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EXISTENCE RESULTS FOR DELAY SECOND ORDER DIFFERENTIAL INCLUSIONS

Dalila Azzam-Laouir and Tahar Haddad

Laboratoire de Mathématiques Pures et Appliquées
Université de Jijel, Algérie

e-mail: azzam_d@yahoo.com

Abstract

In this paper, some fixed point principle is applied to prove the existence of solutions for delay second order differential inclusions with three-point boundary conditions in the context of a separable Banach space. A topological property of the solutions set is also established.

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1. Introduction, notation and preliminaries

Let $(E, \|\cdot\|)$ be a separable Banach space with a topological dual E'. $\overline{\mathbf{B}}(0,\rho)$ is the closed ball of E of center 0 and radius $\rho > 0$. By $\mathbf{L}^1_E([0,1])$ we denote the space of all Lebesgue-Bochner integrable E-valued functions defined on $[0,1]$. Let $\mathbf{C}_E([0,1])$ be the space of all continuous mappings $u : [0,1] \to E$, endowed with the sup norm.

Recall that a mapping $v : [0,1] \to E$ is said to be scalarly derivable when there exists some mapping $\dot{v} : [0,1] \to E$ (called the weak derivative of v) such that, for every $x' \in E'$, the scalar function $\langle x', v(\cdot)\rangle$ is derivable and its derivative is equal to $\langle x', \dot{v}(\cdot)\rangle$. The weak derivative $\ddot{v}$ of $\dot{v}$ when it exists is the weak second derivative.

By $\mathbf{W}^{2,1}_E([0,1])$ we denote the space of all continuous mappings $u \in \mathbf{C}_E([0,1])$ such that their first usual derivatives $\dot{u}$ are continuous and scalarly derivable and such that $\ddot{u} \in \mathbf{L}^1_E([0,1])$.

For closed subsets A and B of E, the Hausdorff distance $\mathcal{H}(A,B)$ between A and B is defined by

$$\mathcal{H}(A,B) = \max\Big[\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\Big],$$

where

$$d(a,B) = \inf_{b\in B} \|a-b\|.$$

Let $r>0$ and θ be a given number in $[0,1[$. The aim of our paper is to provide existence of solutions for the second order delay-differential inclusion:

$$(\mathcal{P}_r)\begin{cases} \ddot{u}(t) \in F(t,u(t),u(h(t)),\dot{u}(t)), & a.e.\ t\in[0,1] \\ u(t) = \varphi(t), \quad \forall t\in[-r,0] \\ u(0)=0; \quad u(\theta)=u(1). \end{cases}$$

We consider $F:[0,1]\times E\times E\times E \rightrightarrows E$, $h:[0,1]\to[-r,1]$, $t-r\le h(t)\le t$, and $\varphi:[-r,0]\to E$. The given mappings h and φ are continuous and F is a convex compact valued multifunction Lebesgue-measurable on $[0,1]$ and upper semi-contiuous on $E\times E\times E$.

A solution u of $(\mathcal{P}_r)$ is a mapping $u:[-r,1]\to E$ satisfying $\ddot{u}(t)\in F(t,u(t),\ u(h(t)),\ \dot{u}(t))$ for almost every $t\in[0,1]$, $u(t)=\varphi(t)$, for all $t\in[-r,0]$ and $u(0)=0$; $u(\theta)=u(1)$, with $u\in\mathbf{X}:=\mathbf{C}_E([-r,1])\cap\mathbf{W}_E^{2,1}([0,1])$ equipped with the norm

$$\|u\|_{\mathbf{X}} = \max\Big\{\sup_{t\in[-r,1]}\|u(t)\|,\ \sup_{t\in[0,1]}\|\dot{u}(t)\|\Big\}.$$

In the second order evolution inclusions some related results are given in [1, 12, 15, 16, 17] and [18].

