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ON IDEALS IN REGULAR TERNARY SEMIGROUPS

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Abstract

In this paper we study some interesting properties of regular ternary semigroups, completely regular ternary semigroups, intra-regular ternary semigroups and characterize them by using various ideals of ternary semigroups.

Keywords: ternary semigroup, ternary group, regular ternary semigroup, completely regular ternary semigroup, intra-regular ternary semigroup, semiprime ideal, bi-ideal.

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1. Introduction

In [5], J. Los studied some properties of ternary semigroups and proved that every ternary semigroup can be embedded in a semigroup. In [9], F.M. Sioson studied ideal theory in ternary semigroups. He also introduced the notion of regular ternary semigroups and characterized them by using the notion of quasi-ideals. In [8], M.L. Santiago developed the theory of ternary semigroups and semiheaps. The notion of regularity was introduced and studied by J. von Neumann [7] in 1936. Subsequently the notion of regular semiring was also introduced and studied as a generalization of regular ring. In [10], Vasile introduced and studied the notion of regular ternary rings.

In [2, 3], Dutta and Kar introduced and studied the notion of regular ternary semirings. Recently, a number of mathematicians have worked on ternary structures, see ([1, 6]) and references therein.

In this paper we study some interesting properties of regular ternary semigroup, completely regular ternary semigroup and intra-regular ternary semigroup.

2. Preliminaries

Definition 2.1. A non-empty set S together with a ternary multiplication, denoted by juxtaposition, is said to be a ternary semigroup if $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in S$.

Example 2.2. Let $\mathbf{Z}^-$ be the set of all negative integers. Then together with usual ternary multiplication of negative integers, $\mathbf{Z}^-$ forms a ternary semigroup.

Example 2.3. Let S be the set of all odd polynomials in one variable with negative integral coefficients. Then S forms a ternary semigroup with respect to ternary multiplication of polynomials.

Example 2.4. Let S be the set of all real numbers and k be a fixed number in S. If we define a ternary multiplication in S by $abc = a + b + c + k$ for all $a, b, c \in S$, then with this ternary multiplication, S forms a ternary semigroup.

Example 2.5. Let S be the set of all continuous functions $f : X \longrightarrow \mathbf{R}^-$, where X is a topological space and $\mathbf{R}^-$ is the set of all negative real numbers. Now we define a ternary multiplication on S by

$$(fgh)(x) = f(x)g(x)h(x) \text{ for all } f, g, h \in S \text{ and } x \in X.$$

Then together with this ternary multiplication, S forms a ternary semigroup.

Definition 2.6. A ternary semigroup S is said to be commutative if $x_1x_2x_3 = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ for every permutation σ of $\{1, 2, 3\}$ and $x_1, x_2, x_3 \in S$.

Definition 2.7. A ternary semigroup S is said to be

(i) left cancellative (LC) if $abx = aby \Longrightarrow x = y$ for all $a, b, x, y \in S$;

(ii) right cancellative (RC) if $xab = yab \Longrightarrow x = y$ for all $a, b, x, y \in S$;

(iii) laterally cancellative (LLC) if $axb = ayb \Longrightarrow x = y$ for all $a, b, x, y \in S$;

(iv) cancellative if S is left, right and laterally cancellative.

Definition 2.8 [8]**.** A pair (a, b) of elements in a ternary semigroup S is said to be an idempotent pair if $ab(abx) = abx$ and $(xab)ab = xab$ for all $x \in S$.

Definition 2.9 [8]**.** Two idempotent pairs (a, b) and (c, d) of a ternary semigroup S are said to be equivalent, in notation we write $(a, b) \sim (c, d)$, if $abx = cdx$ and $xab = xcd$ for all $x \in S$.

Definition 2.10. A non-empty subset I of a ternary semigroup S is called

(i) a left ideal of S if $SSI \subseteq I$

(ii) a lateral ideal of S if $SIS \subseteq I$

(iii) a right ideal of S if $ISS \subseteq I$

(iv) an ideal of S if I is a left, a right, a lateral ideal of S. An ideal I of a ternary semigroup S is called a proper ideal if $I \neq S$.

Proposition 2.11 [9]**.** *Let S be a ternary semigroup and $a \in S$. Then the principal*

(i) *left ideal generated by 'a' is given by $< a >_l = SSa \cup \{a\}$*

(ii) *right ideal generated by 'a' is given by $< a >_r = aSS \cup \{a\}$*

(iii) *lateral ideal generated by 'a' is given by $< a >_m = SaS \cup SSaSS \cup \{a\}$*

(iv) *ideal generated by 'a' is given by $< a > = SSa \cup aSS \cup SaS \cup SSaSS \cup \{a\}$.*

Definition 2.12. An ideal I of a ternary semigroup S is called idempotent if $I^3 = I$.

