Discussiones Mathematicae
Probability and Statistics 28 (2008) 143–155

SOME REMARKS ON OPERATORS PRESERVING PARTIAL ORDERS OF MATRICES

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Abstract

Stępniak [Linear Algebra Appl. 151 (1991)] considered the problem of equivalence of the Löwner partial order of nonnegative definite matrices and the Löwner partial order of squares of those matrices. The paper was an important starting point for investigations of the problem of how an order between two matrices $\mathbf{A}$ and $\mathbf{B}$ from different sets of matrices can be preserved for the squares of the corresponding matrices $\mathbf{A}^2$ and $\mathbf{B}^2$, in the sense of the Löwner partial ordering, the star partial ordering, the minus partial ordering, and the sharp partial ordering. Many papers have since been published (mostly coauthored by **J.K. Baksalary - to whom the present paper is dedicated**) that generalize the results in two directions: by widening the class of matrices considered and by replacing the squares by arbitrary powers. In the present paper we make a résumé of some of these results and suggest some further generalizations for polynomials of the matrices considered.

Keywords: star partial ordering, minus partial ordering, löwner partial ordering, sharp partial order, nonnegative definite matrix, Hermitian matrix, EP-matrix, power of a matrix, polynomial of a matrix.

2000 Mathematics Subject Classification: 15A45, 15A57.

1. Introduction and preliminaries

Let $\mathbb{C}_{m,n}$ be the set of $m \times n$ complex matrices. The symbols $\mathbf{K}^*$, $\mathcal{R}(\mathbf{K})$, and $\mathrm{r}(\mathbf{K})$ will denote the conjugate transpose, range, and rank, respectively, of $\mathbf{K} \in \mathbb{C}_{m,n}$. Further, $\mathbf{K}^+$ will stand for the Moore-Penrose inverse of $\mathbf{K}$, i.e., the unique matrix satisfying the equations

$$\mathbf{KK}^+\mathbf{K} = \mathbf{K},\ \mathbf{K}^+\mathbf{KK}^+ = \mathbf{K}^+,\ \mathbf{KK}^+ = (\mathbf{KK}^+)^*,\ \mathbf{K}^+\mathbf{K} = (\mathbf{K}^+\mathbf{K})^*, \tag{1.1}$$

$\mathbf{K}^\#$ will stand for the group inverse of $\mathbf{K}$, i.e., the unique (if it exists) matrix satisfying the equations

$$\mathbf{KK}^\#\mathbf{K} = \mathbf{K},\ \mathbf{K}^\#\mathbf{KK}^\# = \mathbf{K}^\#,\ \mathbf{KK}^\# = \mathbf{K}^\#\mathbf{K}, \tag{1.2}$$

and $\mathbf{I}_n$ will be the identity matrix of order n. Moreover, $\mathbb{C}_n^{\mathsf{GP}}$, $\mathbb{C}_n^{\mathsf{EP}}$, $\mathbb{C}_n^{\mathsf{N}}$, $\mathbb{C}_n^{\mathsf{H}}$, and $\mathbb{C}_n^{\geq}$ will denote the subsets of $\mathbb{C}_{n,n}$ consisting of GP, EP, normal, Hermitian, and Hermitian nonnegative definite matrices, respectively, i.e.,

$\mathbb{C}_n^{\mathsf{GP}} = \{\mathbf{K} \in \mathbb{C}_{n,n}\colon\ \mathrm{r}(\mathbf{K}) = \mathrm{r}(\mathbf{K}^2)\}, \mathbb{C}_n^{\mathsf{EP}} = \{\mathbf{K} \in \mathbb{C}_{n,n}\colon\ \mathcal{R}(\mathbf{K}) = \mathcal{R}(\mathbf{K}^*)\}$,

$\mathbb{C}_n^{\mathsf{N}} = \{\mathbf{K} \in \mathbb{C}_{n,n}\colon\ \mathbf{KK}^* = \mathbf{K}^*\mathbf{K}\}, \mathbb{C}_n^{\mathsf{H}} = \{\mathbf{K} \in \mathbb{C}_{n,n}\colon\ \mathbf{K} = \mathbf{K}^*\}$, and

$\mathbb{C}_n^{\geq} = \{\mathbf{K} \in \mathbb{C}_{n,n}\colon\ \mathbf{K} = \mathbf{LL}^* \text{ for some } \mathbf{L} \in \mathbb{C}_{n,p}\}$.

