ elements divide them into two halves with $(n-1)/2$ elements each. They can be depicted as follows:

$$R_i = \big((1)(11...11...1)(-1-1...-1-1)\big) \text{ of size } n,$$

$$R_j = \big((1)(1-1...-1-1...11)(11...1-11-1...11)\big) \text{ of size } n,$$

where in R_j let $+1$'s be k in number and hence -1's are $(n-1)/2-k$ in the first half, and in the other half $+1$'s are $(n-1)/2-k$ and -1's are k in number.

Now we evaluate the formula for the orthogonal number. Let k be the number of $+1$'s, excluding the first element $+1$. In the two given rows, (i) the first elements of the two rows 1 coincides with 1, in the first two sets of the given rows, (ii) the k number of $+1$'s in the row R_j correspond with the k number of $+1$'s in R_i. Now in the second sets of the given rows, (iii) the

$(n-1)/2-k$ number of $+1$'s in R_j correspond with the same number of -1's in R_i, and (iv) the k number of -1's in R_j correspond with the same number of -1's in R_i. Hence $g = (R_i \cdot R_j) = 1 \times 1 + k \times 1 \times 1 + ((n-1)/2-k) \times 1 \times (-1) + ((n-1)/2-k) \times 1 \times (-1) + k \times (-1) \times (-1) = 4k+2-n$. If we see for different values of k, when $k=0$ then $g = 2-n$, when $k=1$ then $g=6-n$, and so on. And when $k=(n-1)/2$ then $g=n$. Thus there should be $(n-1)/2+1=(n+1)/2$ orthogonal numbers in an M-matrix. ■

Result 2.1. It holds that $(R_1 \cdot R_j) = 1, (R_i \cdot R_i) = n$ and $(R_{2+i} \cdot R_{n-i}) = 2-n$ for $i = 0, 1, ..., (n-3)/2$.

Proof. Since R_1 consists of all $+1$'s, the inner product with any row consequently is 1, i.e., $(R_1 \cdot R_j) = 1$. This situation has not been counted in the formula $4k+2-n$. And for $(R_i \cdot R_i) = n$, the inner product of any row with itself, gives out n only. Besides when $k=(n-1)/2$, we also get $(R_i \cdot R_i) = n$. They are called trivial orthogonal numbers. The third equality is obvious from the proof of Proposition 2.2. ■

Note 2.1. There are some orthogonal numbers which occur in pairs only. They are called orthogonal pairs, which will be discussed in later sections.

Note 2.2. There are certain exceptions depending on the coincidences. Sometimes the theoretically obtained g_i's may not exist in numerical problems. For example, when $n = 11$, orthogonal numbers should be $-9, -5, -1, 3, 7, 11$, but in the numerical problem we get these orthogonal numbers as $-9, -1, 3$. The orthogonal numbers -5 and 7 do not exist. Of course, 1 and 11 will be there, which are trivial orthogonal numbers, when $(R_1 \cdot R_j) = 1, (R_i \cdot R_i) = 11$ are considered. Besides the trivial orthogonal number this 1 may again exist as an orthogonal number in the list, like in the case of $n=5$ and $k=1, g=1$.

Proposition 2.3. *In an M-matrix the sum of the orthogonal numbers is $(n+1)/2$. If 1 of $(R_1 \cdot R_j) = 1$ is also added, then it will be $(n+3)/2$.*

Proof. Let g_i's be orthogonal numbers in the given M-matrix. Then

$$\sum_{i=1}^{(n+1)/2} g_i = \sum_{k=0}^{(n-1)/2} (4k+2-n) = (n+1)/2.$$

The orthogonal numbers are in an arithmetic progression with arithmetic mean 4. The orthogonal number obtained from the first row R_1 with any other row is not considered in this. Note that in considering the sum, all possible orthogonal numbers by the formula are taken. Hence the proof. ∎

Note 2.3. If M_1 and M_2 are two given M-matrices of order m and n respectively, then the Kronecker product $M_1 \otimes M_2$ is not an M-matrix. Since the M-matrix of order n exists for n being a prime, when two matrices of order m, n are taken, their product $m \times n$ ceases to be a prime and hence the resulting matrix is not an M-matrix of Type I.

