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# OPTIMAL CONTROL OF SYSTEMS DETERMINED BY STRONGLY NONLINEAR OPERATOR VALUED MEASURES

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**Abstract**

In this paper we consider a class of distributed parameter systems (partial differential equations) determined by strongly nonlinear operator valued measures in the setting of the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ with continuous and dense embeddings where $H$ is a separable Hilbert space and $V$ is a reflexive Banach space with dual $V^*$. The system is given by

$$dx + A(dt, x) = f(t, x)\gamma(dt) + B(t)u(dt),\ x(0) = \xi,\ t \in I \equiv [0, T]$$

where $A$ is a strongly nonlinear operator valued measure mapping $\Sigma \times V$ to $V^*$ with $\Sigma$ denoting the sigma algebra of subsets of the set $I$ and $f$ is a nonlinear operator mapping $I \times H$ to $H$, $\gamma$ is a countably additive bounded positive measure and the control $u$ is a suitable vector measure. We present existence, uniqueness and regularity properties of weak solutions and then prove the existence of optimal controls (vector valued measures) for a class of control problems.

**Keywords:** evolution equations, strongly nonlinear operator valued measures, existence of solutions, regularity properties, optimal control.

**2000 Mathematics Subject Classification:** 35K10, 35L10, 34G20, 34K30, 35A05, 93C20.

## 1. Introduction

In a series of papers [1, 2, 3, 4, 5, 9] we considered general evolution equations on Banach spaces determined by operator valued measures and controlled by vector measures. We studied the questions of existence and regularity properties of mild solutions of semilinear problems [1–3] and weak solutions for strongly nonlinear parabolic and hyperbolic problems [4–6]. For semilinear problems, the substantial theory of optimal controls has been developed [1–3]. However, for the strongly nonlinear problems, determined by nonlinear operator valued measures, no such control theory exists. The major goal of this paper is to initiate such development.

A closely related topic, yet very distinct, is the subject of relaxed controls. This has been studied extensively in the literature and well documented in the recent book of Fattorini [13] where the reader will find extensive references. These systems are governed by differential equations on Banach spaces with controls which are probability measure valued functions, while the systems considered in this paper are determined by operator valued measures and controlled by vector measures. The first distinction is in the structure of the system dynamics and the second is in the space of controls used.

The rest of the paper is organized as follows: We present relevant notations and terminologies in Section 2. In Section 3, a brief review of recent results on existence and regularity properties of weak solutions for strongly nonlinear parabolic systems determined by nonlinear operator valued functions and measures is presented. In Section 4, we consider control systems and study the questions of existence of solutions and their continuous dependence on controls. In Section 5, we study optimal control problems. The paper is concluded with an example in Section 6.

## 2. Preliminaries

**Function Spaces:** Let $H$ be a real separable Hilbert space with the scalar product and norms denoted by $(v,w)$ and $|v| \equiv \sqrt{(v,v)}$ respectively for $v,w \in H$. Let $V$ be a linear subspace of the Hilbert space $H$ carrying the structure of a Hilbert space with the scalar product denoted by $(v,w) \equiv (v,w)_V$ and norm denoted by $\parallel v \parallel_V$ with $V^*$ denoting its topological dual. Identifying $H$ with its own dual and assuming that $V$ is dense in $H$, we have the inclusion

$$V \hookrightarrow H \hookrightarrow V^*$$

where the injections are continuous and dense. The duality pairing between

$v \in V$ and $w \in V^*$ is denoted by

$$\langle v, w \rangle \equiv \langle v, w \rangle_{V,V^*} \equiv \langle w, v \rangle_{V^*,V}.$$

In case $w \in H$, this reduces to the scalar product in $H$. We assume that there exists a complete system of basis vectors $\{v_i\} \subset V$ which is orthogonal in $V$ and $V^*$ and ortho-normal in $H$ and that it spans all the three spaces $\{V, H, V^*\}$ known as the Gelfand triple. For more details on these spaces see [6, 7] and the references therein.

Let $I \equiv [0, T]$ be an interval with $T < \infty$ and let $\Sigma \equiv \sigma(I)$ denote the sigma algebra of subsets of the set $I$. Let $B(I, H)$ denote the vector space of bounded $\Sigma$ measurable functions on $I$ with values in $H$. Furnished with the sup norm topology, this is a Banach space. Let $\mu$ be any countably additive positive measure on $\Sigma$ having bounded total variation on $I$. For any of the spaces $X \equiv \{V, H, V^*\}$ and $1 \leq p < \infty$, we let $L_p(\mu, X)$ denote the Lebesgue-Bochner space of measurable functions on $I$ with values in $X$ satisfying

$$\int_I \| f(s) \|_X^p \mu(ds) < \infty.$$

Strictly speaking, this is the equivalence class of $\mu$ measurable $X$ valued functions whose $X$-norms are $p - th$ power integrable. If $\mu$ is a Lebesgue measure, we use the standard symbol $L_p(I, X)$, $1 \leq p < \infty$, and $L_\infty(I, X)$ to denote the standard Lebesgue-Bochner spaces. Furnished with the standard norm topology, $L_p(\mu, X)$ is a Banach space. By $L_p(\mu)$ we denote the Banach space of scalar valued $p$-th power $\mu$ integrable functions defined on the interval $I$. By $BV(I, X)$ we denote the vector space of functions, defined on $I$ and taking values from the Banach space $X$, having bounded total variation. Furnished with total variation norm this is a Banach space.

**Vector Measures:** Let $F$ be a Banach space and $I \equiv [0, T]$ a bounded interval with $\Sigma \equiv \sigma(I)$ the sigma algebra of subsets of the set $I$. Let $\mathcal{M}_c(\Sigma, F)$ denote the space of countably additive bounded vector measures defined on the sigma algebra $\Sigma$ with values in the Banach space $F$. Let $\mathcal{M}_{cbv}(\Sigma, F)$ be a proper subspace of the space $\mathcal{M}_c(\Sigma, F)$ consisting of countably additive $F$-valued vector measures having bounded total variation (on $I$). This is furnished with the topology induced by the total variation norm. That is, for each $\nu \in \mathcal{M}_{cbv}(\Sigma, F)$, we write

$$|\nu| \equiv |\nu|(I) \equiv \sup_\pi \left( \sum_{\sigma \in \pi} \| \nu(\sigma) \|_F \right) \tag{1}$$

where the supremum is taken over all partitions $\pi$ of the interval $I$ into a finite number of disjoint members of $\Sigma$. With respect to this topology, $\mathcal{M}_{cbv}(\Sigma, F)$ is a Banach space. For any $\sigma \in \Sigma$, denote the variation of $\nu$ on $\sigma$ by $|\nu|(\sigma)$. Since $\nu$ is countably additive and bounded, this defines a countably additive bounded positive measure on $\Sigma$ [10, Proposition 9, p. 3]. In case $F = R$, the real line, we have the space of real valued signed measures which we denote by $\mathcal{M}_c(\Sigma)$ and if they are nonnegative we use $\mathcal{M}_c^+(\Sigma)$.

We introduce two other topologies which are used later. Let $1 \leq q < \infty$, $\pi$ any finite partition of the interval $I$ by disjoint members of $\Sigma$ and $\nu \in \mathcal{M}_c(\Sigma, F)$. The vector measure $\nu$ is said to have $q$-variation if

$$\sup_{\pi}\left(\sum_{\sigma\in\pi} \parallel \nu(\sigma) \parallel^q\right)^{1/q} < \infty$$

where the supremum is taken over all such partitions $\pi$. We denote this vector space by $BV_q(\Sigma, F)$. It is easy to verify that this is a Banach space with respect to the norm topology

$$\parallel \nu \parallel_{BV_q(\Sigma,F)} \equiv \sup_{\pi}\left(\sum_{\sigma\in\pi} \parallel \nu(\sigma) \parallel_F^q\right)^{1/q}. \tag{2}$$

Clearly, $BV_1(\Sigma, F) \equiv \mathcal{M}_{cbv}(\Sigma, F)$.