The existence of solutions for the second order delay differential problems have been discussed in the literature. For example, the problem described by the delay differential equation

$$\ddot{u}(t) = f(t,u(t),u(h(t)),\dot{u}(t)), \quad t\in[0,T]$$

with the boundary conditions

$$\begin{aligned} u(t) &= \varphi(t),\ \forall t\in[-r,0]; \\ u(T) &= B \end{aligned}$$

has been studied in [10] (see also the references therein). Another type of delay differential inclusions of the form

$$\dot{u}(t) \in H(t, \tau(t)u), \quad a.e.\ t \in [0,1]$$

with the boundary conditions

$$\begin{aligned} u(t) &= \varphi(t),\ \forall t \in [-r,0]; \\ u(0) &= u_0, \end{aligned}$$

where, for any $t \in [0,1]$, $\tau(t) : \mathbf{C}_E([-r,t]) \to \mathbf{C}_E([-r,0])$ is defined by $(\tau(t)u)(s) = u(t+s)$ for all $s \in [-r,0]$, $H : [0,1] \times \mathbf{C}_E([0,1]) \rightrightarrows \mathbb{R}^n$, has been studied among others in [6, 7, 8] and [13].

In this paper, we apply the multivalued analogue of Shaefer continuous principle to prove the existence of solutions to our problem $(\mathcal{P}_r)$. In particular, if F is uniformly Lipschitz in the sense

$$(*) \qquad \begin{aligned} &\mathcal{H}(F(t,x_1,y_1,z_1), F(t,x_2,y_2,z_2)) \\ &\le k_1\|x_1 - x_2\| + k_2\|y_1 - y_2\| + k_3\|z_1 - z_2\| \end{aligned}$$

where $k_1, k_2,\ k_3$ are positive constants satisfying $k_1 + k_2 + k_3 < 1$, then we show that the solution set of $(\mathcal{P}_r)$ is a retract of $\mathbf{X} := \mathbf{C}_E([-r,1]) \cap \mathbf{W}_E^{2,1}([0,1])$.

2. Existence result

In the sequel, we need the following results from [1]. See also [14] for the two point boundary value problems for second order differential equations.

Lemma 2.1. *Let E be a separable Banach space and let $G : [0,1]\times[0,1] \to \mathbb{R}$ be the function defined by the formula*

$$(1) \qquad G(t,s) = \begin{cases} -s & if \quad 0 \le s \le t, \\ -t & if \quad t < s \le \theta, \\ t(s-1)/(1-\theta) & if \quad \theta < s \le 1; \end{cases}$$

for $0 \le t < \theta$ and by

$$(2) \qquad G(t,s) = \begin{cases} -s & if \quad 0 \le s < \theta, \\ (\theta(s-t) + s(t-1))/(1-\theta) & if \quad \theta \le s \le t, \\ t(s-1)/(1-\theta) & if \quad t < s \le 1; \end{cases}$$

for $\theta \le t \le 1$.

Then the following assertions hold.

1) *If* $u \in \mathbf{W}_E^{2,1}([0,1])$ *with* $u(0)=0$ *and* $u(\theta)=u(1)$*, then*

$$u(t) = \int_0^1 G(t,s)\ddot{u}(s)ds, \forall t \in [0,1]. \tag{3}$$

2) $G(\cdot,s)$ *is derivable on* $[0,1]$*, for every* $s \in [0,1]$*, its derivative is given by the formula*

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} 0 & if \quad 0 \le s \le t, \\ -1 & if \quad t < s \le \theta, \\ (s-1)/(1-\theta) & if \quad \theta < s \le 1; \end{cases} \tag{4}$$

for $0 \le t < \theta$ *and by*

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} 0 & if \quad 0 \le s < \theta, \\ (s-\theta)/(1-\theta) & if \quad \theta \le s \le t, \\ (s-1)/(1-\theta) & if \quad t < s \le 1; \end{cases} \tag{5}$$

for $\theta \le t \le 1$.

3) $G(\cdot,\cdot)$ *and* $\frac{\partial G}{\partial t}(\cdot,\cdot)$ *satisfies*

$$\sup_{t,s\in[0,1]} |G(t,s)| \le 1, \quad \sup_{t,s\in[0,1]} \left|\frac{\partial G}{\partial t}(t,s)\right| \le 1. \tag{6}$$

4) *For* $f \in \mathbf{L}_E^1([0,1])$ *and for the mapping* $u_f : [0,1] \to E$ *defined by*

$$u_f(t) = \int_0^1 G(t,s)f(s)ds, \forall t \in [0,1], \tag{7}$$

one has $u_f(0)=0$ *and* $u_f(\theta)=u_f(1)$.

