

INDEPENDENT CYCLES AND PATHS IN BIPARTITE BALANCED GRAPHS

BEATA ORCHEL AND A. PAWEŁ WOJDA

Faculty of Applied Mathematics
AGH University of Science and Technology
Al. Mickiewicza 30, 30–059 Kraków, Poland

e-mail: orchel@uci.agh.edu.pl

e-mail: wojda@uci.agh.edu.pl

Abstract

Bipartite graphs $G = (L, R; E)$ and $H = (L', R'; E')$ are bi-placeable if there is a bijection $f : L \cup R \rightarrow L' \cup R'$ such that $f(L) = L'$ and $f(u)f(v) \notin E'$ for every edge $uv \in E$. We prove that if G and H are two bipartite balanced graphs of order $|G| = |H| = 2p \geq 4$ such that the sizes of G and H satisfy $\|G\| \leq 2p - 3$ and $\|H\| \leq 2p - 2$, and the maximum degree of H is at most 2, then G and H are bi-placeable, unless G and H is one of easily recognizable couples of graphs.

This result implies easily that for integers p and k_1, k_2, \dots, k_l such that $k_i \geq 2$ for $i = 1, \dots, l$ and $k_1 + \dots + k_l \leq p - 1$ every bipartite balanced graph G of order $2p$ and size at least $p^2 - 2p + 3$ contains mutually vertex disjoint cycles $C_{2k_1}, \dots, C_{2k_l}$, unless $G = K_{3,3} - 3K_{1,1}$.

Keywords: bipartite graphs, bi-placing, path, cycle.

2000 Mathematics Subject Classification: 05C38, 05C35.

1. PRELIMINARIES

Let $G = (L, R; E)$ and $G' = (L', R'; E')$ be two bipartite graphs. $|G|$ denotes the order of G and by $\|G\|$ its size ($|G| = |L \cup R|$, $\|G\| = |E|$). $\Delta_R(G)$ is the maximum vertex degree $d_G(x)$, when $x \in R$ and $\Delta_L(G)$ the maximum degree $d(y, G)$ when $y \in L$. The maximum vertex degree in G is denoted

by $\Delta(G)$ ($\Delta(G) = \max\{\Delta_L(G), \Delta_R(G)\}$). The corresponding minimum degrees are denoted by $\delta_R(G), \delta_L(G)$ and $\delta(G)$, respectively. A vertex x with $d(x, G) = 1$ is said to be *pendent*. The set $L(G) = L$ is called the *left hand side set*, and $R(G) = R$ the *right hand side set* of bipartition of the vertex set $V(G) = L \cup R$.

For $x \in V(G)$, $N(x; G)$ denotes the set of the neighbors of the vertex x in G . C_k denotes a cycle of the length k .

G is called (p, q) -*bipartite* if $|L(G)| = p$ and $|R(G)| = q$. If $p = q$ then G is said to be *balanced*. $K_{p,q}$ stands for the complete bipartite graph with $|L(K_{p,q})| = p$ and $|R(K_{p,q})| = q$.

Bi-placement of G and G' is a bijection $f : L \cup R \rightarrow L' \cup R'$ such that $f(L) = L'$ and $f(u)f(v) \notin E'$ for every edge $uv \in E$. If there is a bi-placement of G and G' then we say that G and G' are *bi-placeable*.

Note that the bipartite graphs $H = (\{a, b\}, \{c, d, e\}; \{ac, ad, be\})$ and $H' = (\{a', b'\}, \{c', d', e'\}; \{a'c', b'c'\})$ are not bi-placeable, while it is very easy to find a bi-placement of H and $H'' = (\{a'', b''\}, \{c'', d'', e''\}; \{a''c'', a''d''\})$ (see Figure 1).

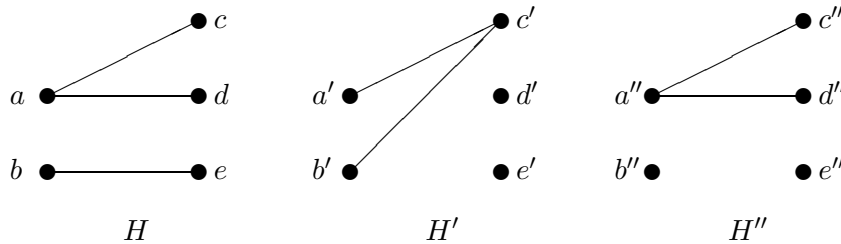


Figure 1. H bi-placeable with H'' and non bi-placeable with H' .

The notion of bi-placeability of bipartite graphs appeared in [7]. To say that G and G' are *bi-placeable* is equivalent to saying that the bipartite graph $G = (L, R; E)$ is a subgraph of the bipartite graph $\overline{G'} = (L, R; \overline{E'})$ in the sense of [4] ($\overline{E'} = \{xy : x \in L', y \in R', xy \notin E'\}$). The problem of existence of a matching or a hamiltonian cycle in a bipartite graph is, in fact, a problem of bi-placeability of some bipartite graphs. For a survey of results concerning placing of graphs and bi-placing of bipartite graphs we refer the reader to [3, 11] or [12].