The second topology is very much related to the preceding one and is dependent on a given countably additive bounded nonnegative measure, say $\gamma$. Let $BV_q(\gamma, F)$ denote the class of vector measures $\mu \in \mathcal{M}_c(\Sigma, F)$ for which

$$\parallel \mu \parallel_{BV_q(\gamma,F)} \equiv \sup_{\pi}\left\{\sum_{\sigma\in\pi}\left(\frac{\parallel \mu(\sigma) \parallel_F}{\gamma(\sigma)}\right)^q \gamma(\sigma)\right\}^{1/q} < \infty, \tag{3}$$

where we use the convention $0/0 = 0$. With respect to this norm topology, $BV_q(\gamma, F)$ is a Banach space. Since $\gamma$ is a countably additive bounded positive measure and $q \geq 1$, it is easy to verify that the embeddings

$$BV_q(\gamma, F) \hookrightarrow BV_q(\Sigma, F), BV_q(\gamma, F) \hookrightarrow \mathcal{M}_{cbv}(\Sigma, F),$$

are continuous.

**Operator Valued Measures:** Let $E$ and $F$ be any pair of Banach spaces and $\mathcal{L}(E, F)$ the space of bounded linear operators from $E$ to $F$.

A set function $\Phi$ mapping $\Sigma \times E$ to $F$ is said to be an operator valued measure if for each $\sigma \in \Sigma, e \in E$, $\Phi(\sigma, e) \in F$ and $\Phi(\emptyset, e) = 0$ the zero operator. Here we are interested in countable additivity in the weak sense only. That is, the operator $\Phi$ is said to be weakly countably additive if for any family of disjoint sets $\sigma_i \in \Sigma$ and any pair $(e, f^*) \in E \times F^*$, we have

$$\Big\langle \Phi\Big(\bigcup \sigma_i, e\Big), f^* \Big\rangle_{F,F^*} = \sum \langle \Phi(\sigma_i, e), f^* \rangle_{F,F^*}.$$

If $e \longrightarrow \Phi(\sigma, e)$ is linear we may write $\Phi : \Sigma \longrightarrow \mathcal{L}(E, F)$. Further notations will be introduced as and when required.

## 3. NONLINEAR OPERATOR VALUED FUNCTIONS AND MEASURES

In this section, we review some recent results on systems governed by strongly nonlinear parabolic equations determined by operator valued functions coupled with scalar measures, and also operator valued measures. Consider the system

$$dx + A(t, x)\alpha(dt) = f(t)\alpha(dt),\ x(0) = x_0,\ t \in I, \tag{4}$$

where $A : I \times V \longrightarrow V^*$ is an operator valued function, $\alpha$ is a countably additive bounded positive measure and $f$ is a $V^*$ valued function. Without further notice, we assume that $V$ and $V^*$ have the structure of separable reflexive Banach spaces with the embeddings $V \hookrightarrow H \hookrightarrow V^*$ being continuous and dense. Let

$$1 < q \leq 2 \leq p < \infty \ \text{ with } \ 1/p + 1/q = 1.$$

Throughout the paper we assume, unless otherwise stated, that the pair of numbers $\{p, q\}$ satisfy these assumptions.

Now we are prepared to consider the question of existence of solutions for the system (4). We assume that the operator $A$ satisfies the following properties:

(B1): $A(t, \cdot) : V \longrightarrow V^*$ is monotone and hemicontinuous for $\alpha$-a.a $t \in I$; and for every $u, v \in V$, $t \rightarrow \langle A(t, u), v \rangle_{V^*,V}$ is continuous.

(B2): there exist $a > 0, b \geq 0$ so that $\langle A(t, v), v \rangle_{V^*,V} + b|v|_H^2 \geq a \parallel v \parallel_V^p$ for $\alpha$-a.a $t \in I$.

(B3): there exist constants $c_1, c_2 \geq 0$ so that $|A(t,v)|_{V^*} \leq c_1 + c_2 \parallel v \parallel_V^{p/q}$ $\alpha$-a.a. $t \in I$.

The result below was proved in a recent paper of the author.

**Theorem 3.1.** *Consider the evolution equation* (4) *and suppose the assumptions* (B1)–(B3) *hold and* $f \in L_q(\alpha, V^*)$. *Then for each* $x_0 \in H$, *equation* (4) *has a unique weak solution* $x \in L_\infty(I,H) \cap L_p(\alpha, V) \cap BV_q(\alpha, V^*)$.

***Proof.*** See [4, Theorem 4.4]. ∎

**Remark 3.2.** In case $\alpha$ is the Lebesgue measure or it is absolutely continuous with respect to the Lebesgue measure, we recover the classical result [6, Theorem 4.1, p. 96].

Note that $A$ is considered to be a nonlinear operator valued function mapping $I \times V$ to $V^*$. Recently this result has been further extended covering nonlinear operator valued measures [5]. That is, $A : \Sigma \times V \longrightarrow V^*$ is an operator valued set function. The system model considered is given by

$$dx + A(dt, x(t)) = f(t)\gamma(dt), \; t \in I, \; x(0) = x_0. \tag{5}$$

Basic assumptions used are as follows:

(C1): The map $A : \Sigma \times V \longrightarrow V^*$ is maximal monotone and hemicontinuous in the second argument satisfying

$$\langle A(\sigma, u) - A(\sigma, v), u - v \rangle_{V^*,V} \geq 0, \forall \; \sigma \in \Sigma, \quad \text{and} \quad \forall \; u, v \in V.$$

There exist two countably additive nonnegative measures $\gamma, \beta \in M_c^+(\Sigma)$ having bounded variations on $I$ with $\gamma$ being positive; and two real numbers $c_1 \geq 0, c_2 \geq 0$, such that

(C2): $\langle A(\sigma, v), v \rangle_{V^*,V} + \beta(\sigma)|v|_H^2 \geq \gamma(\sigma) \parallel v \parallel_V^p \quad \forall \; \sigma \in \Sigma$,

(C3): $\parallel A(\sigma, v) \parallel_{V^*} \leq \gamma(\sigma)\{c_1 + c_2 \parallel v \parallel_V^{p/q}\} \quad \forall \; \sigma \in \Sigma$.

**Note:** We wish to point out that the measure $\gamma$ is not assumed to be nonatomic.

We are concerned with the question of existence of solutions for the system (5). By a solution, we mean a weak solution as defined below. Let $C_T^1(0,T)$ denote the class of $C^1$ functions on $I \equiv [0,T]$ vanishing at $T$.

**Definition 3.3.** An element $x \in L_\infty(I,H) \cap B(I,H) \cap L_p(\gamma, V)$ is said to be a weak solution of the problem (5) if for every $v \in V$ and $\varphi \in C_T^1(0,T)$, it satisfies the following identity

$$\begin{aligned} -(x_0, \varphi(0)v) - \int_I (x(t), \dot{\varphi}(t)v)dt + \int_I \langle A(dt, x(t)), \varphi(t)v\rangle_{V^*,V} \\ = \int_I \langle f(t), \varphi(t)v\rangle_{V^*,V}\gamma(dt). \end{aligned} \tag{6}$$

Now we present some recent results from [5] on the questions of existence of solutions and their regularity properties.

**Theorem 3.4.** *Suppose $\gamma$ is a countably additive bounded positive measure having finite variation on $I$ and the operator valued measure $A$ satisfies the assumptions* (C1)–(C3) *and $f \in L_q(\gamma, V^*)$. Then for each $x_0 \in H$, the system* (5) *has a unique weak solution $x \in L_\infty(I,H) \cap B(I,H) \cap L_p(\gamma, V)$ and further $x \in BV_q(\Sigma, V^*)$.*

***Proof.*** See [5, Theorem 5.3]. ■

**Remark 3.5.** In the system model (5), the same measure $\gamma(\cdot)$ has been used to represent both the external forces as well as the internal ones embodied in the fundamental operator $A$. A more general representation is given by

$$dx(t) + A(dt, x(t)) = \nu(dt),\ x(0) = x_0,\ t \in I, \tag{7}$$

where the operator valued measure $A$ satisfies the assumptions (C1)–(C3) involving the scalar measures $\gamma, \beta$; while $\nu$ is a $V^*$-valued vector measure. The following result shows that under some mild assumptions this general model can be reduced to the one given by (5).

**Corollary 3.6.** *Consider the system* (7) *and suppose the operator valued measure $A$ and the measures $\gamma, \beta$ satisfy the assumptions of Theorem* 3.4. *Let $\nu$ be a countably additive $V^*$-valued vector measure having finite q-variation on $I$ and that it is $\gamma$ continuous. Then for each $x_0 \in H$, the system* (7) *has a unique weak solution $x \in B(I,H) \cap L_p(\gamma, V)$ and further $x \in BV_q(\Sigma, V^*)$.*

***Proof.*** See [5, Corollary 5.6]. ■

**Remark 3.7.** It would be interesting to consider a more general situation where the measure $\nu$ is not $\gamma$ continuous. It is not clear to the author if such an extension is possible. This is an open problem.

