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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF DIFFERENCE EQUATIONS IN BANACH SPACES

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Abstract

In this paper we consider the first order difference equation in a Banach space

$$\Delta x_n = \sum_{i=0}^{\infty} a_n^i f(x_{n+i}).$$

We show that this equation has a solution asymptotically equal to a.

As an application of our result we study the difference equation

$$\Delta x_n = \sum_{i=0}^{\infty} a_n^i g(x_{n+i}) + \sum_{i=0}^{\infty} b_n^i h(x_{n+i}) + y_n$$

and give conditions when this equation has solutions.

In this note we extend the results from [8, 9]. For example, in [9] the function f is a real Lipschitz function. We suppose that f has values in a Banach space and satisfies some conditions with respect to the measure of noncompactness and measure of weak noncompactness.

Keywords: Banach space, difference equation, fixed point, measure of noncompactness, asymptotic behaviour of solutions.

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1. Introduction

Let c be the set of complex numbers. Let R be the set of real numbers and $l_1(C)$ the space of complex valued sequences (c_n) such that:

$$\|(c_n)\|_1 := \sum_{i=1}^{\infty} |c_n| < \infty.$$

Let $(X, \|\cdot\|)$ be a complex or real Banach space and $l_\infty(X)$ denote the space of bounded sequences $x = (x_n)$ in X with the norm

$$\|x\|_\infty = \|(x_n)\|_\infty = \sup_n \|x_n\|.$$

With this norm $l_\infty(X)$ is a Banach space.

Consider the first order difference equation:

$$(*) \qquad \Delta x_n = \sum_{i=0}^{\infty} a_n^i f(x_{n+i}),$$

where $\Delta x_n = x_{n+1} - x_n$ denotes the difference operator, the coefficients a_n^i are complex numbers and f is a function from X to X.

By a solution of equation $(*)$ we understand a sequence $x = (x_n)$ in $l_\infty(X)$ which satisfies the equation $(*)$.

The first order difference equation $(*)$ was considered by C. Gonzalez and A. Jimenez-Melado [9]. They gave sufficient conditions on the coefficients a_n^i and on the function f so that this equation has solutions asymptotically constant. In [9] $f : X \to X$ was a Lipschitz function.

In this note, the results from [9] are extended, the Lipschitz function is replaced by a continuous mapping, which is condensing with respect to the measure of noncompactness.

2. Main results

We give necessary and sufficient conditions for the existence of asymptotically constants solutions.

For each set of coefficients $\{a_n^i\}_{n,i\in N}$ we can create a new set $\{\alpha_n^j\}_{n,j\in N}$ such that

$$\alpha_n^j = \sum_{k=0}^{j} a_{n+k}^{j-k} = a_{n+j}^0 + a_{n+j-1}^1 + a_{n+j-2}^2 + \ldots + a_n^j, \quad n, j \in N.$$

Our result will be proved by the following fixed point theorem

Theorem 1.1 [5]**.** *Let D be a nonempty, closed, convex and bounded subset of a Banach space.*

Let $F : D \to D$ be a continuous mapping, which is condensing with respect to the measure of noncompactness α, i.e.

$$(**) \qquad \alpha(F(V)) \leq L\alpha(V), \quad L < 1.$$

Then F has a fixed point, where α is the Kuratowski's measure of noncompactness (*see* [4]).

We shall use the following properties of the measure of noncompactness α.

Theorem 1.2. *Let A, B be bounded subsets of a Banach space X. Then*

(a) $\alpha(A) = 0 \Rightarrow A$ *is a relatively compact subset of* X

(b) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$

(c) $\alpha(k \cdot A) = |k| \cdot \alpha(A)$

(d) $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$

(e) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$

(f) $\alpha(\operatorname{conv} A) = \alpha(A)$.

For the properties of the measure of noncompactness, see [3] and [4].