Further, the mapping u_f *is derivable, and its derivative* $\dot{u}_f$ *satisfies*

$$\lim_{h\to 0} \frac{u_f(t+h)-u_f(t)}{h} = \dot{u}_f(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds \tag{8}$$

for all $t \in [0,1]$*. Consequently,* $\dot{u}_f$ *is a continuous mapping from* $[0,1]$ *into* E*.*

5) *The mapping* $\dot{u}_f$ *is scalarly derivable, that is, there exists a mapping* $\ddot{u}_f : [0,1] \to E$ *such that, for every* $x' \in E'$*, the scalar function* $\langle x', \dot{u}_f(\cdot)\rangle$ *is derivable with* $\frac{d}{dt}\langle x', \dot{u}_f(t)\rangle = \langle x', \ddot{u}_f(t)\rangle$*; further*

$$\ddot{u}_f = f \quad a.e. \text{ on } [0,1]. \tag{9}$$

Proposition 2.1. *Let* E *be a separable Banach space and let* $f : [0,1] \to E$ *be a continuous mapping (respectively a mapping in* $\mathbf{L}^1_E([0,1])$*). Then the mapping*

$$u_f(t) = \int_0^1 G(t,s)f(s)ds, \forall t \in [0,1]$$

is the unique $\mathbf{C}^2_E([0,1])$*-solution (respectively* $\mathbf{W}^{2,1}_E([0,1])$*-solution) to the differential equation*

$$\begin{cases} \ddot{u}(t) = f(t) \;\; \forall t \in [0,1]; \\ u(0) = 0; \; u(\theta) = u(1). \end{cases}$$

We also need the following fixed point theorem which is the multivalued analogue of the Shaefer continuation principle. For more details for the fixed point theory we refer the reader to [11].

Theorem 2.1. *Let* Y *be a normed linear space and* $A : Y \to 2^Y$ *an upper semicontinuous compact multivalued operator with compact convex values. Suppose that there exists an* $R > 0$ *such that the a priori estimate*

$$x \in \lambda A x \;\; (0 < \lambda \leq 1) \Rightarrow \|x\| \leq R$$

holds. Then A *has a fixed point in the ball* $\overline{\mathbf{B}}(0,R)$*.*

Now, we are ready to prove our main existence theorem.

Theorem 2.2. *Let* E *be a separable Banach space,* $F : [0,1]\times E\times E\times E \rightrightarrows E$ *be a convex compact valued multifunction, Lebesgue-measurable on* $[0,1]$ *and upper semicontinuous on* $E\times E\times E$*. We assume that* $F(t,x,y,z) \subset \Gamma(t)$ *for all* $(t,x,y,z) \in [0,1]\times E\times E\times E$*, for some convex norm-compact valued, and measurable multifunction* $\Gamma : [0,1] \rightrightarrows E$ *which is integrably bounded, that is, there exists a function* $k \in \mathbf{L}^1_{\mathbb{R}}([0,1])$ *such that* $\|v\| \leq |k(t)|$ *a.e.* $t \in [0,1]$ *for all* $v \in \Gamma(t)$*. Let* $h : [0,1] \to [-r,t]$ *be a continuous mapping and* $\varphi \in \mathbf{C}_E([-r,0])$ *with* $\varphi(0) = 0$*. Then the boundary value problem* $(\mathcal{P}_r)$ *has at least one solution in* $\mathbf{X} := \mathbf{C}_E([-r,1]) \cap \mathbf{W}^{2,1}_E([0,1])$*.*

Proof. We transform the problem $(\mathcal{P}_r)$ into a fixed point inclusion in the Banach space $\mathbf{X}$. By Lemma 2.1 and Proposition 2.2, the existence solution of $(\mathcal{P}_r)$ is equivalent to the problem of finding $u \in \mathbf{X}$ such that

$$(10) \qquad \begin{cases} u(t) \in \displaystyle\int_0^1 G(t,s)F(t,u(s),u(h(s)),\dot{u}(s))ds, \quad \forall t \in [0,1] \\ u(t) = \varphi(t), \quad \forall t \in [-r,0]. \end{cases}$$