Theorem 3.4 was extended in reference [5] to cover systems of the form

$$dx(t) + A(dt, x(t)) = f(x(t))\gamma(dt),\ x(0) = x_0,\ t \in I, \tag{8}$$

where $f$ is a suitable nonlinear map from $V$ to $V^*$ or from $H$ to $H$.

**Theorem 3.8.** *Suppose the operator valued measure $A$ and the scalar measures $\gamma, \beta$ satisfy the assumptions of Theorem 3.4. Let $f : V \longrightarrow V^*$ be continuous satisfying the polynomial growth*

$$|\langle f(v), v\rangle_{V^*,V}| \leq K(1+ \parallel v \parallel_V^p)$$

*for some $K \in [0, 1)$. Further, suppose the corresponding Nemytski operator $F$, given by $F(x)(t) \equiv f(x(t)), t \in I$, is continuous from $L_p(\gamma, V)$ to $L_q(\gamma, V^*)$ with respect to the corresponding weak topologies. Then for each $x_0 \in H$ the system (8) has at least one weak solution $x \in B(I, H) \cap L_\infty(I, H) \cap L_p(\gamma, V)$.*

***Proof.*** See [5, Theorem 6.1]. ∎

The assumption on weak-weak continuity of the Nemytski operator $F$, associated with the map $f$, is rather strong. It can be relaxed if the nonlinear operator $f$ is more regular as stated in the following extension.

**Theorem 3.9.** *Suppose the operator valued measure $A$ and the scalar measures $\gamma, \beta$ satisfy the assumptions of Theorem 3.4 and the injection $V \hookrightarrow H$ is compact. Let $f : H \longrightarrow H$ be continuous satisfying the growth condition,*

$$|\langle f(h), h\rangle_H| \leq K(1+ \parallel h \parallel_H^2), \tag{9}$$

*for a finite positive number $K$. Then for each $x_0 \in H$ the system (8) has at least one weak solution $x \in B(I, H) \cap L_\infty(I, H) \cap L_p(\gamma, V)$. Further, if $-f$ is monotone the solution is unique.*

***Proof.*** See [5, Theorem 6.2]. ∎

## 4. Existence and continuous dependence of solutions

We consider the following control system

$$dx + A(dt, x) = f(x)\gamma(dt) + B(t)u(dt),\ t \in I,\ x(0) = \xi, \tag{10}$$

where $B$ is the control operator and the vector measure $u(\cdot)$ is the control.

We wish to consider control problems of the form

$$J(u) \equiv \Psi(x,u) \longrightarrow \inf., \tag{11}$$

where $\Psi$ is a suitable cost functional, $x$ is the (weak) solution (if one exists) corresponding to the control $u$ and the infimum is taken over the class of admissible controls $\mathcal{U}_{ad}$ which is a suitable subset of the space of vector measures.

Now we introduce the class of admissible controls. Let $F$ be a Banach space with dual $F^*$, $\gamma \in \mathcal{M}_c^+(\Sigma)$, a countably additive bounded positive measure having bounded total variation on $I$, and consider the Banach space $BV_q(\gamma, F^*)$. Let $\mathcal{U}_{ad} \subset BV_q(\gamma, F^*)$ be a bounded set denoting the class of admissible controls.

We need the following a-priori bounds.

**Lemma 4.1.** *Suppose the operator valued measure $A$, the scalar measures $\{\gamma,\beta\}$, and the operator $f$ satisfy the assumptions of Theorem 3.9. Let $B \in L_\infty(\gamma, \mathcal{L}(F^*, V^*))$ so that $B^* \in L_\infty(\gamma, \mathcal{L}(V,F)) \subset L_\infty(\gamma, \mathcal{L}(V, F^{**}))$. Let $x_0 = \xi \in H$ and $u \in \mathcal{U}_{ad}$. Then if $x$ is any solution of the system (10), it must be an element of $L_p(\gamma, V) \cap L_\infty(I,H) \cap B(I,H)$.*

***Proof.*** Let $x$ be any solution of equation (10) corresponding to the initial state $x_0 = \xi \in H$ and control $u \in \mathcal{U}_{ad}$. Scalar multiplying equation (10) by $x$ and then integrating by parts over the interval $I_t \equiv [0,t]$, it is easy to verify that for each $t \in I$,

$$\begin{aligned} |x(t)|_H^2 + 2\int_0^t \| x(s) \|_V^p \,\gamma(ds) &\le |\xi|_H^2 + 2\int_0^t |x(s)|_H^2 \beta(ds) \\ &+ 2\int_0^t (f(x(s)), x(s))_H \gamma(ds) + 2\int_0^t \langle B^*(s)x(s), u(ds)\rangle_{F,F^*}. \end{aligned} \tag{12}$$

By virtue of our assumption with respect to the operator $B$ there exists a finite positive number $b$ such that

$$\gamma - \text{ess-sup}\,\{\| B^*(t) \|_{\mathcal{L}(V,F)},\ t \in I\} \le b.$$

Using this bound and the fact that $u \in BV_q(\gamma, F^*)$, we have

$$\Big| \int_0^t \langle B^*(s)x(s), u(ds)\rangle \Big| \le b\bigg(\int_0^t \| x(s) \|_V^p \,\gamma(ds)\bigg)^{1/p} \| u \|_{BV_q(\gamma_t, F^*)} \tag{13}$$

where by $\gamma_t$ we mean the restriction of the measure $\gamma$ to the interval $I_t$. By virtue of Cauchy inequality it follows from this that for any $\varepsilon > 0$,

$$\begin{aligned}(14)\qquad &\Big|\int_0^t \langle B^*(s)x(s), u(ds)\rangle\Big| \\ &\leq\; b(\varepsilon^p/p)\int_0^t \parallel x(s) \parallel_V^p \gamma(ds) + b(\varepsilon^{-q}/q) \parallel u \parallel^q_{BV_q(\gamma_t,F^*)}\,.\end{aligned}$$

Clearly, it follows from the growth condition of $f$ given by (9), that

$$(15)\qquad \int_0^t (f(x(s)), x(s))_H \gamma(ds) \leq K\gamma(I_t) + K\int_0^t |x(s)|^2_H \gamma(ds).$$

Since $\varepsilon > 0$ is arbitrary, one can choose $\varepsilon = \varepsilon_0 > 0$ so that $2b(\varepsilon_0^p/p) = 1$. Using $\varepsilon = \varepsilon_0$, it follows from the expressions (12), (14) and (15) that

$$\begin{aligned}|x(t)|^2_H + \int_0^t \parallel x(s) \parallel_V^p \gamma(ds) \;&\leq \Big(|\xi|^2_H + 2K\gamma(I_t) + 2b(\varepsilon_0^q/q) \parallel u \parallel^q_{BV_q(\gamma_t,F^*)}\Big) \\ &\quad + 2\int_0^t |x(s)|^2_H(\beta(ds) + K\gamma(ds)) \\ (16)\qquad\qquad &\leq \Big(|\xi|^2_H + 2K\gamma(I) + 2b(\varepsilon_0^q/q) \parallel u \parallel^q_{BV_q(\gamma,F^*)}\Big) \\ &\quad + 2\int_0^t |x(s)|^2_H(\beta(ds) + K\gamma(ds)).\end{aligned}$$

By virtue of the generalized Gronwall inequality [Ahmed, 8], it follows from (16) that

$$(17)\qquad |x(t)|^2_H \leq C\exp 2\{\beta(I) + K\gamma(I)\} \equiv \tilde{C}\ \forall\ t \in I,$$

where

$$(18)\qquad C \equiv \Big(|\xi|^2_H + 2K\gamma(I) + 2b(\varepsilon_0^q/q) \parallel u \parallel^q_{BV_q(\gamma,F^*)}\Big)\,.$$

Since the measures $\beta$ and $\gamma$ are finite, $\tilde{C} < \infty$ and hence $x \in L_\infty(I, H)$. More precisely, it follows from (17) that $x \in B(I, H)$. Using these facts, again it follows from (16) that

$$(19)\qquad \int_0^t \parallel x(s) \parallel_V^p \gamma(ds) \leq C + 2\tilde{C}\{\beta(I) + K\gamma(I)\},\ \forall\ t \in I.$$

This shows that $x \in L_p(\gamma, V)$ as well and hence we have $x \in L_p(\gamma, V) \cap L_\infty(I, H) \cap B(I, H)$. This completes the proof. ∎

As a corollary of this lemma we have the following result.