The following theorem was proved in [9].

Theorem 1. *Let $G = (L, R; E)$ and $H = (L', R', ; E')$ be two bipartite balanced graphs of order $2p$ such that $\| G \| \leq p - 1$ and $\| H \| \leq 2p$. Then G and H are bi-placeable unless $\| G \| = p - 1$, $\| H \| = 2p$ and either*

- $\Delta_L(G) \leq 1$ and $H = K_{2,p} \cup K_{p-2,0}$ or
- $\Delta_R(G) \leq 1$ and $H = K_{p,2} \cup K_{0,p-2}$ or
- $G = K_{1,p-1} \cup \overline{K_{p-1,1}}$ and $\Delta_L(H) = 2$ or else
- $G = K_{p-1,1} \cup \overline{K_{1,p-1}}$ and $\Delta_R(H) = 2$.

$G = (L, R; E)$ is said to be $2k$ freely cyclable whenever, for any sequence k_1, \dots, k_l of integers such that $k_i \geq 2$ for $i = 1, \dots, l$ and $k_1 + \dots + k_l \leq k$, G contains mutually vertex disjoint cycles $C_{2k_1}, \dots, C_{2k_l}$. The problem of the existence of a union of independent cycles of prescribed lengths in a graph was considered by many authors (see [1, 5, 6, 8, 10]).

Theorem 1 implies easily the following generalisation of a result of Amar, Fournier and Germa (Theorem 2 in [2]).

Theorem 2. *Let $G = (L, R; E)$ be a bipartite balanced graph of order $2p$ and size at least $p^2 - p + 1$. Then G is $2p$ freely cyclable unless $\| G \| = p^2 - p + 1$ and G contains a pendent vertex.*

In the next section we give a sufficient condition for a (p, p) -bipartite graphs to be $2(p - 1)$ freely cyclable. Namely, we shall prove that the only balanced bipartite graph of order $2p$ and size at least $p^2 - 2p + 3$ which is not $2(p - 1)$ freely cyclable is $K_{3,3}$ minus a perfect matching.

2. RESULTS

Theorem 3. *Let $p \geq 2$ be an integer, and let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, p) -bipartite graphs such that $\| G \| \leq 2p - 3$, $\| H \| \leq 2p - 2$ and $\Delta(H) \leq 2$. Then G and H are bi-placeable unless one of the following occurs:*

- (1) $\Delta_L(G) = p$ and $\delta_L(H) > 0$,
- (2) $\Delta_R(G) = p$ and $\delta_R(H) > 0$,
- (3) $p = 3$, G is a perfect matching $3K_{1,1}$, and $H = K_{2,2} \cup \overline{K_{1,1}}$ (see Figure 2),
- (4) $p = 6$, $G = K_{3,3} \cup \overline{K_{3,3}}$, $H = C_8 \cup 2K_{1,1}$ (see Figure 3).

The couples of graphs G and H described in (1), (2), (3) and (4) will be called *exceptional* or *exceptions* (1), (2), (3) and (4), respectively.

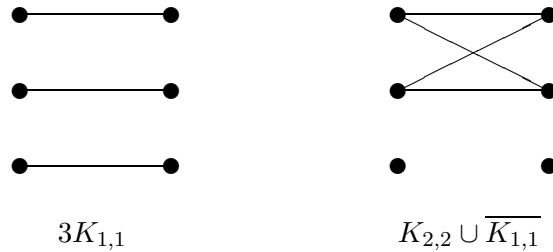


Figure 2. Exceptional couple (3).

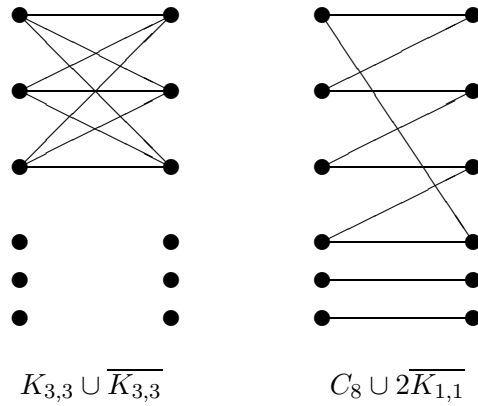


Figure 3. Exceptional couple (4).

Theorem 3 implies easily the following corollary announced already at the end of Section 1.