Definition 2.13. A proper ideal Q of a ternary semigroup S is called a semiprime ideal of S if $I^3 \subseteq Q$ implies $I \subseteq Q$ for any ideal I of S.

Definition 2.14. A proper ideal Q of a ternary semigroup S is called a completely semiprime ideal of S if $x^3 \in Q$ implies that $x \in Q$ for any element x of S.

Definition 2.15. A subsemigroup B of a ternary semigroup S is called a bi-ideal of S if $BSBSB \subseteq B$.

Definition 2.16. An element a of a ternary semigroup S is said to be invertible in S if there exists an element b in S such that $abx = bax = xab = xba = x$ for all $x \in S$.

Definition 2.17. A ternary semigroup S is called a ternary group if for $a, b, c \in S$, the equations $abx = c$, $axb = c$ and $xab = c$ have solutions in S.

Remark 2.18. In a ternary group S, for $a, b, c \in S$, the equations $abx = c$, $axb = c$ and $xab = c$ have unique solutions in S.

Remark 2.19. In a ternary group S, every element has unique inverse in S.

3. Regular ternary semigroups

F.M. Sioson [9] defined the notion of regular ternary semigroup as follows:

Definition 3.1 [9]**.** A ternary semigroup S is said to be regular if for each $a \in S$, there exist elements x, y in S such that $axaya = a$.

Subsequently, M.L. Santiago [8] modified the definition of regular ternary semigroup as follows:

Definition 3.2 [8]**.** An element a in a ternary semigroup S is called regular if there exists an element x in S such that $axa = a$.

A ternary semigroup S is called regular if all of its elements are regular.

Clearly, we see that the above definition of regular ternary semigroup is equivalent to the definition of regular ternary semigroup given by Sioson [9].

Example 3.3.

(i) Every ternary group is a regular ternary semigroup.

(ii) Let $S = \{(m, n) : m, n \in \mathbf{Q_0^-}$ (set of all non-positive rational numbers)$\}$. Then it can be verified that w.r.t. componentwise usual ternary multiplication of non-positive rational numbers S is a regular ternary semigroup.

Remark 3.4. A regular ternary semigroup may not be a ternary group. We see later that under certain conditions a regular ternary semigroup is a ternary group.

We note that every left and right ideal of a regular ternary semigroup may not be a regular ternary semigroup; however, for a lateral ideal of a regular ternary semigroup, we have the following result:

Lemma 3.5. *Every lateral ideal of a regular ternary semigroup S is a regular ternary semigroup.*

Proof. Let L be a lateral ideal of a regular ternary semigroup S. Then for each $a \in L$, there exists $x \in S$ such that $a = axa$. Now $a = axa = axaxa = a(xax)a = aba$, where $b = xax \in L$. This implies that L is a regular ternary semigroup.

Note 3.6. Every ideal of a regular ternary semigroup S is a regular ternary semigroup.

Now we have the following characterization theorem for regular ternary semigroup:

Theorem 3.7. *The following conditions in a ternary semigroup S are equivalent:*

(i) *S is regular*

(ii) *For any right ideal R, lateral ideal M and left ideal L of S, $RML = R \cap M \cap L$*

(iii) *For* $a, b, c \in S$; $< a >_r < b >_m < c >_l = < a >_r \cap < b >_m \cap < c >_l$

(iv) *For* $a \in S$; $< a >_r < a >_m < a >_l = < a >_r \cap < a >_m \cap < a >_l$.

Proof. (i)$\Longrightarrow$(ii).

Suppose S is a regular ternary semigroup.

Let R, M and L be a right ideal, a lateral ideal and a left ideal of S respectively. Then clearly, $RML \subseteq R \cap M \cap L$. Now for $a \in R \cap M \cap L$, we have $a = axa$ for some $x \in S$. This implies that $a = axa = (axa)(xax)(axa) \in RML$. Thus we have $R \cap M \cap L \subseteq RML$. So we find that $RML = R \cap M \cap L$.

Clearly, (ii)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(iv).

To complete the proof, it remains to show that (iv)$\Longrightarrow$(i).