**Corollary 4.2.** *Suppose the assumptions of Lemma* 4.1 *hold and that the set of admissible controls* $\mathcal{U}_{ad}$ *is a bounded subset of* $BV_q(\gamma, F^*)$. *Let* $x(u)$ *denote the weak solution of the system* (10) *corresponding to the control* $u \in \mathcal{U}_{ad}$. *Then the solution set* $X \equiv \{x(u) : u \in \mathcal{U}_{ad}\}$ *is a bounded subset of* $B(I, H) \cap L_\infty(I, H) \cap L_p(\gamma, V)$.

***Proof.*** The proof follows immediately from the estimates (17), (18) and (19) and boundedness of the set $\mathcal{U}_{ad}$. ∎

Now we are prepared to consider the question of existence and regularity properties of solutions of the control system (10).

**Theorem 4.3.** *Suppose the operator valued measure* $A$, *the scalar measures* $\{\gamma, \beta\}$ *and the operator* $f$ *satisfy the assumptions of Lemma* 4.1 *with the injection* $V \hookrightarrow H$ *being compact. Let* $B \in L_\infty(\gamma, \mathcal{L}(F^*, V^*))$ *so that* $B^* \in L_\infty(\gamma, \mathcal{L}(V, F)) \subset L_\infty(\gamma, \mathcal{L}(V, F^{**}))$. *Then for each* $x_0 = \xi \in H$ *and* $u \in \mathcal{U}_{ad}$ *the system* (10) *has at least one weak solution* $x \in B(I, H) \cap L_\infty(I, H) \cap L_p(\gamma, V)$. *The vector measure* $\mu_x$, *given by the relation*

$$\mu_x(\psi) \equiv \int_I \langle \psi(t), \mu_x(dt) \rangle_{V,V^*} = \int_I \langle \psi(t), dx(t) \rangle \forall\ \psi \in L_p(\gamma, V),$$

*is an element of* $BV_q(\gamma, V^*)$. *Further, if* $-f$ *is monotone, the solution is unique.*

***Proof.*** Let $\{v_i\}$ be a complete basis for the Gelfand triple $V \subset H \subset V^*$ so that they are orthogonal in $V$ and $V^*$, and orthonormal in $H$. The proof is based on similar arguments as found in [5, Theorem 5.3 and Theorem 6.2]. It is based on a-priori bounds, finite dimensional projection to an increasing family of (finite dimensional) subspaces determined by $X_n \equiv$ lin.span $\{v_i, 1 \le i \le n\}$, maximal monotonicity of the operator valued measure $A$, Crandall-Liggett generation theorem for nonlinear semigroups corresponding to maximal monotone operators [6, Theorem 4.7, p. 120–121], and continuity of $f$ and, most importantly, compact embedding [7, Theorem 3.2, p. 911] of $M_{p,q} \hookrightarrow L_p(\gamma, H)$ where

$$M_{p,q} \equiv \big\{x : x \in L_p(\gamma, V) \text{ and } \mu_x \in BV_q(\Sigma, V^*)\big\}. \tag{20}$$

Now we use the finite dimensional projection of the system (10) to $X_n$ and denote the corresponding solutions (if they exist) by $x_n \equiv \sum_{i=1}^n z_i^n v_i$ giving the family of finite dimensional systems,

$$\langle dx_n, v_i\rangle + \Big\langle A\Big(dt, \sum_{j=1}^n z_j^n v_j\Big), v_i\Big\rangle = \Big(f\Big(\sum_{j=1}^n z_j^n v_j\Big), v_i\Big)\gamma(dt) + \langle B^*(t)v_i, u(dt)\rangle, \tag{21}$$
$$i = 1, 2, \ldots, n;\ n \in N,$$

with the initial condition given by $x_n(0) \equiv \sum_{i=1}^n (\xi, v_i)v_i$. Since the embedding $V \hookrightarrow H$ is dense, it is clear that

$$x_n(0) \xrightarrow{s} \xi \quad \text{in} \ \ H. \tag{22}$$

Define the maps

$$\begin{aligned} &G_i(\sigma, z) \equiv \Big\langle A\Big(\sigma, \sum_{j=1}^n z_j v_j\Big), v_i\Big\rangle_{V^*,V}, \ \ f_i(z) \equiv \Big(f\Big(\sum_{j=1}^n z_j v_j\Big), v_i\Big)_H, \\ &\quad b_i(t) \equiv B^*(t)v_i,\ \sigma \in \Sigma,\ z \in R^n, \end{aligned} \tag{23}$$

$$\begin{aligned} &G(\sigma, z) \equiv col\,\{G_i(\sigma, z),\ 1 \le i \le n\},\ \tilde{f}(z) \equiv col\,\{f_i(z),\ 1 \le i \le n\}, \\ &\quad b(t) \equiv col\,\{b_i(t), 1 \le i \le n\},\ \sigma \in \Sigma,\ z \in R^n. \end{aligned} \tag{24}$$

Based on these notations, the system (21) takes the form,

$$\begin{aligned} &dz^n + G(dt, z^n) = \tilde{f}(z^n) + \langle b(t), u(dt)\rangle, \\ &z^n(0) = col\,\{(\xi, v_i),\ i = 1, 2, \ldots, n\},\ n \in N. \end{aligned} \tag{25}$$

For each $n \in N$, this is an $n$-dimensional system. Partition the interval $I$ into $m$ disjoint subintervals giving $I = \cup_{i=0}^{m-1}\sigma_i$ where $\sigma_i \equiv (t_i, t_{i+1}], 0 \le i \le m-1$, with $t_0 = 0, t_m = T$. Define the nonlinear operator valued set function $\hat{G}(\sigma)z \equiv G(\sigma, z)$ from $R^n$ to $R^n$. Since for each $\sigma \in \Sigma$, $A(\sigma, \cdot)$ is maximal monotone (from $V$ to $V^*$), $\hat{G}(\sigma)$ is maximal monotone on $R^n$. Hence the range of the operator $(I + \hat{G}(\sigma))$ is all of $R^n$, that is, $R(I + \hat{G}(\sigma)) = R^n$. Thus by use of the implicit difference scheme, one can construct the sequence $\{z_m(t), t \in I\}$ by linear interpolation of the nodes given by

$$\begin{aligned} &z_m(t_{i+1}) \equiv (I + \hat{G}(\sigma_i))^{-1}\big(z_m(t_i) + \tilde{f}(z_m(t_i))\gamma(\sigma_i) + \langle b(t_i), u(\sigma_i)\rangle\big), \\ &\qquad i = 0, 1, \ldots, m-1. \end{aligned} \tag{26}$$

Then it follows from the Crandal-Ligget generation theorem for nonlinear semigroups [6, Theorem 4.7, p. 120–121], that

$$z_m(t) \longrightarrow z(t) \text{ uniformly on } I$$

and that $z$ solves equation (25). We denote this solution by $z = z^n$ solving the $n$-dimensional problem (25). Thus $x_n \equiv \sum_{i=1}^n z_i^n v_i$ solves equation (21). By virtue of the a-priori bounds given by Lemma 4.1, $\{x_n\}$ is contained in a bounded subset of $L_p(\gamma, V) \cap B(I, H) \cap L_\infty(I, H)$. Hence there exist a subsequence, relabeled as the original sequence, and an element $x \in L_p(\gamma, V) \cap L_\infty(I, H) \cap B(I, H)$ such that

$$x_n \xrightarrow{w^*} x \text{ in } L_\infty(I, H) \tag{27}$$

$$x_n \xrightarrow{w} x \text{ in } L_p(\gamma, V). \tag{28}$$

Let $C_T^1[0, T]$ denote the class of $C^1$ functions vanishing at $T$. Multiplying equation (21) by $\varphi \in C_T^1[0, T]$ and integrating by parts we obtain

$$\begin{aligned} &-(x_n(0), \varphi(0)v_i) - \int_I (x_n(t), \dot{\varphi}(t)v_i)dt + \int_I \langle A(dt, x_n(t)), \varphi(t)v_i\rangle \\ &= \int_I (f(x_n(t)), \varphi(t)v_i)\gamma(dt) + \int_I \langle B^*(t)\varphi(t)v_i, u(dt)\rangle_{F,F^*}. \end{aligned} \tag{29}$$