Corollary 4. *Let G be a balanced bipartite graph of order $|G| = 2p$ and size $\|G\| \geq p^2 - 2p + 3$. Then G is $2(p - 1)$ freely cyclable unless $p = 3$ and $G = K_{3,3} - 3K_{1,1} = C_6$. ■*

For $p \geq 3$ the graph $H_{p,p} = K_{2,p-1} \cup \overline{K_{p-2,1}}$ is (p, p) -bipartite of order $2p$ and size $2p - 2$ which is not bi-placeable with any union of vertex disjoint cycles $C_{2l_1} \cup \dots \cup C_{2l_q}$, where $l_1 + \dots + l_q = p - 1$ and $l_1, \dots, l_q \geq 2$. Hence

Theorem 3 may not be improved by a simple rising the size of the graph G . The graph $\overline{H_{p,p}}$ (the complement of $H_{p,p}$ in $K_{p,p}$) proves that also Corollary 4 is sharp.

3. PROOF OF THEOREM 3

The proof is by induction on p . It is easy to verify that the theorem holds for $p = 2, 3$. Suppose that $p \geq 4$ and the theorem holds for p' provided that $2 \leq p' < p$. Note that without loss of generality, we may assume that $\|G\| = 2p - 3$, $\|H\| = 2p - 2$ and $\Delta(H) = 2$. Then the graph H is a union of a number of (even) cycles and exactly two (possibly trivial) paths. Moreover, since H is balanced, either both paths have odd lengths (even order) or each path has an even length. In the later case, if the end vertices of one path are in L' , then the end vertices of the second paths are in R' and vice versa.

We shall consider two cases and several subcases.

Case 1. There is an isolated vertex z in the set $V(H)$.

Without loss of generality we may assume that $z \in R'$. Let x be a vertex of minimal degree in L . It follows immediately that $d(x, G) \leq 1$.

Subcase 1.1. x is an isolated vertex.

Let $y \in R$, $d(y, G) = \Delta_R(G)$, $w \in L'$ and $d(w, H) = 2$. If the graphs $G' = G - \{x, y\}$ and $H' = H - \{w, z\}$ are m.p. then a bi-placement of the graphs G and H is obvious. Hence, we may suppose that the couple G' and H' is one of the exceptions (1)–(4). Note that $H' \neq C_4 \cup \overline{K_{1,1}}$, and that w may be choosen in such a way that $H' \neq C_8 \cup 2K_{1,1}$. Hence we have only two subcases to consider.

Subcase 1.1a. $\Delta_R(G') = p - 1$ and $\delta_R(H') \geq 1$.

There is a vertex $y' \in R(G')$ such that $d(y', G') = p - 1$. Hence $d(y, G) = p - 1$ and we have $e(G) \geq 2p - 2$, a contradiction.

Subcase 1.1b. $\Delta_L(G') = p - 1$ and $\delta_L(H') \geq 1$.

Let $x_1 \in L(G')$ and $d(x_1, G') = p - 1$. If $d(x_1, G) = p$ then the couple G and H form the first exception (1). If $d(x_1, G) = p - 1$ then we can choose the following vertices: y_1 – a pendent vertex in R , w_1 – a pendent vertex in L' . Let $z_1 \in N(w_1, H)$. Then $d(z_1, H) = 2$ and the graphs $G'' = G - \{x, y, x_1, y_1\}$

and $H'' = H - \{w_1, w_2, z, z_1\}$, where w_2 is the second neighbour of z_1 , are bi-placeable by Theorem 1.

Let f be a bi-placement of G'' and H'' . Then we can extend f to a packing f_* of G and H by letting $f_*(v) = f(v)$, for $v \in V(H'')$, $f_*(w_1) = x_1$, $f_*(w_2) = x$, $f_*(z) = y_1$ and $f_*(z_1) = y$.

Subcase 1.2. $d(x, G) = 1$ and the neighbor y of x is not pendent ($d(y, G) \geq 2$).

So, we can apply the induction hypothesis to the graphs $G'_1 = G - \{x, y\}$ and H'_1 , where H'_1 is the graph H' defined in Subcase 1.1. If G'_1 and H'_1 are bi-placeable then it is easy to check that G and H are bi-placeable too.

So we may suppose that the couple G'_1, H'_1 is one of the exceptions. Note that since $\delta_L(G) > 0$, we have $G'_1 \neq K_{3,3} \cup \overline{K_{3,3}}$, and since $\Delta(H) = 2$, we have $H'_1 \neq C_4 \cup \overline{K_{1,1}}$. Hence G'_1 and H'_2 may be the only one of exceptions (1)–(2).

Subcase 1.2a. $\Delta_R(G'_1) = p - 1$ and $\delta_R(H'_1) \geq 1$.

Let $y_1 \in R(G'_1)$, $d(y_1, G) = p - 1$, $x_1 \in N(y_1, G)$ and $d(x_1, G) = 1$. Observe that we may apply the induction hypothesis to the graphs $G_2 = G - \{x_1, y_1\}$ and $H_2 = H'$. From this, we can now map the vertex w to the vertex x_1 and the vertex z to y_1 and we can extend a bi-placement of G_2 and H_2 to a bi-placement of G and H .