Theorem 1.3. *Let $V \subset C(N^+, X)$ be a family of functions. Then*

$$\alpha(V) = \alpha(V(N^+)) = \sup\{\alpha(V(i)) : i \in N^+\}$$

where $\alpha(V)$ denotes the measure of noncompactness in $C(N^+, X)$.

A similar theorem as Theorem 1.1 was proved by O. Arino, S. Gautier, J.P. Penot [1] (see also [10]) when f is weakly-weakly sequentially continuous, i.e., if $x_n \xrightarrow{w} x_0$ (weakly), then $f(x_n) \xrightarrow{w} f(x_0)$ for each sequence (x_n).

Theorem 1.4. *Let $f : X \to X$ be a bounded and continuous function. Let $a \in X$ and for each $n \in N, \alpha_n \in l_1(C)$ and $\|\alpha_n\|_1 \to 0$ as $n \to \infty$. We assume that $k_1 = \sup_{n \geq n_0}\{\sum_{j=0}^{\infty} \|\alpha_n^j\|\}$, $\|f(h_{n+j})\| \leq k$, $k_1 < \frac{1}{k}$ for some $n_0 \in N$ and $n > n_0$. Moreover,*

$$\alpha(F(V)) \leq k\alpha(V)$$

for any bounded subset of X.

Then for each $a \in X$ there exists a solution $x = (x_n) \in l_\infty(X)$ of the equation $(*)$ such that $x_n \to a$.

Proof. Define the operator $T : D \to D$, where $D = \{y : y = (y_1, y_2, \ldots), \|y - b\|_\infty \le kk_1\}, b = (a, a, a, \ldots)$.

For $h \in D$, let $Th = (Th)_n$ be given by

$$(Th)_n = \begin{cases} a & \text{if} \quad n \le n_0 \\ a - \sum_{j=0}^{\infty} \alpha_n^j f(h_{n+j}) & \text{if} \quad n > n_0\,. \end{cases}$$

For $n \le n_0$ and for $x = (x_1, x_2, \ldots) = (a, a, a, \ldots)$ we have

$$\|Th - x\|_X = 0.$$

For $n > n_0$ w have

$$\begin{aligned} \|Th_n - x_n\| &= \Big\|a - \sum_{j=0}^{\infty} \alpha_n^j f(h_{n+j}) - a\Big\| = \Big\|\sum_{j=0}^{\infty} \alpha_n^j f(h_{n+j})\Big\| \\ &\le \sum_{j=0}^{\infty} \|\alpha_n^j\|\,\|f(h_{n+j})\| \le k \sum_{j=0}^{\infty} \|\alpha_n^j\| \le kk_1. \end{aligned}$$

So the operator $T : D \to D$.

Now, we show that the operator T is continuous. Let $h_n = (h_n^1, h_n^2, \ldots)$, $h_0 = (h_0^1, h_0^1, \ldots)$ and $h_n \to h_0$ if $n \to \infty$. Then

$$\|Th_n - Th_0\| = \sup_i \|Th_n^i - Th_0^i\| = 0 \quad \text{for} \ \ n \le n_0.$$

For $n > n_0$ we have:

$$\begin{aligned} \|Th_n - Th_0\| &= \sup_i \|Th_n^i - Th_0^i\| \\ &\le \sup_i \sum_{j=0}^{\infty} \|\alpha_n^j\|\,\|f(h_{n+j}^i) - f(h_j^i)\| \\ &\le \sup_i \sum_{j=0}^{s} \|\alpha_n^j\|\,\|f(h_{n+j}^i) - f(h_j^i)\| \\ &\quad + \sup_i \sum_{j=s+1}^{\infty} \|\alpha_n^j\|\,\|f(h_{n+j}^i)\| + \sup_i \sum_{j=s+1}^{\infty} \|\alpha_n^j\|\,\|f(h_j^i)\|. \end{aligned}$$

Because

$$\sup_{n\geq n_0}\sum_{j=0}^{\infty}\|\alpha_n^j\| = k_1,$$

f is continuous and bounded by k so $Th_n \to Th_0$ if $n\to\infty$.