It is not possible to derive any other M-matrix from either the given M-matrix or from the given set of M-matrices by any other means, because the order of the resulting matrix ceases to be a prime number.

Result 2.2. For the given M-matrix, the value of $|M|$ is -4 if $n = 3$, and 0 if $n \geq 5$.

Proof. For $n = 3$ it is trivial. When $n \geq 5$, the matrix consists of pairs of rows with just opposite signs, and by leaving the first row, consider $R_2 + R_n$ which will be a vector (2 0 0...0), and $R_3 + R_{n-1}$ will also be the same vector (2 0 0...0). In any matrix, if any two rows are equal, then its determinant is zero, i.e., $|M| = 0$. ∎

Now we consider MM', which is an $n \times n$ symmetric matrix. By the properties of orthogonal numbers discussed above, we have $(R_1 \cdot R_j) = 1, (R_i \cdot R_i) = n$ and $(R_{2+i} \cdot R_{n-i}) = 2 - n$. The elements in the principal diagonal are n, the rest of the $n - 1$ elements in the first row and the first column are unities. By considering all the other elements in this matrix, they are some orthogonal numbers, which cannot be fixed theoretically. The number of orthogonal numbers is $(n+1)/2$, and in a row of n elements $(n-1)$ orthogonal numbers have to be listed and hence some orthogonal numbers repeat, when all the possible orthogonal numbers exist. Otherwise, if some do not exist, then the available numbers will repeat in the row.

Proposition 2.4. *When n is a prime, the existence of an M-matrix of Type I of order n implies the existence of an SPBIB design with parameters $v = b = n - 1, r = k = (n - 1)/2$ and λ_i's vary from 0 to $(n - 3)/2$.*

Furthermore, n_i and $p^i_{j\ell}$ cannot be said theoretically, but can be evaluated in the numerical problem.

Proof. Given an M-matrix of order n, where n is a prime. Leaving the first row and the first column we get a matrix D of order $n-1$. Let the matrix be of the form

$$M = \begin{bmatrix} 1 & \mathbf{1}'_{n-1} \\ \mathbf{1}_{n-1} & D \end{bmatrix},$$

where $\mathbf{1}_c$ is the column vector of size c with all 1's. Let D_1 be the matrix obtained from D, by replacing -1's by 0's. Then it follows that D_1 is the incidence matrix of a PBIB design with parameters $v = b = n-1$, $r = k = (n-1)/2$. Regarding λ_i values in the given two rows R_i and R_j if they have exactly opposite signs then $\lambda_i = 0$. In the given two rows R_i and R_j if they have x coincidences then $\lambda_i = x$. Then in each row the number of $+1$'s are $(n-1)/2$ and the number of zeros is also $(n-1)/2$. All $(n-1)/2$ number of $+1$'s cannot coincide with $(n-1)/2$ number of $+1$'s in another row, if so the two rows are identical, which is not possible. Hence at the maximum there may be $(n-1)/2 - 1 = (n-3)/2$ coincidences. Hence λ_i values vary from 0 to $(n-3)/2$. Consequently, n_i's and $p^i_{j\ell}$'s can be evaluated. ■

Proposition 2.5. *The existence of an M-matrix of Type I of order n implies the existence of a regular bipartite graph with parameters $|V| = 2n$ $(|V_1| = |V_2| = n), |E| = 2n$ and valence $(n-1)/2$.*

Procedure for the construction. Given an M-matrix, treating -1's as 0's and by considering the resulting matrix as an adjacency matrix of a graph with elements as one set and the columns as another set, if an element a_{ij} is in column C_k then (a_{ij}, C_k) will be an edge, otherwise not. We have in the graph obtained $|V| = 2n$ $(|V_1| = |V_2| = n)$ and its valence is $(n-1)/2$, since each column and each row has $(n-1)/2$ unities. Thus a regular bipartite graph is obtained. ■