This paper is concerned with four important matrix partial orderings. The first of them introduced by Löwner [22] is defined by

$$\mathbf{A} \leq_{\mathsf{L}} \mathbf{B}\ \Leftrightarrow\ \mathbf{B} - \mathbf{A} \in \mathbb{C}_n^{\geq}.$$

Its generalization for rectangular matrices defined by Hauke and Markiewicz [13,15] is characterized by

$$\mathbf{A} \leq_{\mathsf{GL}} \mathbf{B}\ \Leftrightarrow\ |\mathbf{B}| - |\mathbf{A}| \in \mathbb{C}_n^{\geq} \quad \text{and} \quad \mathbf{AB}^* = |\mathbf{A}||\mathbf{B}|,$$

where $|\mathbf{A}| = \mathbf{AA}^*$.

The second one of them is the star ordering introduced by Drazin [11], which is defined by

$$\mathbf{A} \overset{*}{\leq} \mathbf{B} \ \Leftrightarrow \ \mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B} \quad \text{and} \quad \mathbf{A}\mathbf{A}^* = \mathbf{B}\mathbf{A}^*, \tag{1.3}$$

and can alternatively be specified as

$$\mathbf{A} \overset{*}{\leq} \mathbf{B} \ \Leftrightarrow \ \mathbf{A}^+\mathbf{A} = \mathbf{A}^+\mathbf{B} \quad \text{and} \quad \mathbf{A}\mathbf{A}^+ = \mathbf{B}\mathbf{A}^+. \tag{1.4}$$

The third partial ordering of interest is the minus (rank subtractivity) ordering devised by Hartwig [16] and independently by Nambooripad [25]. It can be characterized as

$$\mathbf{A} \overset{-}{\leq} \mathbf{B} \ \Leftrightarrow \ \mathrm{r}(\mathbf{B} - \mathbf{A}) = \mathrm{r}(\mathbf{B}) - \mathrm{r}(\mathbf{A}) \tag{1.5}$$

or as

$$\mathbf{A} \overset{-}{\leq} \mathbf{B} \ \Leftrightarrow \ \mathbf{A}\mathbf{B}^+\mathbf{B} = \mathbf{A}, \ \ \mathbf{B}\mathbf{B}^+\mathbf{A} = \mathbf{A}, \ \text{ and } \ \mathbf{A}\mathbf{B}^+\mathbf{A} = \mathbf{A}. \tag{1.6}$$

The fourth partial ordering introduced by Mitra [24] can be characterized as

$$\mathbf{A} \overset{\#}{\leq} \mathbf{B} \ \Leftrightarrow \ \mathbf{A} \in \mathbb{C}_n^{\mathsf{GP}}, \quad \mathbf{B} \in \mathbb{C}_n^{\mathsf{GP}}, \quad \mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} = \mathbf{A}^2, \tag{1.7}$$

How difficult the analysis of orderings on functions of matrices can be is shown by the following three representations which are some modifications of results of Hartwig and Styan [17] and Groß [13].

Lemma 1. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$ *and let* $a = \mathrm{r}(\mathbf{A}) < \mathrm{r}(\mathbf{B}) = b$. *Then* $\mathbf{A} \overset{*}{\leq} \mathbf{B}$ *if and only if there exist* $\mathbf{U} \in \mathbb{C}_{m,b}$, $\mathbf{V} \in \mathbb{C}_{n,b}$ *such that* $\mathbf{U}^*\mathbf{U} = \mathbf{I}_b = \mathbf{V}^*\mathbf{V}$, *for which*

$$\mathbf{A} = \mathbf{U} \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^* \quad \textit{and} \quad \mathbf{B} = \mathbf{U} \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{pmatrix} \mathbf{V}^*, \tag{1.8}$$

where $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *are positive definite diagonal matrices of degrees* a *and* $b - a$, *respectively. For* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\mathsf{N}}$ $(\mathbb{C}_n^{\mathsf{H}})$ *the matrix* $\mathbf{U}$ *in* (1.8) *may be replaced by* $\mathbf{V}$, *but then* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *represent any nonsingular (real) diagonal matrices (not necessarily positive definite).*