Define the operator $\mathcal{A}$ on $\mathbf{X}$ by

$$(11) \qquad \begin{aligned} \mathcal{A}u = \{v \in \mathbf{X}/\ v = \varphi \ \text{ on } [-r,0] \ \text{ and} \\ v(t) = \int_0^1 G(t,s)f(s)ds, \ \forall t \in [0,1], \ f \in \mathbf{S}_F^1(u)\} \end{aligned}$$

where

$$(12) \qquad \begin{aligned} &\mathbf{S}_F^1(u) = \\ &= \left\{\vartheta \in \mathbf{L}_E^1([0,1])/\ \vartheta(t) \in F(t,u(t),u(h(t)),\dot{u}(t)), \quad \text{a.e.} \quad t \in [0,1]\right\}. \end{aligned}$$

Then, the integral inclusion (10) is equivalent to the operator inclusion

$$(13) \qquad u(t) \in \mathcal{A}u(t), \quad \forall\, t \in [-r,1].$$

It is clear that $\mathcal{A}$ has its values in $\mathbf{X}$, using Lemma 2.1 and the assumption $\varphi(0) = 0$.

Step 1. First, let us recall that the set $\mathbf{S}_\Gamma^1$ of all measurable selections of Γ is included in $\mathbf{L}_E^1([0,1])$ and it is convex and compact for the weak topology $\sigma(\mathbf{L}_E^1([0,1]), \mathbf{L}_{E'}^\infty([0,1]))$. Furthermore, the set-valued integral

$$\int_0^1 \Gamma(t)dt = \left\{\int_0^1 f(t)dt, \quad f \in \mathbf{S}_\Gamma^1\right\}$$

is convex and norm-compact. (See [4, 5, 9] for a more general result). On the other hand, let us observe that, for any Lebesgue measurable mappings $u, w : [0,1] \to E$, $v : [-r,1] \to E$, there is a Lebesgue-measurable selection $s \in \mathbf{S}_\Gamma^1$ such that $s(t) \in F(t,u(t),v(h(t)),w(t))$ a.e. Indeed, there exist sequences (u_n), (v_n) and (w_n) of simple E-valued mappings which converge pointwise to u, v and w respectively, for E endowed with the norm topology. Notice that the multifunctions $F(.,u_n(.),v_n(h(.)),w_n(.))$ are Lebesgue-measurable. In view of the existence theorem of measurable selection (see [9]), for each n, there is a Lebesgue-measurable selection s_n of $F(.,u_n(.),v_n(h(.)),w_n(.))$. As $s_n(t)\in F(t,u_n(t),v_n(h(t)),w_n(t))\subset\Gamma(t)$, for all $t\in[0,1]$ and as $\mathbf{S}^1_\Gamma$ is weakly compact in $\mathbf{L}^1_E([0,1])$, by Eberlein-Smŭlian theorem, we may extract from (s_n) a subsequence (s'_n) which converges $\sigma(\mathbf{L}^1_E([0,1]),\mathbf{L}^\infty_{E'}([0,1]))$ to a mapping $s\in S^1_\Gamma$. An application of the Banach-Mazur's trick to (s'_n) provides a sequence (z_n) with $z_n\in co\{s_k: k\ge n\}$ such that (z_n) converges pointwise almost everywhere to s. Using this fact and the pointwise convergence of the sequences (u_n), (v_n) and (w_n) and the upper semicontinuity of $F(t,.,.,.)$ it is not difficult to see that $s(t)\in F(t,u(t),v(h(t)),w(t))$ a.e. Consequently, $\mathbf{S}^1_F(u)\neq\emptyset$ for all $u\in\mathbf{X}$. This shows that $\mathcal{A}$ is well defined.