Example 2.1. Take $n = 5$. In $M_n = (a_{ij})$ with $a_{ij} = 1 + (i-1)(j-1)$ mod n for $i, j = 1, 2, 3, 4, 5,$ we have

$$M_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \\ 1 & 4 & 2 & 5 & 3 \\ 1 & 5 & 4 & 3 & 2 \end{bmatrix},$$

by substituting for even numbers 1 and for odd numbers -1 and keeping 1 as it is, we get

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

Leaving the first row and the first column of unities we have

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

Now consider all -1's as zeros then we get the matrix N as

$$N = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

which is considered as an incidence matrix of a design. Then this yields a 2-associate L_2 PBIB design with parameters $v = b = 4, r = k = 2, \lambda_1 = 1, \lambda_2 = 0$ and $p^1_{11} = 0, p^1_{12} = 0, p^1_{21} = 0, p^1_{22} = 2, p^2_{11} = 0, p^2_{12} = 1, p^2_{21} = 1, p^2_{22} = 0, n_1 = 1, n_2 = 2$. If we consider N also as the adjacency matrix of a graph, then we get

	5	6	7	8
1	1	0	1	0
2	0	0	1	1
3	1	1	0	0
4	0	1	0	1

which yields a regular bipartite graph with $V_1 = \{1, 2, 3, 4\}$ (elements) and $V_2 = \{5, 6, 7, 8\}$ (columns) having parameters $(4, 4; 8)$.

These types of graphs form a new family of graphs, which are highly usable in routing problems of salesmen, transportation or communication problems. Suppose there are n nodes (points) say $(a_1, a_2, ..., a_n)$. Consider any point a_1 as a source node and then all the remaining $n - 1$ points are destination nodes. By starting from the source node the message has to be passed on to any other node with the condition that it should not be passed on to the nodes of the same set twice continuously at any time, that is, alternatively the message should be passed on to the nodes of different sets V_1 and V_2 with an exception to the source node at the final stage if needed. Iteratively this goes on in succession and reaches back to the first source node. All the points should be covered. And for each node have two ways of going in and coming out only. Hence each node has only two edge connections. These types of graphs are specifically useful, in passing information very confidentially and routing the messenger to go without informing the persons in the vicinity, but covering all the members (nodes) and reporting back to the source node. The maximum number of lines required is n. But the minimum number of lines required varies as per the task. And how many ways are there for having this function to be performed with the minimum number of hops can be evaluated. This is a richly connected graph to act as a network. For further details refer to Bhuyan and Agarwal [2], Hawkes [14], Mohan and Kulkarni [22], and Nguyen, Vo and Lee [23].

Example 2.2. Especially when $n = 7$ we can construct a 3-associate SPBIB design. By taking $a_{ij} = 1 + (i-1)(j-1) \bmod n$ for $i, j = 1, 2, 3, 4, 5, 6, 7$, the M_n-matrix is obtained as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 7 & 2 & 4 & 6 \\ 1 & 4 & 7 & 3 & 6 & 2 & 5 \\ 1 & 5 & 2 & 6 & 3 & 7 & 4 \\ 1 & 6 & 4 & 2 & 7 & 5 & 3 \\ 1 & 7 & 6 & 5 & 4 & 3 & 2 \end{bmatrix}.$$

Now by substituting for even numbers 1 and for odd numbers -1 and keeping 1 as it is, we get the M-matrix of Type I as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

By leaving the first row and the first column of unities and treating -1's as 0's in the resulting matrix we get an incidence matrix of an SPBIB design, whose incidence matrix is given by

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

This gives a solution of the concerned PBIB design, which will be dealt with in Example 2.5 later.