Lemma 2. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$ *and let* $a = \mathrm{r}(\mathbf{A}) < \mathrm{r}(\mathbf{B}) = b$. *Then* $\mathbf{A} \overset{-}{\leq} \mathbf{B}$ *if and only if there exist* $\mathbf{U} \in \mathbb{C}_{m,b}$, $\mathbf{V} \in \mathbb{C}_{n,b}$ *such that* $\mathbf{U}^*\mathbf{U} = \mathbf{I}_b = \mathbf{V}^*\mathbf{V}$, *for which*

$$\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^* \quad \text{and} \quad \mathbf{B} = \mathbf{U}\begin{pmatrix} \mathbf{D}_1 + \mathbf{R}\mathbf{D}_2\mathbf{S} & \mathbf{R}\mathbf{D}_2 \\ \mathbf{D}_2\mathbf{S} & \mathbf{D}_2 \end{pmatrix}\mathbf{V}^*, \tag{1.9}$$

where $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *are positive definite diagonal matrices of degrees* a *and* $b - a$, *while* $\mathbf{R} \in \mathbb{C}_{a,b-a}$ *and* $\mathbf{S} \in \mathbb{C}_{b-a,a}$ *are arbitrary. For* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\mathsf{H}}$ *the matrices* $\mathbf{U}$ *and* $\mathbf{S}$ *in* (1.9) *may be replaced by* $\mathbf{V}$ *and* $\mathbf{R}^*$, *respectively, but then* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *represent any nonsingular real diagonal matrices (not necessarily positive definite).*

Lemma 3. *Let* $\mathbf{A} \in \mathbb{C}_n^{\mathsf{GP}}$, $\mathbf{B} \in \mathbb{C}_{n,n}$ *and let* $a = \mathrm{r}(\mathbf{A}) < n$. *Then* $\mathbf{A} \overset{\#}{\leq} \mathbf{B}$ *if and only if there exist* $\mathbf{U} \in \mathbb{C}_{n,n}$ *such that* $\mathbf{U}^*\mathbf{U} = \mathbf{I}_n = \mathbf{U}\mathbf{U}^*$, *for which*

$$\begin{aligned} \mathbf{A} &= \mathbf{U}\begin{pmatrix} \mathbf{D}_1\mathbf{K} & \mathbf{D}_1\mathbf{L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^* \quad \text{and} \\ \mathbf{B} &= \mathbf{U}\begin{pmatrix} \mathbf{D}_1\mathbf{K} & \mathbf{D}_1\mathbf{L} - \mathbf{K}^{-1}\mathbf{L}\mathbf{D}_2 \\ \mathbf{0} & \mathbf{D}_2 \end{pmatrix}\mathbf{U}^*, \end{aligned} \tag{1.10}$$

where $\mathbf{D}_1$ *is positive definite diagonal matrix of degree* a, $\mathbf{D}_2$ *is an arbitrary matrix of degree* $n - a$, *while* $\mathbf{K} \in \mathbb{C}_{a,a} = \mathbf{I}_a$ *and* $\mathbf{L} \in \mathbb{C}_{n-a,n-a}$ *are arbitrary such that* $\mathbf{K}\mathbf{K}^* + \mathbf{L}\mathbf{L}^* = n$. *Moreover, if* $\mathbf{B} \in \mathbb{C}_n^{\mathsf{GP}}$, *then* $\mathbf{D}_2 \in \mathbb{C}_{n-a}^{\mathsf{GP}}$

From Lemmas 1 and 2 it is seen that, for any $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$,

$$\mathbf{A} \overset{*}{\leq} \mathbf{B} \;\Rightarrow\; \mathbf{A} \overset{-}{\leq} \mathbf{B}. \tag{1.11}$$

For $\mathbf{B} \in \mathbb{C}_n^{\geq}$, it is also true that

$$\mathbf{A} \overset{-}{\leq} \mathbf{B} \;\Rightarrow\; \mathbf{A} \leq_{\mathsf{L}} \mathbf{B}. \tag{1.12}$$

In the set of all Hermitian matrices (1.12) is not true, even when the minus ordering on the left-hand side is strengthened to $\mathbf{A} \overset{*}{\leq} \mathbf{B}$. An additional necessary condition is given by Baksalary *et al.* ([6], Lemma 1.3).

Lemma 4. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\mathsf{H}}$ be star-ordered as $\mathbf{A} \overset{*}{\leq} \mathbf{B}$. Then $\mathbf{A} \leq_{\mathsf{L}} \mathbf{B}$ if and only if $\nu(\mathbf{A}) = \nu(\mathbf{B})$, where $\nu(.)$ denotes the number of negative eigenvalues of a given matrix.*

Many papers (partly quoted in References) has been published that analyze properties of orderings of different functions of matrices $\mathbf{A}$ and $\mathbf{B}$, taking into account the above-defined orderings (and some others). In the present paper we only collect the results connected with orderings of powers of the matrices considered.