Subcase 1.2b. $\Delta_L(G'_1) = p - 1$ and $\delta_L(H'_1) \geq 1$.

Since $\Delta_L(G) \geq p - 1$ and $\delta_L(G) \geq 1$, we have $\|G\| \geq 2p - 2$, a contradiction.

Subcase 1.3. There is no isolated vertex in L and the neighbors of pendent vertices of L are pendent.

Let xy be an isolated edge of G , $x \in L, y \in R$.

Subcase 1.3.1. There is an isolated vertex w in L' .

Note that $H - \{w, z\}$ is a union of vertex disjoint even cycles. Let $x' \in L - \{x\}$ and $y' \in R' - \{y\}$ be chosen in such a way that the sum of degrees $d(x', G) + d(y', G)$ is maximum. One may check easily that $d(x', G) + d(y', G) \geq 4$. Since $p \geq 4$, there exist two nonadjacent vertices $w' \in L' - \{w\}$ and $z' \in R' - \{z\}$. Observe that $d(w', H) = d(z', H) = 2$. The graphs $G'_3 = G - \{x, y, x', y'\}$ and $H'_3 = H - \{w, z, w', z'\}$ verify the induction hypothesis. Moreover, an easy computation shows that $\Delta(G'_3) < p - 2$. It is also clear that $H'_3 \neq K_{2,2} \cup \overline{K_{1,1}}$ and $H'_3 \neq C_8 \cup 2\overline{K_{1,1}}$. Hence there is a bi-placement,

say f , of H'_3 and G'_3 . The function f_* defined by $f_*(v) = f(v)$, for $v \in V(H'_3)$, $f_*(w) = x'$, $f_*(z) = y'$, $f_*(w') = x$ and $f_*(z') = y$ is a bi-placement of H and G .

Subcase 1.3.2. The minimal vertex degree in L' is equal to one.

Let w be such a vertex of L' that $d(w, H) = 1$ and let $z' \in R'$ be the neighbour of w . Note that $H - \{w, z\}$ is a union of a path of odd length and a number of even cycles.

We have $d(z', H) = 2$. Since $p \geq 4$ we may choose $w' \in L'$ such that $d(w', H) = 2$ and $(w', z') \notin E'$. We set $G'_4 = G'_3$, where G'_3 is defined in Subcase 1.3.1, and $H'_4 = H - \{w, z, w', z'\}$. The graphs G'_4 and H'_4 are bi-placeable, by the induction hypothesis. Every bi-placement of H'_4 and G'_4 may be extended to a bi-placement of H and G by mapping the vertex w to x' , z to y' , w' to x and z' to y .

Case 2. There is no isolated vertex in $V(H)$.

Then the graph H is a sum of two non trivial paths P_1, P_2 and independent cycles.

Subcase 2.1. The paths P_1 and P_2 have length 1.

Let $P_1 = (w, z)$ and $P_2 = (w', z')$, where $w, w' \in L'$ and $z, z' \in R'$, and let $w_1 \in L'$ and $z_1 \in R'$ be two vertices of degree 2 in H .

Subcase 2.1.1. $\delta_L(G) = \delta_R(G) = 0$.

Let $x \in L$ and $y \in R$ be two isolated vertices of G and let $x_1 \in L$ and $y_1 \in R$ be two nonadjacent vertices of G chosen such that the degree sum

$$(1) \quad d(x_1, G) + d(y_1, G)$$

is maximal.

Under the hypothesis of Subcase 2.1.1 we shall prove two claims.

Claim 1. If there is in G a vertex of degree $p - 1$ then G and H are bi-placeable.

Proof of Claim 1. Suppose that $x_0 \in L$ is a vertex of degree $p - 1$ in G . Then $\|G - \{x_0, y\}\| = 2p - 3 - (p - 1) = (p - 1) - 1$. Hence, by Theorem 1, there is a bi-placement f_* of $G - \{x_0, y\}$ and $H - \{w, z\}$ which may be easily extended to a bi-placement of G and H . ■

Claim 2. If $d(x_1, G) + d(y_1, G) \geq 4$ then G and H are bi-placeable.

Proof of Claim 2. If $G' = G - \{x, y, x_1, y_1\}$ and $H' = H - \{w, z, w_1, z_1\}$ are bi-placeable, then we extend a bi-placement of G' and H' to the bi-placement of G and H mapping $x_1 \mapsto w$, $y_1 \mapsto z$, $x \mapsto w_1$, $y \mapsto z_1$.