Clearly, it follows from (22) and (27)that

$$\begin{aligned} &-(x_n(0), \varphi(0)v_i) - \int_I (x_n(t), \dot{\varphi}(t)v_i)dt \\ &\longrightarrow -(\xi, \varphi(0)v_i) - \int_I (x(t), \dot{\varphi}(t)v_i)dt. \end{aligned} \tag{30}$$

The convergence of the third term on the left and the first term on the right of equation (29) are nontrivial. Here we follow a similar approach as given in [5, Theorem 5.3 and Theorem 6.2]. First, consider the third term on the left. Define the sequence of vector measures,

$$a_n(\sigma) \equiv \int_\sigma A(ds, x_n(s)),\ \sigma \in \Sigma. \tag{31}$$

Clearly, it follows from the assumption (C3) and countable additivity of the measure $\gamma$ that for each $v \in V$ and $\sigma \in \Sigma$,

$$\langle a_n(\sigma), v \rangle_{V^*,V} \equiv \int_\sigma \langle A(ds, x_n(s)), v \rangle_{V^*,V}$$

is well defined and the set function $\sigma \longrightarrow \langle a_n(\sigma), v \rangle_{V^*,V}$ is countably additive. Thus $\{a_n\}$ is a sequence of weakly countably additive $V^*$ valued vector measures and that it vanishes on $\gamma$ null sets. Hence it follows from Pettis theorem [12, Theorem 1V.10.1, p. 318] that it is countably additive and $\gamma$ continuous. These facts along with the a-priori bounds of $\{x_n\}$ (see Lemma 4.1) imply that $\{a_n\}$ is contained in a bounded subset of $\mathcal{M}_c(\Sigma, V^*)$, and that

$$\lim_{\gamma(\sigma)\to 0} |a_n|(\sigma) = 0, \text{ uniformly in } n \in N. \tag{32}$$

Since $V^*$ is a reflexive Banach space, it follows from this (boundedness of the set $\{a_n\}$) that for each $\sigma \in \Sigma$, $\{a_n(\sigma), n \in N\}$ is a relatively weakly sequentially compact subset of $V^*$. Recall that $\{V, V^*\}$ are reflexive and hence they have the Radon-Nikodym property (RNP). Thus it follows from the Bartle-Dunford-Schwartz compactness theorem for vector measures [11, Theorem 5, p. 105] that there exists an $a \in \mathcal{M}_c(\Sigma, V^*)$ such that, along a subsequence if necessary, $a_n \longrightarrow a$ weakly. Hence for the third term on the left of the expression (29), we have

$$\int_I \langle A(dt, x_n(t)), \varphi(t)v_i \rangle \equiv \int_I \langle a_n(dt), \varphi(t)v_i \rangle \longrightarrow \int_I \langle a(dt), \varphi(t)v_i \rangle. \tag{33}$$

Now we use the monotonicity and hemicontinuity assumption (C1) to verify that

$$a(\sigma) = \int_\sigma A(ds, x(s)) \ \forall \ \sigma \in \Sigma. \tag{34}$$

By assumption (C1) we have

$$\int_I \langle A(dt, y(t)) - A(dt, x_n(t)), y(t) - x_n(t) \rangle_{V^*,V} \geq 0,$$

for all $y \in L_p(\gamma, V)$. By virtue of Mazur's theorem we can construct a sequence $y_n \in L_p(\gamma, V)$ from the sequence $\{x_n\}$ so that $y_n$ converges strongly to $x$ in $L_p(\gamma, V)$. Again by virtue of $A$ being monotone, we have

$$\int_I \langle A(dt, y(t)) - A(dt, y_n(t)), y(t) - y_n(t) \rangle_{V^*,V} \geq 0. \tag{35}$$

This inequality is valid for all $n \in N$ and hence the limit must also satisfy the inequality,

$$(36) \qquad \int_I \langle A(dt, y(t)) - a(dt), y(t) - x(t)\rangle_{V^*,V} \geq 0, \ \forall\, y \in L_p(\gamma, V).$$

For any $\theta > 0$, and $w \in L_p(\gamma, V)$ arbitrary, take $y = x + \theta w$. Using this $y$ in the above inequality we obtain,

$$(37) \quad \int_I \langle A(dt, x(t) + \theta w(t)) - a(dt), \theta w(t)\rangle_{V^*,V} \geq 0, \ \forall\, \theta > 0, \ w \in L_p(\gamma, V).$$

Dividing by $\theta$ and letting $\theta \downarrow 0$ in the above expression, it follows from the hemicontinuity of $A$ in its second argument that

$$(38) \qquad \int_I \langle A(dt, x(t)) - a(dt), w(t)\rangle_{V^*,V} \geq 0, \ \forall\, w \in L_p(\gamma, V).$$

Since $w \in L_p(\gamma, V)$ is arbitrary, it follows from (38) that

$$a(\sigma) = \int_\sigma A(ds, x(s)), \forall\, \sigma \in \Sigma.$$

This justifies (34). Thus we conclude that along a subsequence, if necessary,

$$(39) \quad \int_I \langle A(dt, x_n(t)), \varphi(t)v_i\rangle \equiv \int_I \langle a_n(dt), \varphi(t)v_i\rangle \longrightarrow \int_I \langle A(dt, x(t)), \varphi(t)v_i\rangle.$$

It remains to verify that the first term on the right hand side of equation (29) converges to the desired limit, that is,

$$(40) \qquad \int_I (f(x_n(t)), \varphi(t)v_i)\gamma(dt) \longrightarrow \int_I (f(x(t)), \varphi(t)v_i)\gamma(dt).$$

Here we use the compact embedding theorem [7, Theorem 3.2, p. 911]. Define the sequence of $V^*$ valued vector measure $\mu_{x_n}$ by

$$\mu_{x_n}(\psi) \equiv \int_I \langle \psi(t), \mu_{x_n}(dt)\rangle_{V,V^*} \equiv \int_I \langle \psi(t), dx_n(t)\rangle_{V,V^*}, \psi \in L_p(\gamma, V).$$

Since $\gamma$ is a countably additive bounded positive measure and $\{x_n\}$ is contained in a bounded subset of $B(I, H) \cap L_\infty(I, H) \cap L_p(\gamma, V)$, it follows from straightforward computation, using equation (21) and the assumptions (C3) and boundedness of $\mathcal{U}_{ad}$ as a subset of $BV_q(\gamma, F^*)$, that $\{\mu_{x_n}\} \subset BV_q(\gamma, V^*)$. Then it follows from the embedding $BV_q(\gamma, V^*) \hookrightarrow BV_q(\Sigma, V^*)$, as seen in section 2, that $\{\mu_{x_n}\} \subset BV_q(\Sigma, V^*)$. Thus by definition of $M_{p,q}$ (see (20)) we have $\{x_n\} \subset M_{p,q}$. Since the embedding $M_{p,q} \hookrightarrow L_p(\gamma, H)$ is compact we can extract a subsequence of the sequence $\{x_n\}$, relabeled as $\{x_n\}$, so that $x_n \xrightarrow{s} x$ in $L_p(\gamma, H)$. Hence there exists a subsequence of the sequence $\{x_n\}$, relabeled as $\{x_n\}$, such that

$$x_n(t) \xrightarrow{s} x(t) \text{ in } H \text{ for } \gamma - a.a \ \ t \in I.$$

By our assumption $f : H \longrightarrow H$ is continuous and hence

$$f(x_n(t)) \xrightarrow{s} f(x(t)) \text{ in } H \text{ for } \gamma - a.a \ \ t \in I.$$

From the growth assumption for $f$ (see equation (9)) and the fact that $\{x_n, x\}$ satisfy the same bounds as stated in Corollary 4.2, $\{f(x_n)\}$ is dominated by an element from $L_p(\gamma, H)$. In fact, it is $\gamma$-essentially bounded in $H$. Thus by the Lebesgue dominated convergence theorem,

$$f(x_n(\cdot)) \xrightarrow{s} f(x(\cdot)) \quad \text{in } L_p(\gamma, H)$$

and consequently

$$\int_I (f(x_n(t)), \varphi(t)v_i)_H \gamma(dt) \longrightarrow \int_I (f(x(t)), \varphi(t)v_i)\gamma(dt) \tag{41}$$

proving (40). Combining these results, in particular, (22), (30), (39) and (41), we arrive at the following identity

$$\begin{aligned} &- (\xi, \varphi(0)v_i) - \int_I (x(t), \dot{\varphi} v_i)dt + \int_I \langle A(dt, x(t)), \varphi(t)v_i\rangle_{V^*,V} \\ &= \int_I (f(x(t)), \varphi(t)v_i)\gamma(dt) + \int_I \langle B^*(t)\varphi(t)v_i, u(dt)\rangle_{F,F^*} \end{aligned} \tag{42}$$

for all $\varphi \in C^1_T(I)$ and for all $i \in N$. Since $\{v_i\}$ is a basis for $V$, this identity holds for all $v \in V$. This shows that $x$ is a weak solution of equation (10) satisfying all the properties as stated. Uniqueness of the solution follows from monotonicity of the operators $A$ and $-f$. This completes the proof. ■

The following result is useful for proving the existence of optimal controls.