Now, we will prove that T satisfies condition $(**)$ of Theorem 1.1. Let $V\subset D$, where $V=\{v : v=(v_1,v_2,\ldots)\}$ and $T(V)=\{T(v) : v\in V\}$. Let $V_k=\{v_k : v\in V, v=(v_1,v_2,\ldots,v_k,\ldots)\}$. For $n\leq n_0$, we obtain:

$$\alpha(T(V))=\alpha(a)=0.$$

For $n>n_0$, by Theorem 1.3 we have

$$\begin{aligned}\alpha(T(V)) &= \sup_n \alpha\left(\left\{a-\sum_{j=0}^{\infty}\alpha_n^j f(V_n)\right\}\right)\\ &\leq \sup_n\left[\left\{\alpha(a)+\alpha\left(\sum_{j=0}^{\infty}\alpha_n^j f(V_n)\right)\right\}\right]\\ &\leq \sup_n \alpha(a)+\sup_n \alpha\left(\left\{\sum_{j=0}^{\infty}\alpha_n^j f(V_n)\right\}\right)\\ &\leq \sup_n\sum_{j=0}^{\infty}\alpha(\alpha_n^j f(V_n))\,.\end{aligned}$$

So:

$$\begin{aligned}\alpha(T(V)) &\leq \sup_n\sum_{j=0}^{\infty}\alpha(\alpha_n^j f(V_n))\leq\sum_{j=0}^{\infty}\alpha(\alpha_n^j f(V))\\ &\leq \sum_{j=0}^{\infty}|\alpha_n^j|\alpha(f(V)).\end{aligned}$$

Using the inequality $\alpha(f(V))\leq k\alpha(V)$ we obtain

$$\alpha(T(V))\leq k\alpha(V)\sum_{j=0}^{\infty}|\alpha_n^j|\leq kk_1\alpha(V),$$

where $kk_1<1$.

By Theorem 1.1 T has a fixed point.

Now, we show that the fixed point of the operator T is asymptotically equal to a.

We know that $Th = h$.

For $n > n_0$, we obtain

$$\begin{aligned}\Delta h_n &= (Th)_{n+1} - (Th)_n \\ &= -\sum_{j=0}^{\infty} \alpha_n^j f(h_{n+j}) + \sum_{j=0}^{\infty} \alpha_{n+1}^j f(h_{n+1+j}).\end{aligned}$$

Using the equation:

$$\alpha_n^j = a_{n+j}^0 + a_{n+j-1}^0 + a_{n+j-2}^0 + \cdots + a_n^j, \quad n, j \in N$$

we have:

$$\begin{aligned}\Delta h_n &= h_{n+1} - h_n = (Th)_{n+1} - (Th)_n \\ &= a_n^0 f(h_n) + \sum_{j=0}^{\infty} \alpha_n^{j+1} f(h_{n+1+j}) - \sum_{j=0}^{\infty} \alpha_{n+1}^j f(h_{n+1+j}) \\ &= a_n^0 f(h_n) + \sum_{j=0}^{\infty} (\alpha_n^{j+1} - \alpha_{n+1}^j) f(h_{n+1+j}) \\ &= a_n^0 f(h_n) + \sum_{j=0}^{\infty} a_n^{j+1} f(h_{n+j+1}) \\ &= \sum_{j=0}^{\infty} a_n^j f(h_{n+j}),\end{aligned}$$

which implies that $h = (h_n)$ satisfies the difference equations $(*)$.

And by the definition of T the solution $h_n \to a$. ■

We can generalize Theorem 1.1 for measure of the weak noncompactness β.

Let $B = \{x \in X : \|x\| \leq 1\}$ and let V be a bounded subset of X. The $\beta(V)$ measure of weak noncompactness $\beta(V)$ of V is defined by:

$$\beta(V) = \inf\{t \geq 0 : V \subset K + tB \quad \text{for some weakly compact} \quad K \subset X\}.$$

Properties of the measure of weak noncompactness are analogously to properties of the measure of noncompactness [4, 7]. We shall use the following properties of the measure of weak noncompactness β.