The basis of our investigations are the following three conditions:

(O1) $$\mathbf{A} \overset{?}{\leq} \mathbf{B}$$

(O2) $$\mathbf{A}^2 \overset{??}{\leq} \mathbf{B}^2$$

(C) $$\mathbf{AB} = \mathbf{BA}$$

analyzed for different sets of matrices $\mathbf{A}$ and $\mathbf{B}$ with orderings $\overset{?}{\leq}$ and $\overset{??}{\leq}$ taken from those defined above.

It is known (Löwner [22]) that

(CR) $$(\mathrm{O2}) \Rightarrow (\mathrm{O1})$$

for $\overset{?}{\leq}$ and $\overset{??}{\leq}$ being Löwner ordering and matrices $\mathbf{A}$ and $\mathbf{B}$ being nonnegative defiinte.

Stępniak [27] considered the problem of equivalence

(CLR) $$(\mathrm{O2}) \Leftrightarrow (\mathrm{O1})$$

of the Löwner partial order for nonnegative definite matrices. He conjectured that (CLR) was true only on the cone of commutative nonnegative definite matrices, i.e., under the condition (C), and proved it for a special case (for $n \leq 2$ - see, Corollary 5.9 in [27]). Mathias [23] constructed a counterexample showing that (CLR) was true also for other matrices and characterized the convex cones of positive semidefinite matrices on which (CLR) was true. However, the paper [27] was an important starting point for investigations of how the problem of the equivalence (CLR) can be preserved on different sets of matrices, in the sense of the Löwner partial ordering, the star partial ordering, the minus partial ordering, and the sharp partial ordering. Many papers have since been published (mostly coauthored by J.K. Baksalary) that generalize the results. In the present paper we make a résumé of some of these results and suggest some further generalizations for polynomials of the matrices considered.

Baksalary and Pukelsheim [9] provided a complete solution to the problem of how an order between two Hermitian nonnegative definite matrices $\mathbf{A}$ and $\mathbf{B}$ is related to the corresponding order between the squares $\mathbf{A}^2$ and $\mathbf{B}^2$, in the sense of the star, minus, and Löwner partial orderings. In further papers, possibilities of generalizing their results were studied from two points of view: by widening the class of matrices considered and by replacing the squares by arbitrary powers.

2. Inheritance of orderings - known results

Theorem 3 of Baksalary and Pukelsheim [9] asserts that,

Theorem 2.1. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\geq}$. *Then*

$$\mathbf{A} \overset{*}{\leq} \mathbf{B} \;\Leftrightarrow\; \mathbf{A}^2 \overset{*}{\leq} \mathbf{B}^2 \quad \textit{and} \quad \mathbf{AB} = \mathbf{BA}. \tag{2.1}$$

This result is revisited in next two theorems Baksalary *et al.* ([6], Theorems 2.1 and 2.2) with the emphasis laid on the question which (if whichever) from among four implications comprised in (2.1) continues to be valid for matrices not necessarily being Hermitian nonnegative definite.

Theorem 2.2. *Let* $\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}}$ *and* $\mathbf{B} \in \mathbb{C}_{n,n}$. *Then*

$$\mathbf{A} \overset{*}{\leq} \mathbf{B} \;\Rightarrow\; \mathbf{A}^2 \overset{*}{\leq} \mathbf{B}^2 \quad \textit{and} \quad \mathbf{AB} = \mathbf{BA}. \tag{2.2}$$

Implication (2.2) is not reversible. However, the combination of the order $\mathbf{A} \overset{*}{\leq} \mathbf{B}$ with the commutativity condition appears sufficient for $\mathbf{A}^2 \overset{*}{\leq} \mathbf{B}^2$ for all quadratic matrices.

Theorem 2.3. *Let* $\mathbf{A}$, $\mathbf{B} \in \mathbb{C}_{n,n}$. *Then*

$$\mathbf{A} \overset{*}{\leq} \mathbf{B} \ \textit{and} \ \mathbf{AB} = \mathbf{BA} \ \Rightarrow \ \mathbf{A}^2 \overset{*}{\leq} \mathbf{B}^2. \tag{2.3}$$

Merikoski and Liu ([24], Theorem 3.1) anlyzed (2.1) for normal matrices with an additional condition presented in the following theorem.