Step 2. In this step we will show that the multivalued operator $\mathcal{A}$ satisfies all the conditions of Theorem 2.1. Clearly, $\mathcal{A}u$ is convex for each $u\in\mathbf{X}$. First, we show that $\mathcal{A}$ has compact values on $\mathbf{X}$. For each $u\in\mathbf{X}$, let (v_n) be a sequence in $\mathcal{A}u$, then by (11), for every n there exists $f_n\in\mathbf{S}^1_F(u)\subset\mathbf{S}^1_\Gamma$ such that

$$v_n(t)=\int_0^1 G(t,s)f_n(s)ds,\quad \forall t\in[0,1]$$

and $v_n(t)=\varphi t)$ for all $t\in[-r,0]$. Since $\mathbf{S}^1_\Gamma$ is weakly compact in $\mathbf{L}^1_E([0,1])$, we may extract from (f_n) a subsequence (that we do not relabel) converging $\sigma(\mathbf{L}^1_E,\mathbf{L}^\infty_{E'})$ to a mapping $f\in\mathbf{S}^1_\Gamma$. Since $F(t,.,.,.)$ is upper semicontinuous and has convex compact values, we get $f(t)\in F(t,u(t),u(h(t)),\dot u(t))$ for almost every $t\in[0,1]$. In particular, for every $x'\in E'$ and for every $t\in[0,1]$, we have

$$\begin{aligned}\lim_{n\to\infty}\langle x',\int_0^1 G(t,s)f_n(s)ds\rangle &= \lim_{n\to\infty}\int_0^1\langle G(t,s)x',f_n(s)\rangle ds\\ &=\int_0^1\langle G(t,s)x',f(s)\rangle ds\\ &=\langle x',\int_0^1 G(t,s)f(s)ds\rangle.\end{aligned}\tag{14}$$

As the set-valued integral $\int_0^1 G(t,s)\Gamma(s)ds$ $(t\in[0,1])$ is norm compact, (14) shows that the sequence $(v_n(.))=(\int_0^1 G(\cdot,s)f_n(s)ds)$ converges pointwise to $v(.) = \int_0^1 G(., s)f(s)ds$, for E endowed with the strong topology. At this point, it is worth mentioning that the sequence $(\dot{v}_n(.)) = (\int_0^1 \frac{\partial G}{\partial t}(., s)f_n(s)ds)$ converges pointwise to $\dot{v}(.)$, for E endowed with the strong topology, using as above the weak convergence of (f_n) and the norm compactness of the set-valued integral $\int_0^1 \frac{\partial G}{\partial t}(t, s)\Gamma(s)ds$. Hence (v_n) converges in $\mathbf{X}$ to a mapping w where

$$w(t) = \int_0^1 G(t, s)f(s)ds, \quad \forall t \in [0, 1]$$

and $w(t) = \varphi(t)$ for all $t \in [-r, 0]$. This says that $\mathcal{A}u$ is compact in $\mathbf{X}$.

Next, we show that $\mathcal{A}$ is a compact operator, that is, $\mathcal{A}$ maps bounded sets into relatively compact sets in $\mathbf{X}$. Let S be a bounded set in $\mathbf{X}$ and let $u \in S$, for each $v \in \mathcal{A}u$ there exists $f \in \mathbf{S}^1_F(u)$ such that

$$v(t) = \int_0^1 G(t, s)f(s)ds, \quad \forall t \in [0, 1]$$

and $v(t) = \varphi(t)$ for all $t \in [-r, 0]$. Observe that for all $t, t' \in [0, 1]$

$$\begin{aligned} \|v(t) - v(t')\| &\leq \int_0^1 |G(t, s) - G(t', s)| \, \|f(s)\| ds \\ &\leq \int_0^1 |G(t, s) - G(t', s)| \, |k(s)| ds, \end{aligned}$$

and

$$\|\dot{v}(t) - \dot{v}(t')\| \leq \int_0^1 \left| \frac{\partial G}{\partial t}(t, s) - \frac{\partial G}{\partial t}(t', s) \right| |k(s)| ds.$$

The function G is continuous on the compact set $[0, 1] \times [0, 1]$, so it is uniformly continuous there. In addition, $k \in \mathbf{L}^1_{\mathbb{R}}([0, 1])$, then, the right-hand side of the above inequalities tends to 0 as $t \to t'$. We conclude that $\mathcal{A}(S)$ and $\{\dot{v} : \ v \in \mathcal{A}(S)\}$ are equicontinuous in $\mathbf{C}_E([0, 1])$. Since $\varphi \in \mathbf{C}_E([-r, 0])$ we get the equicontinuity of $\mathcal{A}(S)$ in $\mathbf{X}$. Further, for each $t \in [-r, 1]$ and each $\tau \in [0, 1]$, the sets $\mathcal{A}(S)(t) = \{v(t)/ \ v \in \mathcal{A}(S)\}$ and $\{\dot{v}(\tau)/ \ v \in \mathcal{A}(S)\}$ are relatively compact in E because they are included in the norm compact sets $\int_0^1 G(t, s)\Gamma(s)ds$ and $\int_0^1 \frac{\partial G}{\partial t}(t, s)\Gamma(s)ds$, respectively. An application of the Arzelà-Ascoli theorem implies that $\mathcal{A}(S)$ is relatively compact in $\mathbf{X}$ and hence $\mathcal{A}$ is compact.