Let $a \in S$. Clearly, $a \in < a >_r \cap < a >_m \cap < a >_l = < a >_r < a >_m < a >_l$. Then we have, $a \in (aSS \cup \{a\})(SaS \cup SSaSS \cup \{a\})(SSa \cup \{a\}) \subseteq aSa$. So we find that $a \in aSa$ and hence there exists an element $x \in S$ such that $a = axa$. This implies that a is regular and hence S is regular.
From Theorem 3.7, we have the following corollary:

Corollary 3.8. *The following conditions on a ternary semigroup S are equivalent:*

(i) *S is regular*

(ii) *$A \cap B = ASB$ for every right ideal A and every left ideal B of S*

(iii) *For* $a, b \in S$; $< a >_r \cap < b >_l = < a >_r S < b >_l$

(iv) *For* $a \in S$; $< a >_r \cap < a >_l = < a >_r S < a >_l$.

Theorem 3.9 [8]**.** *The following conditions in a ternary semigroup S are equivalent:*

(i) *S is regular and cancellative;*

(ii) *S is regular and the idempotent pairs in S are all equivalent;*

(iii) *Every element of S is invertible in S;*

(iv) *S is a ternary group;*

(v) *S contains no proper one-sided ideals.*

Theorem 3.10. *A ternary semigroup S is regular if and only if every ideal of S is idempotent.*

Proof. Let S be a regular ternary semigroup and I be any ideal of S. Then $I^3 = III \subseteq SSI \subseteq I$. Let $a \in I$. Then there exists $x \in S$ such that $a = axa = axaxa$. Since I is an ideal and $a \in I$, $xax \in I$. Thus $a = axa = axaxa \in I^3$. Consequently, $I \subseteq I^3$ and hence $I^3 = III = I$ i.e., I is idempotent.

Conversely, suppose that every ideal of S is idempotent. Let A, B and C be three ideals of S. Then $ABC \subseteq ASS \subseteq A$, $ABC \subseteq SBS \subseteq B$ and $ABC \subseteq SSC \subseteq C$. This implies that $ABC \subseteq A \cap B \cap C$. Also, $(A \cap B \cap C)(A \cap B \cap C)(A \cap B \cap C) \subseteq ABC$. Again, since $A \cap B \cap C$ is an ideal of S, $(A \cap B \cap C)(A \cap B \cap C)(A \cap B \cap C) = A \cap B \cap C$. Thus $A \cap B \cap C \subseteq ABC$ and hence $A \cap B \cap C = ABC$. Therefore, by Theorem 3.7, S is a regular ternary semigroup.

Theorem 3.11. *A commutative ternary semigroup S is regular if and only if every ideal of S is semiprime.*

Proof. Let S be a commutative regular ternary semigroup and Q be any ideal of S such that $A^3 \subseteq Q$ for any ideal A of S. From Theorem 3.7, it follows that $A^3 = A$. Consequently, $A \subseteq Q$ and hence Q is a semiprime ideal of S.

Conversely, suppose every ideal of a commutative ternary semigroup S is semiprime. Let $a \in S$. Then aSa is an ideal of S. Now by hypothesis, aSa is a semiprime ideal of S. If $aSa = S$, then we are done. Now suppose that $aSa \neq S$.

Then

$$< a >< a >< a >$$

$$= (SSa \cup aSS \cup SaS \cup SSaSS \cup \{a\})(SSa \cup aSS \cup SaS \cup SSaSS \cup \{a\})$$

$$(SSa \cup aSS \cup SaS \cup SSaSS \cup \{a\}) \in aSa \text{ i.e., } < a >< a >< a > \subseteq aSa.$$

This implies that $< a > \subseteq aSa$, since aSa is a semiprime ideal of S. Consequently, $a = axa$ for some $x \in S$ and hence S is a regular ternary semigroup.

Definition 3.12. An element a in a ternary semigroup S is said to be left (resp. right) regular if there exists an element $x \in S$ such that $xaa = a$ (resp. $aax = a$).

If all the elements of a ternary semigroup S are left (resp. right) regular, then S is called left (resp. right) regular.

The following theorem gives a characterization of left (resp. right) regularity of a ternary semigroup S in terms of completely semiprime ideals of S.

Theorem 3.13. *A ternary semigroup S is left (resp. right) regular if and only if every left (resp. right) ideal of S is completely semiprime.*

Proof. Let S be a left regular ternary semigroup and L be any left ideal of S. Suppose $a^3 = aaa \in L$ for $a \in S$. Since S is left regular, there exists an element $x \in S$ such that $a = xaa = x(xaa)a = xx(aaa) \in SSL \subseteq L$. Thus L is completely semiprime.