**Corollary 4.4.** *Suppose the assumptions of Theorem* 4.3 *hold. Then the control to solution map $u \longrightarrow x$ is continuous in the sense that whenever $u_n \xrightarrow{w^*} u^o$ in $BV_q(\gamma, F^*)$, along a subsequence if necessary, $x_n \xrightarrow{w^*} x^o$ in $L_\infty(I, H)$ and $x_n \xrightarrow{w} x^o$ in $L_p(\gamma, V)$, where $x^o$ is the solution corresponding to $u^o$.*

***Proof.*** Let $\{u_n\} \subset BV_q(\gamma, F^*)$ with $\{x_n\}$ being the corresponding sequence of weak solutions of the system equation (10). Then by definition, for every $v \in V$ and $\varphi \in C^1_T(I)$, the following identity holds

$$\begin{aligned}
&- (\xi, \varphi(0)v) - \int_I (x_n(t), \dot{\varphi} v)dt + \int_I \langle A(dt, x_n(t)), \varphi(t)v\rangle_{V^*,V} \\
&= \int_I (f(x_n(t)), \varphi(t)v)\gamma(dt) + \int_I \langle B^*(t)\varphi(t)v, u_n(dt)\rangle_{F,F^*}.
\end{aligned} \tag{43}$$

Suppose $u_n \xrightarrow{w^*} u^o$ and let $x^o$ be the weak solution of (10) corresponding to the control $u^o$. Since $u_n \xrightarrow{w^*} u^o$, the set $\{u_n\}$ is contained in a bounded subset of $BV_q(\gamma, F^*)$. Thus by virtue of the a-priori bounds (see Corollary 4.2) the corresponding sequence of solutions $\{x_n\}$ is contained in a bounded subset of $L_\infty(I, H) \cap L_p(\gamma, V) \cap B(I, H)$. Hence there exists a subsequence of the sequence $\{x_n\}$, relabeled as $\{x_n\}$, and an element $x^o \in L_\infty(I, H) \cap L_p(\gamma, V) \cap B(I, H)$ such that

$$x_n \xrightarrow{w^*} x^o \quad \text{in } L_\infty(I, H) \tag{44}$$

$$x_n \xrightarrow{w} x^o \quad \text{in } L_p(\gamma, V). \tag{45}$$

Then by virtue of the same arguments as in the proof of Theorem 4.3, along a subsequence if necessary, we have

$$\int_I (x_n(t), \dot{\varphi}(t)v)_H \longrightarrow \int_I (x^o(t), \dot{\varphi}(t)v)_H \tag{46}$$

$$\int_I \langle A(dt, x_n(t)), \varphi(t)v\rangle_{V^*,V} \longrightarrow \int_I \langle A(dt, x^o(t)), \varphi(t)v\rangle_{V^*,V} \tag{47}$$

$$\int_I (f(x_n(t)), \varphi(t)v)_H \gamma(dt) \longrightarrow \int_I (f(x^o(t)), \varphi(t)v)_H \gamma(dt) \tag{48}$$

for every $v \in V$ and $\varphi \in C^1_T(I)$. Letting $n \longrightarrow \infty$ in (43), it follows from (46)–(48) that

$$
\begin{aligned}
&-(\xi,\varphi(0)v)-\int_I (x^o(t),\dot{\varphi}v)dt+\int_I \langle A(dt,x^o(t)),\varphi(t)v\rangle_{V^*,V} \\
(49)\qquad &=\int_I (f(x^o(t)),\varphi(t)v)\gamma(dt)+\int_I \langle B^*(t)\varphi(t)v,u^o(dt)\rangle_{F,F^*}.
\end{aligned}
$$

This shows that $x^o$ is a weak solution of equation (10) corresponding to the control $u^o$. This completes the proof. ∎

## 5. Existence of optimal controls

In this section, we wish to consider two typical control problems.

$$
(50)\qquad \mathbf{P1}: \quad J(u)=\Phi\big(\| x \|_{L_p(\gamma,V)}\big)+\varphi(u) \longrightarrow \inf,
$$

where both $\Phi$ and $\varphi$ are certain functions to be defined shortly. The infimum is taken over the class of admissible controls $\mathcal{U}_{ad}\subset BV_q(\gamma,F^*)$.
The second problem we consider is given by,

$$
(51)\qquad \mathbf{P2}: \quad J(u)\equiv \int_I \ell(t,x(t))\gamma(dt)+L(x(T))+\varphi(u) \longrightarrow \inf,
$$

where $\ell : I\times H \longrightarrow R$ and $L : H \longrightarrow R$, and $\varphi$ is a nonnegative function mapping $BV_q(\gamma,F^*)$ to the set of extended nonnegative real numbers $R_+\equiv [0,\infty]$. More specific examples of $\varphi$ are given later.

First we consider the problem (P1).

**Theorem 5.1.** *Consider the system* (10) *with the cost functional given by* (50). *Suppose $\mathcal{U}_{ad}$ is a $w^*$ sequentially compact subset of $BV_q(\gamma,F^*)$, the assumptions of Theorem* 4.2 *hold, $\Phi$ is a nondecreasing lower semicontinuous function from $[0,\infty]$ to $R$ satisfying $\Phi(s)\geq c>-\infty$ for all $s\in[0,\infty]$, and the functional $\varphi$ is $w^*$ lower semicontinuous on $BV_q(\gamma,F^*)$. Then there exists an optimal control.*

***Proof.*** Since $\mathcal{U}_{ad}$ is weak star compact, it suffices to verify that $J$ is weak star lower semicontinuous on it and bounded away from $-\infty$. Let $\{u_n\}\subset\mathcal{U}_{ad}$ and suppose $u_n \xrightarrow{w^*} u_o$, and let $\{x_n,x_o\}$ denote the corresponding (weak) solutions of the system (10). Since $\mathcal{U}_{ad}$ is bounded, it follows from Corollary 4.2 that $\{x_n\}$ is contained in a bounded subset of $B(I,H)\cap L_\infty(I,H)\cap$

$L_p(\gamma, V)$. Then by virtue of Corollary 4.4, along a subsequence if necessary, $x_n \xrightarrow{w} x_o$ in $L_p(\gamma, V)$ where $x_o$ is the solution corresponding to the control $u_o$. As an easy consequence of the Hahn-Banach theorem,

$$\| x_o \|_{L_p(\gamma,V)} \leq \liminf_{n\to\infty} \| x_n \|_{L_p(\gamma,V)} .$$

Since $\Phi$ is a nondecreasing lower semicontinuous function, it follows from the above inequality that

$$\Phi(\| x_o \|_{L_p(\gamma,V)}) \leq \liminf_{n\to\infty} \Phi(\| x_n \|_{L_p(\gamma,V)}).$$

Combining this with the assumption that $\varphi$ is weak star lower semicontinuous on $BV_q(\gamma, F^*)$, we have

$$J(u_o) \leq \liminf_{n\to\infty} J(u_n)$$

proving $w^*$ lower semicontinuity of $J$. Further, it follows from the lower bound of $\Phi$, that $\inf\{J(u), u \in \mathcal{U}_{ad}\} > -\infty$. Thus there exists an admissible control at which $J$ attains its infimum. ■