Example 2.3. When $n = 11$ we can get the M_n-matrix as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 \\ 1 & 4 & 7 & 10 & 2 & 5 & 8 & 11 & 3 & 6 & 9 \\ 1 & 5 & 9 & 2 & 6 & 10 & 3 & 7 & 11 & 4 & 8 \\ 1 & 6 & 11 & 5 & 10 & 4 & 9 & 3 & 8 & 2 & 7 \\ 1 & 7 & 2 & 8 & 3 & 9 & 4 & 10 & 5 & 11 & 6 \\ 1 & 8 & 4 & 11 & 7 & 3 & 10 & 6 & 2 & 9 & 5 \\ 1 & 9 & 6 & 3 & 11 & 8 & 5 & 2 & 10 & 7 & 4 \\ 1 & 10 & 8 & 6 & 4 & 2 & 11 & 9 & 7 & 5 & 3 \\ 1 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \end{bmatrix},$$

which yields an M-matrix as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

Let $(R_i \cdot R_j)$ be the inner product of rows R_i and R_j. Then it holds that $(R_1 \cdot R_j) = 1$ for $j \geq 2$ and $(R_i \cdot R_i) = 11$, which are orthogonal numbers, furthermore, $(R_2 \cdot R_3) = (R_2 \cdot R_7) = (R_2 \cdot R_8) = (R_2 \cdot R_9) = (R_3 \cdot R_4) = (R_3 \cdot R_5) = (R_3 \cdot R_6) = (R_4 \cdot R_7) = (R_4 \cdot R_8) = (R_4 \cdot R_{11}) = (R_5 \cdot R_9) = (R_5 \cdot R_{11}) = (R_6 \cdot R_8) = (R_6 \cdot R_9) = (R_6 \cdot R_{11}) = (R_7 \cdot R_{10}) = (R_8 \cdot R_{10}) = (R_9 \cdot R_{10}) = (R_{10} \cdot R_{11}) = -1$, $(R_2 \cdot R_4) = (R_2 \cdot R_5) = (R_2 \cdot R_6) = (R_2 \cdot R_{10}) = (R_3 \cdot R_7) = (R_3 \cdot R_8) = (R_3 \cdot R_9) = (R_3 \cdot R_{11}) = (R_4 \cdot R_5) = (R_4 \cdot R_6) = (R_4 \cdot R_{10}) = (R_5 \cdot R_6) = (R_5 \cdot R_{10}) = (R_6 \cdot R_{10}) = (R_7 \cdot R_8) = (R_7 \cdot R_9) = (R_7 \cdot R_{11}) = (R_8 \cdot R_9) = (R_8 \cdot R_{11}) = (R_9 \cdot R_{11}) = 3$, $(R_2 \cdot R_{11}) = (R_3 \cdot R_{10}) = (R_4 \cdot R_9) = (R_5 \cdot R_8) = (R_6 \cdot R_7) = -9$. Here the orthogonal numbers are $-1, 3, -9$ only. But as per our formula $g = 4k + 2 - n$, we get the orthogonal numbers as $-9, -5, -1, +3, +7, +11$. Because when the first set with $+1$ is selected in R_i and in R_j there is no row in it having only one $+1$, i.e., $k = 1$ does not exist, and hence $k = 4$ also does not exist. Consequently the pair of orthogonal numbers -5 and 11 are missing. The orthogonal number $(R_1 \cdot R_j) = 1$ exists.

By leaving the first row and the first column of unities, and by taking -1's as 0's the incidence matrix of an SPBIB design is obtained as

	11	12	13	14	15	16	17	18	19	20
1	1	0	1	0	1	0	1	0	1	0
2	0	0	0	0	0	1	1	1	1	1
3	1	0	1	1	0	1	0	0	1	0
4	0	0	1	1	1	0	0	0	1	1
5	1	0	0	1	1	0	0	1	1	0
6	0	1	1	0	0	1	1	0	0	1
7	1	1	0	0	0	1	1	1	0	0
8	0	1	0	0	1	0	1	1	0	1
9	1	1	1	1	1	0	0	0	0	0
10	0	1	0	1	0	1	0	1	0	1

which yields a 3-associate SPBIB design with parameters $v = b = 10, r = k = 5$, $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 3, n_1 = 1, n_2 = 4, n_3 = 4$ and for $i, j = 1, 2, 3$,

$$(p^1_{ij}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}, \quad (p^2_{ij}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}, \quad (p^3_{ij}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

By considering the above incidence matrix as an adjacency matrix, a graph called an M-graph can be obtained as the regular bipartite graph with $V_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (due to the row-numbering) and $V_2 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$ (due to the column-numbering).