Conversely, suppose that every left ideal of S is completely semiprime. Now for any $a \in S$, Saa is a left ideal of S. Then by hypothesis, Saa is a completely semiprime ideal of S. Now $a^3 = aaa \in Saa$. Since Saa is completely semiprime, it follows that $a \in Saa$. So there exists an element $x \in S$ such that $a = xaa$. Consequently, a is left regular. Since a is arbitrary, it follows that S is left regular.

Similarly, we can prove the theorem for right regularity.

Definition 3.14. An element a of a ternary semigroup S is said to be completely regular if there exists an element $x \in S$ such that $axa = a$ and the idempotent pairs (a, x) and (x, a) are equivalent.

If all the elements of S are completely regular, then S is called completely regular.

We have the following characterization theorem for completely regular ternary semigroup:

Theorem 3.15. *The following conditions in a ternary semigroup S are equivalent:*

(i) *S is completely regular*

(ii) *S is left and right regular i.e., $a \in a^2S \cap Sa^2$ for all $a \in S$.*

(iii) *$a \in a^2Sa^2$ for all $a \in S$.*

Proof. (i)$\Longrightarrow$(ii).

Suppose S is a completely regular ternary semigroup.

Let $a \in S$. Then there exists an element $x \in S$ such that $axa = a$ and the idempotent pairs (a, x) and (x, a) are equivalent i.e., $axy = xay$ and $yax = yxa$ for all $y \in S$. Now in particular, putting $y = a$ we find that $axa = xaa$ and $aax = axa$. This implies that $a \in a^2S$ and $a \in Sa^2$ i.e., $a \in a^2S \cap Sa^2$.

(ii)$\Longrightarrow$(iii).

Suppose that S is both left and right regular.

Let $a \in S$. Then there exist $x, y \in S$ such that $a = aax$ and $a = yaa$. This implies that $axz = yaaxz = yaz$ for all $z \in S$. Now $a = aax = a(aax)x = a^2(axx) = a^2(yax) = a^2y(yaa)x = a^2y^2(aax) = a^2y^2a = a^2y^2(yaa) = a^2y^3a^2 \in a^2Sa^2$.

(iii)$\Longrightarrow$(iv).

Suppose $a \in a^2Sa^2$ for all $a \in S$. Then there exists $x \in S$ such that $a = a^2xa^2$. Now $a = a^2xa^2 = a(axa)a = aya$, where $y = axa \in S$. This implies that S is regular. Also $ayz = a(axa)z = a^2xa^2xa^2z$ and $yaz = (axa)az = a^2xa^2xa^2z$ for all $z \in S$. This shows that the idempotent pairs (a, y) and (y, a) are equivalent. Consequently, S is a completely regular ternary semigroup.

Theorem 3.16. *A ternary semigroup S is completely regular if and only if every bi-ideal of S is completely semiprime.*

Proof. First suppose that S is a completely regular ternary semigroup. Let B be any bi-ideal of S. Let $b^3 \in B$ for $b \in S$. Since S is completely regular, from Theorem 3.15, it follows that $b \in b^2Sb^2$. This implies that there exists $x \in S$ such that $b = b^2xb^2 = b(b^2xb^2)x(b^2xb^2)b = b^3(xb^2x)b(b^2xb^2)xb^3 = b^3(xb^2x)b^3(xb^2x)b^3 \in BSBSB \subseteq B$. This shows that B is completely semiprime.

Conversely, suppose that every bi-ideal of S is completely semiprime. Since every left and right ideal of a ternary semigroup S is a bi-ideal of S, it follows that every left and right ideal of S is completely semiprime. Consequently, we have from Theorem 3.13 that S is both left and right regular. Now by using Theorem 3.15, we find that S is a completely regular ternary semigroup.

Theorem 3.17. *If S is a completely regular ternary semigroup, then every bi-ideal of S is idempotent.*

Proof. Let S be a completely regular ternary semigroup and B be a bi-ideal of S. Since S is a completely regular ternary semigroup, it is also a regular ternary semigroup. Let $b \in B$. Then there exists $x \in S$ such that $b = bxb$. This implies that $b \in BSB$ and hence $B \subseteq BSB$. Also $BSB \subseteq BSBSB \subseteq B$. Thus we find that $B = BSB$. Again, we have from Theorem 3.15 that $b \in b^2Sb^2 \subseteq B^2SB^2$. This implies that $B \subseteq B^2SB^2 = B(BSB)B = BBB \subseteq B$. Consequently, $B^3 = B$.

Definition 3.18. A ternary semigroup S is called intra-regular if for each element $a \in S$, there exist elements $x, y \in S$ such that $xa^3y = a$.