For the functional $\varphi$, one may like to choose $\varphi(u) = g(\| u \|_{BV_q(\gamma,F^*)})$ where $g$ is a nonnegative, nondecreasing scalar function possibly lower semicontinuous on $[0, \infty]$. Physical motivation for such a choice is that the norm functional in this case is a measure of oscillation of controls or, equivalently, frequency of changes in control. Frequent changes in control are costly and also undesirable from stability point of view and so must be contained. Unfortunately, $\varphi$ may not be $w^*$ lower semicontinuous even though $g$ is lower semicontinuous in its argument. This is because the norm functional $u \longrightarrow \| u \| \equiv N(u)$ is not always lower semicontinuous. However, by virtue of the theory of lifting (see [11], p. 115), every element of $BV_q(\gamma, F^*)$ admits a weak star density. That is for each $u \in BV_q(\gamma, F^*)$ there exists an $f \in L_q^w(\gamma, F^*)$ such that $du = f d\gamma$, where $L_q^w(\gamma, F^*)$ denotes the class of scalarly measurable functions whose scalar product with an element from $L_p(\gamma, F)$ is measurable and Lebesgue integrable. In case both $F$ and $F^*$ satisfy RNP (Radon-Nikodym Property), $f$ can be chosen from the strongly measurable class $L_q(\gamma, F^*)$ and the norm is given by

$$N(u) = \| f \|_{L_q(\gamma,F^*)} .$$

Thus $u \to f$ is an isomorphism and $BV_q(\gamma, F^*)$ and $L_q(\gamma, F^*)$ are isometrically isomorphic. If $F$ is reflexive, the dual of $L_q(\gamma, F^*)$ is precisely $L_p(\gamma, F)$. In this case $u \longrightarrow N(u)$ is weak star lower semicontinuous and consequently we can have

$$\varphi(u) \equiv g(\| u \|_{BV_q(\gamma,F^*)})$$

with any lower semicontinuous, nonnegative, nondecreasing function $g$.

Next we consider problem (P2). We prove the following result.

**Theorem 5.2.** *Consider the system* (10) *with the cost functional given by* (51). *Suppose $\mathcal{U}_{ad}$ is a $w^*$ sequentially compact subset of $BV_q(\gamma, F^*)$, the assumptions of Theorem* 4.3 *hold, $\ell : I \times H \longrightarrow \overline{R}$ is measurable in the first argument and lower semicontinuous in the second and there exist $h_0, h_1 \in L_1(\gamma)$ such that*

$$h_0(t) \leq \ell(t, \xi) \leq h_1(t) + c_1 |\xi|_H^p \quad \gamma - a.a \;\; t \in I$$

*and $L : H \longrightarrow R$ is weakly lower semicontinuous and there exists a constant $c_2 \in R$ such that $L(\xi) \geq c_2$ for all $\xi \in H$, and the functional $\varphi$ is $w^*$ lower semicontinuous on $BV_q(\gamma, F^*)$. Then there exists an optimal control.*

***Proof.*** We rewrite $J(u)$ as

$$J(u) \equiv \int_I \ell(t, x(t))\gamma(dt) + L(x(T)) + \varphi(u) \equiv J_1(u) + \varphi(u), \tag{52}$$

where $J_1(u)$ denotes the sum of the first two terms of the objective functional. Since $\varphi$ is $w^*$ (weak star) lower semicontinuous, it suffices to verify that $J_1$ is weak star lower semicontinuous and bounded away from $-\infty$. The fact that $J_1(u) > -\infty$ follows from the lower bounds of $\ell$ and $L$ as stated in the theorem. We prove $w^*$-lower semicontinuity. Let $u_n \xrightarrow{w^*} u_o$ in $BV_q(\gamma, F^*)$ and let $\{x_n, x_o\} \in L_\infty(I, H) \cap L_p(\gamma, V) \cap B(I, H)$ denote the corresponding sequence of (weak) solutions of the system (10). Then by Corollary 4.4, there exists a subsequence, relabeled as the original sequence, such that $x_n \xrightarrow{w} x_o$ in $L_p(\gamma, V)$. Again, by virtue of the compact embedding, $M_{p,q} \hookrightarrow L_p(\gamma, H)$, as stated in the proof of Theorem 4.3, we can extract a subsequence of the sequence $\{x_n\}$, relabeled as the original sequence, such that $x_n \xrightarrow{s} x_o$ in $L_p(\gamma, H)$. Hence one can extract a subsequence of this sequence that converges pointwise to the same limit for $\gamma$ almost all $t \in I$. Since $\ell$ is lower semicontinuous in the second argument on $H$, we have

$$\ell(t, x_o(t)) \leq \liminf_{n\to\infty} \ell(t, x_n(t)), \gamma - a.a.\ t \in I.$$

Hence by extended Fatou's lemma we have

$$\int_I \ell(t, x_o(t))\gamma(dt) \leq \liminf_{n\to\infty} \int_I \ell(t, x_n(t))\gamma(dt). \tag{53}$$

For the terminal cost, first note that by virtue of boundedness of the admissible controls $\mathcal{U}_{ad}$, it follows from (18) and (17) that $\{x_n(T)\}$ is contained in a bounded subset of $H$ which is weakly relatively compact. Thus along a subsequence, if necessary, it follows from weak lower semicontinuity of $L$ that we have

$$L(x_o(T)) \leq \liminf_{n\to\infty} L(x_n(T)). \tag{54}$$

Clearly, using a common subsequence if necessary, it follows from (53) and (54) that $u \to J_1(u)$ is $w^*$ lower semicontinuous. Since by our assumption, the map $u \longrightarrow \varphi(u)$ is weak star lower semicontinuous, we conclude that $u \longrightarrow J(u)$ is $w^*$ lower semicontinuous. Hence due to $w^*$-sequential compactness of the set $\mathcal{U}_{ad}$, $J$ attains its infimum on it. This proves the existence of an optimal control. ∎

## 6. An example

A classical example of a strongly nonlinear parabolic problem representing nonlinear diffusion (for example heat flow) with the homogeneous Dirichlet boundary condition is given by

$$\partial\psi(t,\xi)/\partial t - div\ \Phi(t, \nabla\psi) + c(t,\xi)\psi = h(t,\xi), (t,\xi) \in I \times \Omega,$$
$$\psi(0,\xi) = \phi(\xi), \xi \in \Omega,\ \psi|_{\partial\Omega}(t,\xi) = 0,\ \xi \in \partial\Omega,$$

where $\Omega$ is a bounded open connected domain with smooth boundary $\partial\Omega$. We are interested in the measure version of this example including controls. This occurs when the fundamental parameters are vector valued measures. In that case the model is written as

$$\partial\psi(t,\xi) - div\ \Phi(dt, \nabla\psi) + c(dt,\xi)\psi = div(b(t,\xi)u(dt,\xi)), (t,\xi) \in I \times \Omega,$$
$$\psi(0,\xi) = \phi(\xi), \xi \in \Omega,\ \psi|_{\partial\Omega}(t,\xi) = 0,\ \xi \in \partial\Omega, \tag{55}$$

where $c : \Sigma \times \Omega \longrightarrow R$, and $u : \Sigma \times \Omega \longrightarrow R$ are set functions in the first argument defined on $\Sigma \equiv \sigma(I)$ and measurable in the second, and $b : I \times \Omega \longrightarrow R^n$ is measurable. The operator $\Phi : \Sigma \times R^n \longrightarrow R^n$ is a set function with respect to the first argument and a point function in the second argument and continuous from $R^n$ to $R^n$ satisfying the following properties:

There exist two countably additive bounded nonnegative measures $\gamma(\cdot), \beta(\cdot)$ (not necessarily nonatomic) and nonnegative constants $c_1, c_2$ such that

(1): $(\Phi(\sigma,\zeta),\zeta) + \beta(\sigma)|\zeta|^2 \geq \gamma(\sigma)|\zeta|^p$, for all $\sigma \in \Sigma,\ \zeta \in R^n$

(2): $|\Phi(\sigma,\zeta)| \leq \gamma(\sigma)\{c_1 + c_2|\zeta|^{p-1}\}$, for all $\sigma \in \Sigma,\ \zeta \in R^n$

(3): $(\Phi(\sigma,\zeta) - \Phi(\sigma,\eta), \zeta - \eta) \geq 0$, for all $\sigma \in \Sigma,\ \zeta, \eta \in R^n$

(4): $c(\sigma,\xi) \geq \beta(\sigma)\ \forall\ \sigma \in \Sigma,\ \xi \in \Omega.$