M-matrix of Type II. Here we consider the case, where $n+1$ is a prime, of the form $M_n = (a_{ij})$ with $a_{ij} = ij \bmod (n+1)$ for $i, j = 1, 2, ..., n$. The elements of the principal diagonal of this matrix has a property, i.e., if D_n is the principal diagonal of the $n \times n$ M-matrix of Type II, then we get its elements by $x^2 \bmod (n+1)$, quadratic residue mod $(n+1)$ elements, for some positive integer x. Thus, for example, $D_2 = (1\ 1), D_4 = (1\ 4\ 4\ 1), D_6 = (1\ 4\ 2\ 2\ 4\ 1)$, where $n/2$ elements repeat in the reverse order.

Proposition 2.6. *In an M-matrix of Type II of order n, the orthogonal number between any two distinct rows is given by $4k - n$, where k is the number of unities in the selected set and $0 \le k \le n/2$.*

Proof. Let R_i and R_j be the two given rows in the M-matrix. Their inner products $(R_i \cdot R_j)$, where $i \neq j$, have to be calculated. By elementary transformations we can make the row R_i with the first $n/2$ elements as $+1$'s and the next $n/2$ elements as -1's. The orthogonal numbers of the matrix remain invariant by such elementary transformations. Now consider the other row R_j which has n elements. Let these n elements be divided into two halves with $n/2$ elements each. They can be depicted as follows:

$$R_i = \big((11...11...1)(-1-1...-1-1)\big) \text{ of size } n,$$

$$R_j = \big((11-1...-1-1...11)(11...1-11-1...11)\big) \text{ of size } n,$$

where in R_j let $+1$'s be k in number and hence -1's are $n/2 - k$ in the first half, and in the other half $+1$'s are $n/2 - k$ and -1's are k in number. Now we evaluate the formula for the orthogonal number. In the first two sets of the two rows, (i) the k number of $+1$'s in the first set of R_j correspond with the k number of $+1$'s in the first set of R_i, (ii) the $n/2-k$ number of -1's in the first set of R_j correspond with the same number of -1's in the first set of R_i. Next, in the second sets of the given two rows, (iii) the $n/2-k$ number of $+1$'s in the second set of R_j correspond with the same number of -1's in the second set of R_i, and (iv) the k number of -1's in R_j correspond with the same number of -1's in R_i. Hence $g=(R_i\cdot R_j)=k\times1\times1+(n/2-k)\times(-1)\times1+(n/2-k)\times1\times(-1)+k\times(-1)\times(-1)=4k-n$. If we see for different values of k, when $k=0$ then $g=-n$, when $k=1$ then $g=4-n$, and so on. And when $k=n/2$ then $g=n$. Thus there should be $n/2+1=(n+2)/2$ orthogonal numbers in an M-matrix of Type II. ■

Proposition 2.7. *In an M-matrix of Type II of order n the sum of the orthogonal numbers is 0.*

Proof. Let g_i's be orthogonal numbers in the given M-matrix. Then

$$\sum_{i=1}^{(n+2)/2} g_i=\sum_{k=0}^{n/2}(4k-n)=0.$$

The orthogonal numbers are in an arithmetic progression with arithmetic mean 4. Hence the proof. ■

If M is the M-matrix of Type II of order n, then the matrix MM' is also an $n\times n$ symmetric matrix, and then the elements in its principal diagonal are n only as $(R_i\cdot R_i)=n$, which is a trivial orthogonal number. It has no second trivial orthogonal number as in the case of Type I, and the rest of its elements are the orthogonal numbers.