So, by the induction hypothesis, G' and H' is one of exceptions (1)–(4) described in the theorem. Note that $H' \neq K_{2,2} \cup \overline{K_{1,1}}$ and $H' \neq C_8 \cup 2K_{1,1}$. So let us suppose that $\Delta(G') = p - 2$. Without loss of generality we may assume that there is a vertex $x' \in L - \{x, x_1\}$, such that $d(x', G') = p - 2$. If $x'y_1 \in E$ then $d(x', G) = p - 1$ and we apply Claim 1. If $x'y_1 \notin E$ then, by the maximality of the sum (1), we have $d(x_1, G) = p - 2$ and the graphs $G'' = G - \{x_1, x', y_1, y\}$ and $H'' = H - \{w, w', z, z'\}$ are bi-placeable, unless $H'' = K_{2,2}$, but then $G = \overline{K_{1,1}} \cup K_{1,1} \cup K_{2,2}$ and $H = 2K_{1,1} \cup K_{2,2}$ are bi-placeable. Any bi-placement of G'' and H'' may be easily extended to a bi-placement of G and H . ■

By Claim 2 we may suppose that $d(x_1, G) + d(y_1, G) < 4$. Consider the following three subcases.

Subcase 2.1.1.1. $d(x_1, G) + d(y_1, G) = 1$.

Without loss of generality we may suppose that $d(x_1, G) = 1$ and $d(y_1, G) = 0$. By the maximality of the sum (1) we have $d(u, G) \leq 1$ for every $u \in L$ and therefore $2p - 3 = \|G\| \leq p - 1$, contrary to $p \geq 4$.

Subcase 2.1.1.2. $d(x_1, G) + d(y_1, G) = 2$.

• $d(x_1, G) = d(y_1, G) = 1$.

Then the degree of each vertex in L which is not a neighbour of y_1 is 1 at the most. Denote by x_2 the neighbor of y_1 . We have $2p - 3 = \|G\| \leq p - 2 + d(x_2, G)$. Hence $d(x_2, G) = p - 1$ and the theorem follows from Claim 1.

• $d(x_1, G) = 0$, $d(y_1, G) = 2$.

Then all the vertices of L which are not the neighbors of y_1 are isolated. Since $\|G\| = 2p - 3$ one of the two neighbors of y_1 has degree at least $p - 1$ and we may apply Claim 1.

Subcase 2.1.1.3. $d(x_1, G) + d(y_1, G) = 3$.

• $d(x_1, G) = 3$, $d(y_1, G) = 0$.

Note that in this subcase we have necessarily $p \geq 5$ (since in R , except of the vertices y and y_1 which are isolated, we have three neighbors of x_1). Let y_2, y_3 and y_4 be the neighbors of x_1 . By the maximality of the sum (1) each vertex of R which is not a neighbor of x_1 is isolated. One of the vertices y_2, y_3, y_4 has the degree equal to 3 otherwise $2p - 3 \leq 6$ and therefore $p \leq 4$, which is a contradiction. Without loss of generality we may suppose $d(y_2, G) = 3$. Note that now the vertices of L which are not the neighbors of y_2 are isolated in G . Hence $2p - 3 = \|G\| \leq 9$ and, in consequence, either $p = 5$ or $p = 6$. If $p = 6$ then $G = K_{3,3} \cup \overline{K_{3,3}}$ and $H = C_8 \cup 2K_{1,1}$ (exceptional couple (4)). If $p = 5$ then $H = C_6 \cup 2K_{1,1}$ and G is one of two graphs G_1, G_2 depicted in Figure 4 (note that in G_2 there are two nonadjacent vertices $u \in L$ and $v \in R$ with degree sum equal to 4).

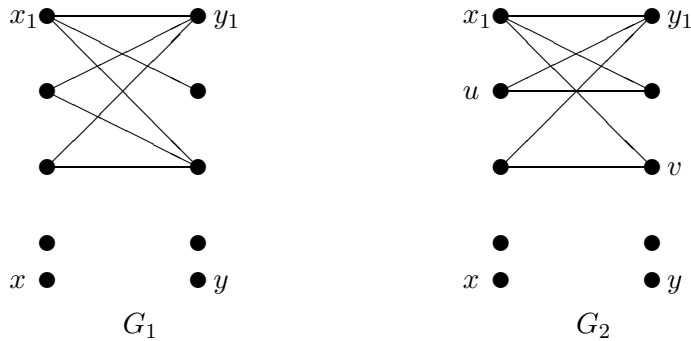


Figure 4. Two bi-placeable graphs with $G = C_6 \cup 2K_{1,1}$.

- $d(x_1, G) = 2, d(y_1, G) = 1$ and there is no vertex of degree greater than 2 in G .

In R there is one isolated vertex (the vertex y), one pendent vertex (the vertex y_1) and all remaining vertices have their degrees equal to 2. Hence $p = 4$ (otherwise there is a vertex $y' \in R$ such that $d(x_1, G) + d(y', G) = 4$ and x_1 and y' are nonadjacent, so Claim 2 is applicable), $G = \overline{K_{1,1}} \cup K_{1,1} \cup C_4$, $H = 2K_{1,1} \cup K_{2,2}$ and G and H are bi-placeable.