Let $\{p,q\}$ be the conjugate pairs as defined in Section 4 and $W_0^{1,p}(\Omega), p \geq 2$, denote the standard $L_p$-Sobolev space with the dual $W^{-1,q}(\Omega)$. For this example, the appropriate vector spaces are $V \equiv W_0^{1,p}(\Omega)$ and $V^* \equiv W^{-1,q}(\Omega)$. Since $p \geq 2$ we can take $H \equiv L_2(\Omega)$. Thus we have the required Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ with continuous, dense and compact embeddings. By use of integration by parts, it is easy to verify that the operator $A$, subject to the homogeneous Dirichlet boundary condition, defined by

$$A(\sigma,\psi) \equiv -div\ \Phi(\sigma,\nabla\psi) + c(\sigma,\cdot)\psi \tag{56}$$

satisfies the following properties:

(a1): $A : \Sigma \times V \longrightarrow V^*$,

(a2): $\langle A(\sigma,w), w\rangle_{V,V^*} + 2\beta(\sigma)|w|_H^2 \geq \gamma(\sigma) \parallel w \parallel_V^p, \forall\ w \in V,\ \sigma \in \Sigma$,

(a3): there exists a nonnegative constant $c$, dependent on $c_1, c_2$ and the Lebesgue measure of the set $\Omega$, and the embedding constants $V \hookrightarrow H$ such that

$$\parallel A(\sigma,w) \parallel_{V^*} \leq c\gamma(\sigma)\{1 + \parallel w \parallel_V^{p/q}\}.$$

(a4): $\langle A(\sigma,w) - A(\sigma,v), w - v\rangle_{V^*,V} \geq 0\ \ \forall\ \ w, v \in V.$

Further, the reader can verify that $A$ is hemicontinuous from $V$ to $V^*$. Note that the operator $A$ defined above contains $p$-Laplacian as a special case.

The control operator is defined as follows:

$$B(t)u(\sigma) \equiv div\,\{b(t,\cdot)u(\sigma,\cdot)\},\ t \in I,\ \sigma \in \Sigma.$$

We assume that $b : I \times \Omega \longrightarrow R^n$, is bounded measurable and that

$$\hat{b} \equiv \sup\{|b(t,\xi)|_{R^n}, (t,\xi) \in I \times \Omega\} < \infty,$$

and choose $F^* \equiv L_q(\Omega)$. It is clear from this choice that $B(t) : F^* \longrightarrow V^*$ for all $t \in I$. Indeed, for any $\eta \in V \equiv W_0^{1,p}(\Omega)$, it follows from integration by parts that

$$\begin{aligned}\langle B(t)u(\sigma),\eta\rangle_{V^*,V} &= \int_\Omega div(b(t,\xi)u(\sigma,\xi))\ \eta(\xi)d\xi \\ &= -\int_\Omega (b(t,\xi)u(\sigma,\xi),\nabla\eta)_{R^n}d\xi.\end{aligned}$$

From this it follows that

$$\begin{aligned}|\langle B(t)u(\sigma),\eta\rangle_{V^*,V}| &\le \hat{b}\ \| u(\sigma) \|_{L_q(\Omega)} \| \eta \|_{W_0^{1,p}(\Omega)} \\ &\equiv \hat{b}\ \| u(\sigma) \|_{L_q(\Omega)} \| \eta \|_V,\ \forall\,\eta \in V.\end{aligned}$$

This verifies that

$$\| B(t)u(\sigma) \|_{V^*} \le \hat{b}\ \| u(\sigma) \|_{L_q(\Omega)}\ \ \forall\,\sigma \in \Sigma.$$

Thus, introducing the vector valued function $x(t) \equiv \psi(t,\cdot)$, the system given by equation (55) can be formulated in the abstract form

$$dx + A(dt,x) = B(t)u(dt), x_0 \equiv \phi(\cdot). \tag{57}$$

Assuming that $u \in BV_q(\gamma,F^*)$, the existence and regularity of solutions of the system (57) and hence those of the homogeneous Dirichlet initial boundary value problem (55) follow from Theorem 4.2.

An interesting control problem is: find a control from the set

$$\mathcal{U}_{ad} \equiv \{u \in BV_q(\gamma,F^*) : \| u \|_{BV_q(\gamma,F^*)} \le r < \infty\}$$

that minimizes the functional

$$J(u) \equiv \int_I |x(t) - x_d(t)|_H^p \gamma(dt) + \| u \|_{BV_q(\gamma,F^*)}, \tag{58}$$

where $x_d \in L_p(\gamma, H)$ is a given (desired) trajectory and $x$ is the weak solution of (57) corresponding to the control $u$. By virtue of the theory of lifting $BV_q(\gamma, F^*) \cong (L_p(\gamma, F))^* = L_q^w(\gamma, F^*)$ where $L_q^w(\gamma, F^*)$ is the class of scalarly measurable $F^*$ valued functions whose scalar product with elements from $L_p(\gamma, F)$ is Lebesgue integrable. Since for this example $F^* = L_q(\Omega)$, $1 < q < \infty$, it is a reflexive Banach space and hence it satisfies RNP( Radon-Nikodym Property) and so $L_q^w(\gamma, F^*) = L_q(\gamma, F^*)$. Further, each element of $BV_q(\gamma, F^*)$ admits a $w^*$ density in the sense that for each $u \in BV_q(\gamma, F^*)$ there exists a unique $f \in L_q(\gamma, F^*)$ such that $du = f d\gamma$. Thus the map $u \longrightarrow f$ is an isometric isomorphism and

$$\| u \|_{BV_q(\gamma,F^*)} = \| f \|_{L_q(\gamma,F^*)} .$$

Hence the set $\mathcal{U}_{ad}$ can be identified with the closed ball $B_r(L_q(\gamma, F^*))$ and the $BV_q(\gamma, F^*)$ norm is weak star lower semicontinuous. Clearly, this set is also weakly compact and since compactness is preserved under isomorphism, $\mathcal{U}_{ad}$ is $w^*$-compact. Thus the cost functional (58) satisfies all the assumptions of Theorem 5.2. Hence an optimal control exists. ∎

**Open Problems:** An interesting problem that remains to be solved is the development of necessary (and possibly sufficient) conditions of optimality. This will certainly require some stronger regularity properties for the maps $v \longrightarrow A(\sigma, v)$ from $V$ to $V^*$ and $f : H \longrightarrow H$.

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## References

[1] N.U. Ahmed, *Differential inclusions, operator valued measures and optimal control*, Special Issue of Dynamic Systems and Applications, Set-Valued Methods in Dynamic Systems, Guest Editors: M. Michta and J. Motyl **16** (2007), 13–36.

[2] N.U. Ahmed, *Evolution equations determined by operator valued measures and optimal control*, Nonlinear Analysis **67** (2007), 3199–3216.

[3] N.U. Ahmed, *Vector and operator valued measures as controls for infinite dimensional systems: optimal control*, Discuss. Math. Differential Inclusions Control and Optimization **28** (2008), 95–131.

[4] N.U. Ahmed, *A class of semilinear parabolic and hyperbolic systems determined by operator valued measures*, DCDIS **14** (4) (2007).

[5] N.U. Ahmed, *Parabolic systems determined by strongly nonlinear operator valued measures*, Nonlinear Analysis, Special Issue (Felicitation of Professor V. Lakshmikantham on his 85th birth date).

[6] N.U. Ahmed, Optimization and Identification of Systems Governed by Evolution Equations on Banach Spaces, Pitman Research Notes in Mathematics Series 184, Longman Scientific and Technical, U.K. and co-publisher John Wiley, New York, 1988.

[7] N.U. Ahmed, K.L. Teo and S.H. Hou, *Nonlinear impulsive systems on infinite dimensional spaces*, Nonlinear Analysis **54** (2003), 907–925.

[8] N.U. Ahmed, *Some remarks on the dynamics of impulsive systems in Banach spaces*, DCDIS **8** (2001), 261–274.

[9] N.U. Ahmed, *Impulsive perturbation of $C_0$-semigroups by operator valued measures*, Nonlinear Functional Analysis & Applications **9** (1) (2004), 127–147.

[10] N.U. Ahmed, *Existence of optimal controls for a general class of impulsive systems on Banach spaces*, SIAM J. Control. Optim. **42** (2) (2003), 669–685.

[11] J. Diestel and J.J. Uhl, Jr., Vector Measures, Mathematical Surveys, no. 15, American Mathematical Society, Providence, Rhode Island, 1977.

[12] N. Dunford and J.T. Schwartz, Linear Operators, Part 1: General Theory, Interscience Publishers, Inc., New York, London, 1958, 1964.

[13] H.O. Fattorini, Infinite Dimensional Optimization and Control Theory, Encyclopedia of Mathematics and its Applications, Vol. 62, Cambridge University, Cambridge, U.K., 1999.

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