Note 2.4. In these orthogonal numbers there exist orthogonal pairs, which either exist together or do not exist at all. The pair formation depends on the number of $+1$'s and the number of -1's in a selected set of a row. The first number of the pair occurs when the first set of a row R_j is considered in which the number of $+1$'s is θ_1 and the number of -1's is θ_2, and the second number of the pair occurs when the first set of a row R_j is considered in which the number of -1's is θ_1 and the number of $+1$'s is θ_2, where $\theta_1+\theta_2=(n-1)/2$ in Type I, and $\theta_1+\theta_2=n/2$ in Type II. For example, when $n=11$, $\theta_1=1$ and $\theta_2=4$, then -5 is the orthogonal number, and again $\theta_2=1$ and $\theta_1=4$, then 7 is the orthogonal number, both of these -5 and 7 form an orthogonal pair, which do not occur. In it the second orthogonal pair is -9 and 11, and the third orthogonal pair is -1 and 3. In a given M-matrix of either of the types, 1 may be an orthogonal number and also $(R_1 \cdot R_j) = 1$, $j = 1, 2, ..., (n-1)/2$, in Type I. In Type II there exists no row with all unities satisfying $(R_1 \cdot R_j) = 1$.

Result 2.3. In an M-matrix of Type I the sum of the numbers in an orthogonal pair is 2, and in Type II it is 0.

Proof. In the M-matrix of Type I the orthogonal numbers are obtained from the formula $4k + 2 - n$. Suppose the orthogonal pair exists for $k = \theta_1$ and $k = \theta_2$, where $\theta_1 + \theta_2 = (n-1)/2$. So it follows that the sum of the two orthogonal numbers in a pair is $(4\theta_1 + 2 - n) + (4\theta_2 + 2 - n) = 4(\theta_1 + \theta_2) + 4 - 2n = 4 \times \{(n-1)/2\} + 4 - 2n = 2$. In the M-matrix of Type II the orthogonal numbers are obtained from the formula $4k - n$. Suppose the orthogonal pair exists for $k = \theta_1$ and $k = \theta_2$, where $\theta_1 + \theta_2 = n/2$. Hence it follows that the sum of the two orthogonal numbers in a pair is $(4\theta_1 - n) + (4\theta_2 - n) = 4(\theta_1 + \theta_2) - 2n = 4 \times (n/2) - 2n = 0$. ■

Proposition 2.8. *When $n+1$ is a prime, the existence of an M-matrix of Type II of order n implies the existence of an SPBIB design with parameters $v = b = n, r = k = n/2$ and λ_i's vary from 0 to $(n-2)/2$. Furthermore, n_i and $p^i_{j\ell}$ can be evaluated in the numerical problem, as it is complex to find them theoretically.*

Proof. Given an M-matrix of Type II of order n, consider all -1's as zeros. There are n rows and n columns and then $v = b = n$. In each row and each column the number of $+1$'s is $n/2$. Hence $r = k = n/2$. And the λ_i values vary from 0 to $(n-2)/2$ depending on the coincidences of $+1$'s in the rows, since the maximum coincidence may be $(n-2)/2$. ■

Proposition 2.9. *When $n+1$ is a prime, the existence of an M-matrix of Type II of order n implies the existence of a regular bipartite graph.*

Procedure for the construction. For the matrix that we have taken, considering that as the adjacency matrix with elements as one set and the columns as another set, the graph is defined with the condition if an element a_{ij} is in column C_k, then (a_{ij}, C_k) will be an edge where $1 \leq i, j, k \leq n$, otherwise not. Here $|V| = n, |V_1| = n/2, |V_2| = n/2$ and the valence is $n/2$ as each row and each column has $n/2$ number of $+1$'s. Hence we will get a regular bipartite graph.

Example 2.4. Take $n = 4$ (then $n + 1 = 5$ a prime). Consider $M_n = (ij)$ mod 5 for $i, j = 1, 2, 3, 4$. Then the M_n-matrix is given as

$$M_n = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

Now substituting 1 for even numbers and -1 for odd numbers and for 1 also in the above matrix we obtain the M-matrix as

$$M = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

In this example $n = 4$ and the formula is $4k - n$. We get the orthogonal number 4, the case of identical rows. And that is the case when $k = 2$ in the formula $4k - n$. The orthogonal numbers are $(R_1 \cdot R_2) = (R_1 \cdot R_3) = (R_2 \cdot R_4) = (R_3 \cdot R_4) = 0$, $(R_1 \cdot R_4) = (R_2 \cdot R_3) = -4$, and $(R_i \cdot R_i) = 4$, $i = 1, 2, 3, 4$. This is the trivial orthogonal number. Now consider -1's as zeros. Then we get the incidence matrix of a design as