Next, we prove that the graph of $\mathcal{A}$, $\mathrm{gph}(\mathcal{A}) = \{(u,v) \in \mathbf{X}\times\mathbf{X} / \ v \in \mathcal{A}u\}$ is closed. Let (u_n, v_n) be a sequence of $\mathrm{gph}(\mathcal{A})$ converging uniformly to $(u,v) \in \mathbf{X}\times\mathbf{X}$ with respect to $\|\cdot\|_{\mathbf{X}}$. Since $v_n \in \mathcal{A}u_n$, for each n there exists $f_n \in \mathbf{S}^1_F(u_n) \subset \mathbf{S}^1_\Gamma$ such that

$$v_n(t) = \int_0^1 G(t,s)f_n(s)ds, \quad \forall t \in [0,1]$$

and $v_n(t) = \varphi(t)$ for all $t \in [-r,0]$. As $\mathbf{S}^1_\Gamma$ is weakly compact in $\mathbf{L}^1_E([0,1])$, we may extract from (f_n) a subsequence (that we do note relabel) converging $\sigma(\mathbf{L}^1_E, \mathbf{L}^\infty_{E'})$ to a mapping $f \in \mathbf{S}^1_\Gamma$.

Observe that $f_n(t) \in F(t, u_n(t), u_n(h(t)), \dot{u}_n(t))$. Since $\|u_n - u\|_{\mathbf{X}} \to 0$ and $F(t,.,.,.)$ is upper semicontinuous on $E\times E\times E$ with convex compact values we conclude that $f(t) \in F(t, u(t), u(h(t)), \dot{u}(t))$, using a closure type theorem (see [9]). Equivalently, $f \in \mathbf{S}^1_F(u)$. On the other hand, repeating the arguments given above, it is not difficult to see that the sequence $(v_n(.)) = (\int_0^1 G(.,s)f_n(s)ds)$ converges pointwise to $\int_0^1 G(.,s)f(s)ds$ and that the sequence $(\dot{v}_n(.)) = (\int_0^1 \frac{\partial G}{\partial t}(.,s)f_n(s)ds)$ converges pointwise to $\int_0^1 \frac{\partial G}{\partial t}(.,s)f(s)ds$, for E endowed with the strong topology. As (v_n) converges to v in $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ we get

$$v(t) = \int_0^1 G(t,s)f(s)ds, \quad \forall t \in [0,1]$$

and $v(t) = \varphi(t)$ for all $t \in [-r,0]$. This shows that $\mathcal{A}$ has a closed graph and hence it is an upper semicontinuous operator on $\mathbf{X}$. Finally, we show that there exists an $R > 0$ such that the a priori estimate

$$u \in \lambda\mathcal{A}u \ \ (0 < \lambda \le 1) \Rightarrow \|u\| \le R$$

holds. We have

$$u \in \lambda\mathcal{A}u \ \ \Leftrightarrow \ \ \text{there exists } f \in \mathbf{S}^1_F(u) \subset \mathbf{S}^1_\Gamma$$

such that

$$\begin{cases} u(t) = \lambda \displaystyle\int_0^1 G(t,s)f(s)ds, & \forall t \in [0,1] \\ u(t) = \lambda\varphi(t), & \forall t \in [-r,0]. \end{cases}$$

For each $t \in [0,1]$, using relation (6) and the assumption over Γ, we have

$$\begin{aligned} \|u(t)\| &\le \int_0^1 |G(t,s)|\,\|f(s)\|ds, \\ &\le \int_0^1 |k(s)|ds = \|k\|_{\mathbf{L}^1_{\mathbb{R}}([0,1])} \end{aligned}$$

and

$$\|\dot{u}(t)\| \le \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right| \|f(s)\|ds \le \|k\|_{\mathbf{L}^1_{\mathbb{R}}([0,1])}.$$