Theorem 1.5 (see [4, 7]). *Let A, B be bounded subsets of X. Then*

(a) $\beta(A) = 0 \Rightarrow A$ *is a relatively weakly compact subset*

(b) $\beta(\overline{\text{conv}}\, A) = \beta(A)$

(c) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$

(d) $\beta(kA) = k\beta(A)$ *for* $k \in (0, \infty)$

(e) $\beta(\{x_0\} \cup A) = \beta(A)$.

Theorem 1.6 [11]. *Let $V \subset C(N^+, X)$ be a family of functions. Then*

$$\beta(V) = \beta(V(N^+)) = \sup\{\beta(V(i)) : i \in N^+\},$$

where $\beta(V)$ denotes the measure of weak noncompactness in $C(N^+, X)$.

Theorem 1.7 (see [1, 10]). *Let D be a nonempty, weakly closed, convex and bounded subset of a Banach space X. Let $F : D \to D$ be a weakly-weakly, sequentially continuous mapping, which is condensing with respect to the measure of weak noncompactness β, i.e.,*

$$\beta(F(V)) \leq k\beta(V)$$

for $\beta(V) > 0, V \in D$, then F has a fixed point.

Similary as Theorem 1.4, we can prove the following theorem:

Theorem 1.8. *Let f be a bounded and weakly-weakly, sequentially continuous function. Let $a \in X$ and $\|\alpha_n\|_1 \to 0$ as $n \to \infty$, where $(\alpha_n) \in l_1(C)$. We assume that $k_1 = \sup_{n \geq n_0} \{\sum_{j=0}^{\infty} \|\beta_n^j\|\}$ and $\|f(h_{n+j})\| \leq k$ and $k_1 < \frac{1}{k}$, for some $n_0 \in N$. Moreover,*

$$\beta(f(V)) \leq k\beta(V).$$

Then for each $a \in X$ there exists a solution $x = (x_n) \in l_\infty(X)$ of the equation $(*)$ such that $x_n \to a$.

As an application of our result we consider the difference equation (see [8])

$$\Delta x_n = \sum_{i=0}^{\infty} a_n^i g(x_{n+i}) + \sum_{i=0}^{\infty} b_n^i h(x_{n+i}) + y_n,$$

where $\sum_{i=0}^{\infty}|a_n^i| < \infty$ and $\sum_{i=0}^{\infty}|b_n^i| < \infty$ for $i = 0, 1, 2, 3, \dots, r$ and $\sum_{n=0}^{\infty} y_n$ is convergent.

Remark 1. We will show that for this equation the condition $(**)$ of Theorem 1.1 are satisfied. Assume that

$$G(x) = \sum_{i=0}^{\infty} a_n^i g(x_{n+i}) \quad \text{and} \quad H(x) = \sum_{i=0}^{\infty} b_n^i h(x_{n+i}).$$

If

$$\|Gx_1 - Hx_2\| \le k\|x_1 - x_2\|$$

then (see [3])

$$\alpha(G(V)) \le k\alpha(V).$$

Let $F = G + H$, where $G : C \to X$ is a Lipschitz mapping and $H : C \to X$ is a compact operator and C is a nonempty, bounded set in a Banach space.

Because $\alpha(H(V)) = 0$ so we obtain

$$\alpha((G + H)(V)) = \alpha(G(V) + H(V)) \le \alpha(G(V)) + \alpha(H(V)) \le k\alpha(V).$$

So, G satisfies the condition $(**)$ of Theorem 1.1.

Remark 2. Observe that the class of continuous functions is different from the class of weakly-weakly sequentially continuous and weakly-weakly continuous functions.

There exist many important examples of mappings, which are weakly sequentially continuous but not weakly continuous.

The relationship between strong-weak and weak sequential continuity for mappings is studied in [2].

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