$$N = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

which is a 2-associate group divisible PBIB design with parameters $v = b = 4, r = k = 2$, $\lambda_1 = 0, \lambda_2 = 1, n_1 = 2, n_2 = 1$, and for $i, j = 1, 2$,

$$(p_{ij}^1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (p_{ij}^2) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Example 2.5. Take $n = 6$ (then $n+1 = 7$ a prime) (as this type of matrix is being considered in Example 2.2). By analogous construction we can get an SPBIB design which is constructed below. By considering the matrix $(a_{ij}) = (ij)$ mod 7 for $i, j = 1, 2, ..., 6$, the M_n-matrix is given as

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \\ 3 & 6 & 2 & 5 & 1 & 4 \\ 4 & 1 & 5 & 2 & 6 & 3 \\ 5 & 3 & 1 & 6 & 4 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix},$$

in which by changing the even numbers as -1's and odd numbers as 1's and keeping 1's in the matrix as it is, we get the following M-matrix as

$$M = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

In this case the orthogonal numbers are given by $(R_1 \cdot R_2) = (R_1 \cdot R_4) = (R_2 \cdot R_4) = (R_3 \cdot R_5) = (R_4 \cdot R_6) = (R_5 \cdot R_6) = -2$, $(R_1 \cdot R_3) = (R_1 \cdot R_5) = (R_2 \cdot R_3) = (R_2 \cdot R_6) = (R_4 \cdot R_5) = (R_4 \cdot R_6) = 2$, $(R_1 \cdot R_6) = (R_2 \cdot R_5) = (R_3 \cdot R_4) = -6$. Also $(R_i \cdot R_i) = 6$, $i = 1, 2, ..., 6$, which is the trivial orthogonal number. This type of matrix is not having the other trivial orthogonal number. The orthogonal pairs form when $\theta_1 + \theta_2 = n/2$.

Now from the formula $4k - n$ it can be seen that the sum of the orthogonal numbers in an orthognal pair is 0. Here the pairs are $(-2, 2), (-6, 6)$. As there is no row with all unities, the question of $(R_1 \cdot R_j)$ does not arise. By changing -1's as 0's in the above M-matrix we get

$$N = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix},$$

which is the incidence matrix of a 3-associate SPBIB design with parameters $v = b = 6, r = k = 3,\ \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0, n_1 = 2, n_2 = 2, n_3 = 1$, and for $i, j = 1, 2, 3$,

$$(p_{ij}^1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad (p_{ij}^2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad (p_{ij}^3) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly by treating the above incidence matrix N of the design as an adjacency matrix of a graph we get a regular bipartite graph with $V_1 = \{1, 2, 3, 4, 5, 6\}$ (due to the row-numbering) and $V_2 = \{7, 8, 9, 10, 11, 12\}$ (due to the column-numbering) as in Example 2.3.

This leads to a network problem, which can be stated as follows. Suppose there are n points and each point is connected to m points, $m < n$. By considering any one point as a source point and the remaining $n - 1$ points as destination points find the maximum possible number of distinct routes from the source point to a destination point, we can as well impose many other restrictions on this system.

Conclusion. In this paper we have considered M-matrices of Type I and Type II, which are structurally different, but serve the same purpose.

In Type I, we delete the first row and the first column of unities, but in Type II we retain the element 1 in the matrix. These matrices are non-orthogonal $(1, -1)$-matrices useful for many applications as mentioned earlier. But some problems remain open. For example, some orthogonal numbers theoretically exist but do not be found to be in the numerical problems. In the case of $n = 11$ in Type I, the orthogonal numbers -5 and 7 do not exist (see Note 2.2). And the matrix MM' could not be formulated and their row sums could not be established by suitable formulae in both the types. It is one way, the diagonal elements are n only and all other entries are the orthogonal numbers.

More technical features of these applications can be seen in a sequel to this paper to appear [22] shortly. And for the other details of these networks refer to Bhuyan and Agarwal [2], Hawkes [14], Nguyen, Vo and Lee [23].

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