Subcase 2.1.2. $\delta_R(G) = 0$ and $\delta_L(G) = 1$.

Let $y \in R$ be an isolated vertex of G , $x_1 \in L$ a vertex of degree 1 and $y_1 \in R$ its neighbor in G . Let $x_2 \in L$ be a vertex not adjacent to y_1 such that the

sum

$$(2) \quad d(x_2, G) + d(y_1, G)$$

is maximum (note, that if $d(y_1, G) = p$ then G and H form an exceptional couple (2)).

Subcase 2.1.2.1. $d(x_2, G) + d(y_1, G) \geq 4$.

Then, by the induction hypothesis, either $G' = G - \{x_1, y, x_2, y_1\}$ and $H' = H - \{w, z, w_1, z_1\}$ are bi-placeable or G' and H' form an exceptional couple (1)–(4).

- If there is a bi-placement of G' and H' , then it may be extended to a bi-placement of G and H by mapping $x_2 \mapsto w, y_1 \mapsto z, x_1 \mapsto w_1, y \mapsto z_1$.

- Suppose that $\Delta_L(G') = p - 2$. Let $x_3 \in L - \{x_1, x_2\}$ be a vertex of degree $p - 2$ in G' . Since $\delta_L(G) = 1$ and $(p - 2) + (p - 1) = \|G\|$, we have $d(x_3, G) = p - 2$. Moreover, since x_3 and y_1 are nonadjacent and, by the maximality of the degree sum (2), we have also $d(x_2, G) = p - 2$ and $2p - 3 = \|G\| \geq 2(p - 2) + p - 2$. This gives $p \leq 3$, a contradiction.

- Suppose that there is a vertex $y_2 \in R(G')$ such that $d(y_2, G') = p - 2$. If $d(y_2, G) = p - 1$ then $G - \{x_1, y_2\}$ and $H - \{w, z\}$ are bi-placeable by Theorem 1, and bi-placeability of G and H follows easily. So we may assume that x_2 and y_2 are nonadjacent. Since $d(x_2, G) \geq 1$, we have $\|G - \{x_2, y_2\}\| \leq p - 2$ and, again by Theorem 1, $G - \{x_2, y_2\}$ and $H - \{w, z\}$ are bi-placeable. $x_2 \mapsto w, y_2 \mapsto z$ extends any bi-placement of $G - \{x_2, y_2\}$ and $H - \{w, z\}$ to a bi-placement of G and H .

Note that, since H contains two independent edges, $H' \neq K_{2,2} \cup \overline{K_{1,1}}$. For $p - 2 = 6$ the vertices w_1 and z_1 may be chosen in such a way that $H' \neq C_8 \cup 2K_{1,1}$. Hence G' and H' may be supposed to form neither the exceptional couple (3) nor the exceptional couple (4).

Subcase 2.1.2.2. If $u, v \in L$ and $t \in R$ are such vertices of G that $d(u, G) = 1$, t is the neighbor of u and the vertices v and t are nonadjacent, then

$$(3) \quad d(v, G) + d(t, G) < 4.$$

- If $d(y_1, G) \geq 3$, then either $d(y_1, G) = p$ and G and H form an excluded couple, or there is a vertex $s \in L$ not adjacent to y_1 . Since $\delta_L(G) \geq 1$, this contradicts (3).

- Suppose that $d(y_1, G) = 2$ and let x_3 denote the second neighbor of y_1 . By (3) we have $d(a, G) \leq 1$ for every $a \in L - \{x_1, x_3\}$. Hence $d(a, G) = 1$ for every $a \in L - \{x_1, x_3\}$ and $2p-3 = \|G\| = 1+d(x_3, G)+p-2 = d(x_3, G)+p-1$ and therefore $d(x_3, G) = p-2$.

Let $y_2 \in R$ be a vertex of the maximum degree in R , such that $y_2 \neq y_1$ (since $p \geq 4$ we check at once that such a vertex exists). We have $\|G - \{x_1, x_3, y, y_2\}\| \leq (2p-3) - p = p-3$, $\|H - \{w, z, w', z'\}\| \leq 2p-4$ and, by Theorem 1, there is a bi-placement of $G - \{x_1, x_3, y, y_2\}$ and $H - \{w, z, w', z'\}$ which may be easily extended to a bi-placement of G and H .

- Hence we may suppose that the neighbor of every pendent vertex $u \in L$ is also pendent.