On the other hand, for each $t \in [-r,0]$ we have

$$\|u(t)\| = \|\lambda\varphi(t)\| \le \|\varphi\|_{\mathbf{C}_E([-r,0])}.$$

Taking the above inequalities into account, we obtain

$$\|u\|_{\mathbf{X}} \le \max\left(\|k\|_{\mathbf{L}^1_{\mathbb{R}}([0,1])}, \|\varphi\|_{\mathbf{C}_E([-r,0])}\right) = R.$$

Hence by the conclusion of Theorem 2.1, $\mathcal{A}$ has a fixed point in the ball $\overline{\mathbf{B}}(0,R)$, what, in turn, means that this point is a solution in $\mathbf{X}$ to our boundary value problem $(\mathcal{P}_r)$. ■

To end the paper, we prove below that under suitable Lipschitz assumption on the second member, the solution set of $(\mathcal{P}_r)$ is a retract of $\mathbf{X}$. Compare with Theorem 1 in [2], and Theorem 5 in [12] in which the authors deal with nonconvex differential inclusions and Theorem 2 in [2] in the convex case. See also [3].

Theorem 2.3. *Under the hypotheses of Theorem 2.2, if we replace the upper semicontinuity assumption on $F(t,\cdot,\cdot,\cdot)$ by the condition*

$$(*) \qquad \begin{aligned} &\mathcal{H}(F(t,x_1,y_1,z_1), F(t,x_2,y_2,z_2)) \\ &\le k_1\|x_1-x_2\| + k_2\|y_1-y_2\| + k_3\|z_1-z_2\| \end{aligned}$$

for all $(t,x_1,y_1,z_1),(t,x_2,y_2,z_2) \in [0,1]\times E\times E\times E$, where k_1,k_2,k_3 are positive constants satisfying $k_1+k_2+k_3<1$. Then the solution set of the problem $(\mathcal{P}_r)$ is a retract of $\mathbf{X}$.

Proof. The idea of proof comes from ([2], Theorem 2). Let us denote by $\mathcal{X}(\varphi)$ the solution set of $(\mathcal{P}_r)$. By virtue of the proof of Theorem 2.2, $u \in \mathcal{X}(\varphi)$ iff $u \in \mathcal{A}u$. Let us prove that $\mathcal{A}$ is a contraction. Let $u_1, u_2 \in \mathbf{X}$ and $v_1 \in \mathcal{A}u_1$, then $v_1 = \varphi$ on $[-r, 0]$ and there exists $f_1 \in \mathbf{S}_F(u_1)$ such that $v_1(t) = \int_0^1 G(t,s) f_1(s) ds$, for all $t \in [0,1]$. We have $f_1(t) \in F(t, u_1(t), u_1(h(t)), \dot{u}_1(t))$, as F is compact valued and $F(\cdot, u_2(\cdot), u_2(h(\cdot)), \dot{u}_2(\cdot))$ is measurable, the multifunction H defined from $[0,1]$ into E by

$$H(t) = \big\{w \in F(t, u_2(t), u_2(h(t)), \dot{u}_2(t)) : \\ \|f_1(t) - w\| = d(f_1(t), F(t, u_2(t), u_2(h(t)), \dot{u}_2(t)) \ a.e\big\}$$

is also measurable with nonempty closed values. In view of the existence theorem of measurable selections (See [9]), there is a measurable mapping $f_2 : [0,1] \to E$ such that $f_2(t) \in H(t)$ for all $t \in [0,1]$. This yields $f_2(t) \in F(t, u_2(t), u_2(h(t)), \dot{u}_2(t))$ and

$$\|f_1(t) - f_2(t)\| = d(f_1(t), F(t, u_2(t), u_2(h(t)), \dot{u}_2(t))) \quad a.e \text{ on } [0,1]. \tag{15}$$

Let us define the mapping v_2 on $[-r, 1]$ by

$$v_2(t) = \begin{cases} \varphi(t), & \forall t \in [-r, 0] \\ \displaystyle\int_0^1 G(t,s) f_2(s) ds, & \forall t \in [0,1]. \end{cases}$$