Theorem 3.19. *If S is an intra-regular ternary semigroup then for any left ideal L, lateral ideal M and right ideal R of S, $L \cap M \cap R \subseteq LMR$.*

Proof. Suppose that S is an intra-regular ternary semigroup. Let L, M and R be a left ideal, a lateral ideal and a right ideal of S respectively. Now for $a \in L \cap M \cap R$, we have $a = xa^3y$ for some $x, y \in S$. This implies that $a = xa^3y = (xxa^3)(yxa^3yx)(a^3yy) \in LMR$. Thus we have $L \cap M \cap R \subseteq LMR$.

Proposition 3.20. *Let S be an intra-regular ternary semigroup. Then a non-empty subset I of S is an ideal of S if and only if I is a lateral ideal of S.*

Proof. Clearly, if I is an ideal of S, then I is a lateral ideal of S.

Conversely, let I be a lateral ideal of an intra-regular ternary semigroup. Let $a \in I$ and $s, t \in S$. Then $a \in S$ and hence there exist elements $x, y \in S$ such that $a = xa^3y$. Now $sta = stxa^3y \in SIS \subseteq I$ and $ast = xa^3yst \in SIS \subseteq I$. This implies that I is both a left ideal and a right ideal of S. Consequently, I is an ideal of S.

Lemma 3.21. *Every lateral ideal of an intra-regular ternary semigroup S is an intra-regular ternary semigroup.*

Proof. Let L be a lateral ideal of an intra-regular ternary semigroup S. Then for each $a \in L$, there exists $x, y \in S$ such that $a = xa^3y$.

Now $a = xa^3y = x(xa^3y)(xa^3y)(xa^3y)y = (xxa^3yx)a^3(yxa^3yy) \in La^3L$. This implies that there exist $u, v \in L$ such that $a = ua^3v$. Consequently, L is an intra-regular ternary semigroup.

From Proposition 3.20, we have the following result:

Corollary 3.22. *Every ideal of an intra-regular ternary semigroup S is an intr-regular ternary semigroup.*

In ring theory, we note that if I is an ideal of a ring R and J is an ideal of I, then J need not be an ideal of the entire ring R. But it is well known that the result is true for regular ring. Like ring theory, we note that if I is an ideal of a ternary semigroup S and J is an ideal of I, then J need not be an ideal of the entire ternary semigroup S. But in particular, for an intra-regular ternary semigroup S, we have the following result:

Theorem 3.23. *Let I be an ideal of an intra-regular ternary semigroup S and J be an ideal of I. Then J is an ideal of the entire ternary semigroup S.*

Proof. It is sufficient to show that J is a lateral ideal of S. Let $a \in J \subseteq I$ and $s, t \in S$. Then $sat \in I$. We have to show that $sat \in J$. From Corollary 3.22, it follows that I is an intra-regular ternary semigroup. Thus there exist $u, v \in I$ such that $sat = u(sat)^3v = u(sat)(sat)(sat)v = (usats)a(tsatv) \in IJI \subseteq J$. Consequently, J is a lateral ideal of S.

Theorem 3.24. *A ternary semigroup S is intra-regular if and only if every ideal of S is completely semiprime.*

Proof. Let S be an intra-regular ternary semigroup and I be an ideal of S. Let $a^3 \in I$ for $a \in S$. Since S is intra-regular, there exist $x, y \in S$ such that $a = xa^3y \in I$. Consequently, I is completely semiprime.

Conversely, suppose that every ideal of S is completely semiprime. Let $a \in S$. Then $a^3 \in < a^3 >$. This implies that $a \in < a^3 >$, since $< a^3 >$ is completely semiprime. Now $< a^3 > = SSa^3 \cup a^3SS \cup Sa^3S \cup SSa^3SS \cup a^3$. So we have the following cases:

If $a \in SSa^3$, then $a^3 \in SSa^3a^2$. Hence $a \in SSSSa^3a^2 \subseteq SSSa^2aS \subseteq Sa^3S$.

If $a \in a^3SS$, then $a^3 \in a^2a^3SS$. Hence $a \in a^2a^3SSSS \subseteq Saa^2SSS \subseteq Sa^3S$.

If $a \in Sa^3S$, then we are done.

If $a \in SSa^3SS$, then $a^3 \in aSSa^3SSa$.

Hence $a \in SSaSSa^3SSaSS \subseteq SSSa^3SSS \subseteq Sa^3S$.

If $a = a^3$, then $a = a^3 = (a^3)(a^3)(a^3) \subseteq Sa^3S$.

So we find that in any case, S is intra-regular.

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