It is clear by (3), that for every $u \in L$ we have $d(u, G) \leq 2$. Since $\delta_L(G) \geq 1$, we have exactly three vertices of degree 1 in $L(G)$ while the remaining $p-3$ vertices have their degree equal to 2. Let $x_1 \in L$, $y_1 \in R$ be two pendent vertices adjacent in G ; $x_2 \in L$ such that $d(x_2, G) = 1$ and $y_3 \in R$ of maximum degree in R (note that $d(y_3, G) \geq 2$). In H we choose the vertices $w, w' \in L'$, $z \in R'$ (each of which has its degree equal to 1) and $z_1 \in R'$ with $d(z_1, H) = 2$. We have $\|G - \{x_1, y_1, x_2, y_3\}\| \leq \|G\| - 4 = 2(p-2) - 3$ and $\|H - \{w, z, w', z_1\}\| \leq 2(p-2)$. By the induction hypothesis $G' = G - \{x_1, y_1, x_2, y_3\}$ and $H' = H - \{w, z, w', z_1\}$ are bi-placeable (note that G' and H' are not an excluded couple). Every bi-placement of G' and H' may be extended to a bi-placement of G and H by mapping $x_1 \mapsto w'$, $x_2 \mapsto w$, $y_1 \mapsto z_1$, $y_3 \mapsto z$.

Subcase 2.1.3. There are no isolated vertices in $V(G)$ ($\delta(G) \geq 1$).

Let $x \in L$, $y \in R$ be nonadjacent pendent vertices in $V(G)$, $y_1 \in N(x, G)$, $x_1 \in N(y, G)$ and let $w, w_1 \in L'$, $z, z_1 \in R'$ be such that wz and w_1z_1 are isolated edges in H .

Subcase 2.1.3.1. We can choose vertices x and y in such a way that $(x_1, y_1) \notin E$.

Put $G'_3 = G - \{x, y\}$ and $H'_3 = H - \{w, z_1\}$. Note that $\Delta(G'_3) < p-1$, otherwise since $\delta(G) \geq 1$ we would have $\|G\| \geq 2(p-1)$. For $p=4$ we may

choose x and y such that $G'_3 \neq \overline{3K_{1,1}}$. It is also clear that $H'_3 \neq C_8 \cup \overline{2K_{1,1}}$. Hence, by the induction hypothesis, there is a bi-placement of G'_3 and H'_3 .

- If $f(x_1) \neq w_1$ and $f(y_1) \neq z$ then we extend f to a bi-placing of G and H by mapping $x \mapsto w$, $y \mapsto z_1$.

- If $f(x_1) = w_1$ and $f(y_1) = z$ then f_* defined by: $f_*(v) = f(v)$ for every $v \in V(G'_3) - \{x_1, y_1\}$, $f_*(x) = w$, $f_*(y_1) = z_1$, $f_*(y) = z$ and $f_*(x_1) = w_1$ is a desired bi-placement of G and H .

- If $f(x_1) = w_1$ and $f(y_1) = z' \neq z$ then there is a vertex $y' \in R(G'_3)$ such that $f(y') = z$. Define f_* by the formula $f_*(v) = f(v)$ for every $v \in V(G'_3) - \{y_1\}$, $f_*(x) = w$, $f_*(y_1) = z_1$ and $f_*(y) = z'$.

Subcase 2.1.3.2. For each choice of vertices x and y we have $(x_1, y_1) \in E$. If $d(x_1, G) = p$ or $d(y_1, G) = p$ then G and H are exceptional and the theorem is proved. So assume that $d(x_1, G) \leq p - 1$ and $d(y_1, G) \leq p - 1$. Note that $G'_3 = G - \{x, y\}$ and $H'_3 = H - \{w, z_1\}$ is not an exceptional couple of graphs hence, by induction hypothesis, there is a bi-placement of G'_3 and H'_3 . If $f(x_1) \neq w_1$ and $f(y_1) \neq z$ we extend f to a bi-placement of G and H easily.

So, we suppose that $f(x_1) = w_1$ or $f(y_1) = z$. Without loss of generality we may assume that $f(x_1) = w_1$. Then there is a vertex $y_2 \in R - N(x_1, G)$ and a vertex $z_2 \in R(H'_3)$ such that $f(y_2) = z_2$. We map $y \mapsto z_2$, $y_2 \mapsto z_1$ and

- if $f(y_1) \neq z$ then $x \mapsto w$,
- if $f(y_1) = z$ then choose $x_2 \in L - N(y_1, G)$. Let $w_2 = f(x_2)$. Map $x \mapsto w_2$, $x_2 \mapsto w$.

Subcase 2.2. $|P_1| \geq 3$ or $|P_2| \geq 3$.

Subcase 2.2.1. There is an isolated vertex, say y , in $V(G)$.