Clearly, $v_2 \in \mathcal{A}u_2$. For every $t \in [0,1]$ we have

$$\|v_1(t) - v_2(t)\| = \left\| \int_0^1 G(t,s)(f_1(s) - f_2(s)) ds \right\| \leq \int_0^1 \|f_1(s) - f_2(s)\| ds.$$

From this, (15) and the assumption $(*)$, for every $t \in [0,1]$ we obtain

$$\begin{aligned} &\|v_1(t) - v_2(t)\| \\ &\leq \int_0^1 \|f_1(s) - f_2(s)\| ds \\ &= \int_0^1 d(f_1(s), F(s, u_2(s), u_2(h(s)), \dot{u}_2(s))) ds \end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 \mathcal{H}(F(s,u_1(s),u_1(h(s)),\dot u_1(s)),F(s,u_2(s),u_2(h(s)),\dot u_2(s)))ds\\
&\le \int_0^1 (k_1\|u_1(s)-u_2(s)\|+k_2\|u_1(h(s))-u_2(h(s))\|+k_3\|\dot u_1(s)-\dot u_2(s)\|)ds\\
&\le \int_0^1 (k_1\|u_1-u_2\|_{\mathbf{X}}+k_2\|u_1-u_2\|_{\mathbf{X}}+k_3\|u_1-u_2\|_{\mathbf{X}})ds\\
&= (k_1+k_2+k_3)\|u_1-u_2\|_{\mathbf{X}}.
\end{aligned}
$$

Consequently,

$$
\|v_1-v_2\|_{\mathbf{C}_E([-r,1])}=\|v_1-v_2\|_{\mathbf{C}_E([0,1])}\le (k_1+k_2+k_3)\|u_1-u_2\|_{\mathbf{X}}. \tag{16}
$$

On the other hand, using Lemma 2.1, we have

$$
\|\dot v_1(t)-\dot v_2(t)\|=\left\|\int_0^1 \frac{\partial G}{\partial t}(t,s)(f_1(s)-f_2(s))ds\right\|\le \int_0^1 \|f_1(s)-f_2(s)\|ds.
$$

By repeating the same arguments, we obtain

$$
\|\dot v_1-\dot v_2\|_{\mathbf{C}_E([0,1])}\le (k_1+k_2+k_3)\|u_1-u_2\|_{\mathbf{X}}. \tag{17}
$$

The inequalities (16) and (17) give

$$
\|v_1-v_2\|_{\mathbf{X}}\le (k_1+k_2+k_3)\|u_1-u_2\|_{\mathbf{X}}.
$$

Then we get

$$
d(v_1,\mathcal{A}u_2)=\inf_{v_2\in\mathcal{A}u_2}\|v_1-v_2\|_{\mathbf{X}}\le (k_1+k_2+k_3)\|u_1-u_2\|_{\mathbf{X}},
$$

and

$$
\sup_{v_1\in\mathcal{A}u_1} d(v_1,\mathcal{A}u_2)\le (k_1+k_2+k_3)\|u_1-u_2\|_{\mathbf{X}}.
$$

By similar computations and by interchanging the role of u_1 and u_2, we have

$$\sup_{v_2 \in \mathcal{A}u_2} d(v_2, \mathcal{A}u_1) \leq (k_1 + k_2 + k_3)\|u_1 - u_2\|_{\mathbf{X}}.$$

Hence we obtain

$$\mathcal{H}(\mathcal{A}u_1, \mathcal{A}u_2) \leq (k_1 + k_2 + k_3)\|u_1 - u_2\|_{\mathbf{X}}$$

with $(k_1 + k_2 + k_3) < 1$, proving that $\mathcal{A}$ is a contraction in $\mathbf{X}$. By a result of Ricceri [19], the set

$$\mathrm{Fix}(\mathcal{A}) = \{u \in \mathbf{X} : \quad u \in \mathcal{A}u\}$$

is a retract of $\mathbf{X}$. It is clear that $\mathrm{Fix}(\mathcal{A}) = \mathcal{X}(\varphi)$. This shows that the solution set of $(\mathcal{P}_r)$ is a retract of $\mathbf{X}$ and the proof of the theorem is complete. ■

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