Without loss of generality we may assume that $y \in R$. Let $x \in L$ and $d(x, G) = \Delta_L(G)$. There is a pendent vertex $w \in L'$ such that, if $z \in N(w, H)$ then $d(z, H) = 2$. If the graphs $G' = \{x, y\}$ and $H' = \{w, z\}$ are bi-placeable, then there is also a bi-placement of G and H . Note also, that the the couple G' and H' is neither exception (3) nor (4) of the theorem. Hence, by the induction hypothesis, $\Delta(G') = p - 1$. Note that since $\Delta_L(G) = d(x, G)$ we have $\Delta(G') = \Delta_R(G')$, otherwise $\|G\| \geq 2(p - 1)$, a contradiction.

Let $y_1 \in R(G')$ be a vertex of degree $p - 1$ in G' . If $d(y_1, G) = p$ then G and H is an exceptional couple of graphs. For $d(y_1, G) = p - 1$ define $G'' = G - \{x, x_1, y, y_1\}$ where $x_1 \in L(G)$ is a pendent vertex of G and $H'' = H - \{w_1, w_2, z_1, z_2\}$, where $w_1, w_2 \in L(H)$, $z_1, z_2 \in R(H)$, z_1 is pendent, w_1 is the neighbor of z_1 , z_2 is a neighbor of w_1 if $d(w_1, G) = 2$, otherwise z_2 is any vertex of $R(G) - \{z_1\}$, and w_2 is any vertex of $L(G) - \{w_1\}$. We have $\|G''\| \leq 2p - 3 - (p - 1 + 2) < p - 3$ and $\|H''\| < 2(p - 2)$ hence, by Theorem 1, G'' and H'' are bi-placeable. The mappings $x \mapsto w_1, x_1 \mapsto w_2, y_1 \mapsto z_1, y \mapsto z_2$ extend any bi-placement of G'' and H'' to a bi-placement of G and H .

Subcase 2.2.2. There is no isolated vertex in $V(G)$.

There are pendent vertices $x \in L$ and $y \in R$ such that $(x, y) \notin E$. Let y_1 be the neighbor of x and x_1 the neighbor of y in G .

It is easily seen that in H there are pendent vertices $w \in L'$ and $z \in R'$, such that their respective neighbors $z' \in R'$ and $w' \in L'$ have their degrees equal to 2. Note that the couple of graphs $G' = G - \{x, y\}$ and $H' = H - \{w, z'\}$ is not exceptional. Hence, by induction hypothesis, G' and H' are bi-placeable.

Let w_1 be the second neighbor of z' in H ($w_1 \neq w$). If there is a bi-placement f of G' and H' such that $f(x_1) \neq w_1$ then f may be extended by the mapping $x \rightarrow w, y \rightarrow z'$ to a bi-placement of G and H . Therefore we may assume that $f(x_1) = w_1$.

We shall prove that $d(x_1, G) = p - 2$ and for every $v \in L - \{x_1\}$ $d(v, G) = 1$ (unless G and H are bi-placeable). It is clear that $d(x_1, G) \leq p - 2$, since there is no isolated vertex in L and $\sum_{v \in L} d(v, G) = 2p - 3$. Moreover, if $d(x_1, G) = p - 2$ then all remaining vertices of L are pendent.

Suppose that $d(x_1, G) \leq p - 3$. Then there is a vertex $y_2 \in R$ such that $y_2 \neq y_1, x_1 y_2 \notin E(G)$ and $f(x_1) f(y_2) \notin E(H)$ (we remember that w_1 has in H at most two neighbors). Let z'' denote the vertex $f(y_2)$ and define $f_* : V \rightarrow V'$ by the following formulas: $f_*(v) = f(v)$ for $v \neq x, y, y_2$, $f_*(x) = w, f_*(y_2) = z'$ and $f_*(y) = z''$. f_* is a bi-placement of G and H .

In the exactly the same way we prove that either G and H are bi-placeable, or $d(y_1, G) = p - 2$.

Observe now that either

- x_1 and y_1 are adjacent and G is the union of two independent edges and two stars $K_{1,p-3}$ and $K_{p-3,1}$ with adjacent centers (see Figure 5a) or else

• x_1 and y_1 are nonadjacent and G is the union of two stars $K_{1,p-2}$, $K_{p-2,1}$ and an isolated edge (see Figure 5(b)).

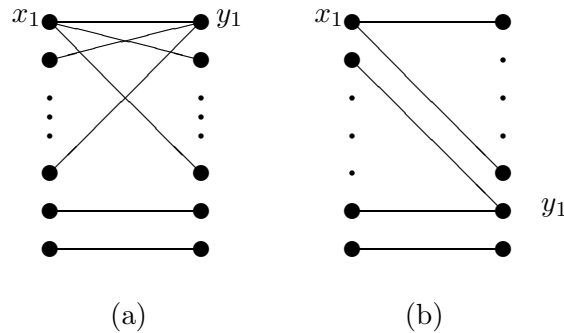


Figure 5

To finish the proof one may verify easily that then G and H (which is a union of two non-trivial paths and some cycles) are bi-placeable. ■

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Received 23 May 2008

Accepted 14 July 2008