Theorem 2.4. *Let* $\mathbf{A}$, $\mathbf{B} \in \mathbb{C}_n^{\mathsf{N}}$. *Then*

$$\mathbf{A} \overset{*}{\leq} \mathbf{B}$$

is equivalent to the following:

$$\mathbf{A}^2 \overset{*}{\leq} \mathbf{B}^2$$

and if $\mathbf{A}$ *and* $\mathbf{B}$ *have nonzero eigenvalues* α *and, respectively,* β *such that* α^2 *and* β^2 *are eigenvalues of* $\mathbf{A}^2$ *and, respectively,* $\mathbf{B}^2$ *with a common eigenvector* $\mathbf{x}$*, then* $\alpha = \beta$ *and* $\mathbf{x}$ *is a common eigenvector of* $\mathbf{A}$ *and* $\mathbf{B}$.

Groß [12], Theorem 5 showed that for $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\geq}$

$$\mathbf{A} \overset{-}{\leq} \mathbf{B} \ \text{and} \ \mathbf{A}^2 \overset{-}{\leq} \mathbf{B}^2 \ \Leftrightarrow \ \mathbf{A} \overset{*}{\leq} \mathbf{B}. \tag{2.4}$$

Its generalization for Hermitian matrices was presented by Baksalary and Hauke [5], Theorem 4. Another modified version of (2.4) under weaker assumptions is given in the next theorem (Baksalary *et al.* [6], Theorem 3.2).

Theorem 2.5. *Let* $\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}}$ *and* $\mathbf{B} \in \mathbb{C}_n^{\mathsf{H}}$. *Then*

$$\mathbf{A} \overset{-}{\leq} \mathbf{B} \ \text{and} \ \mathbf{A}^2 \leq_{\mathsf{L}} \mathbf{B}^2 \ \Leftrightarrow \ \mathbf{A} \overset{*}{\leq} \mathbf{B}. \tag{2.5}$$

Observe that (2.4) follows from (1.12) and the above theorem (Corollary 3.1 of Baksalary *et al.* [6]).

Baksalary *et al.* [6], Theorem 3.1 showed that

Theorem 2.6. *Let* $\mathbf{A}$*,* $\mathbf{B} \in \mathbb{C}_{n,n}$*. Then*

$$\mathbf{A} \overset{-}{\leq} \mathbf{B} \text{ and } \mathbf{AB} = \mathbf{BA} \Rightarrow \mathbf{A}^2 \overset{-}{\leq} \mathbf{B}^2. \tag{2.6}$$

Next theorem [9], Theorem 2 is valid only for nonnegative definite matrices and cannot be extended even to the set of Hermitian matrices (counterexample – matrices (2.3) in [6]).

Theorem 2.7. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\geq}$*. Then*

$$\mathbf{A}^2 \overset{-}{\leq} \mathbf{B}^2 \text{ and } \mathbf{AB} = \mathbf{BA} \;\Rightarrow\; \mathbf{A} \overset{-}{\leq} \mathbf{B}. \tag{2.7}$$

Analyzing sharp ordering let us observe that it follows from Lemma 3 (Groß [13] and Mitra [23]) that for $\mathbf{A}$, $\mathbf{B} \in \mathbb{C}_{n,n}$

$$\mathbf{A} \overset{\#}{\leq} \mathbf{B} \;\Rightarrow\; \mathbf{A}^2 \overset{\#}{\leq} \mathbf{B}^2$$

and

$$\mathbf{A} \overset{\#}{\leq} \mathbf{B} \;\Leftrightarrow\; \mathbf{A}^2 \overset{\#}{\leq} \mathbf{B}^2, \mathbf{AB} = \mathbf{BA}, \quad \textit{and} \;\; \mathbf{B} \in \mathbb{C}_n^{\mathsf{GP}}.$$

A more precize result obtained by Groß [13], Theorem 2 is given by:

Theorem 2.8. *Let* $\mathbf{A}$*,* $\mathbf{B} \in \mathbb{C}_n^{\mathsf{GP}}$*. Then any two of the following three statements imply the third:*

(i) $\mathbf{A} \overset{\#}{\leq} \mathbf{B}$,

(ii) $\mathbf{A}^2 \overset{\#}{\leq} \mathbf{B}^2$,

(iii) $\mathbf{AB} = \mathbf{BA}$.

Finally, replacing squares by arbitrary powers Baksalary and Pukelsheim [9], p. 140 remarked that for $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\geq}$ the statement (2.1) can be generalized to the form

$$\mathbf{A} \overset{*}{\leq} \mathbf{B} \;\Leftrightarrow\; \mathbf{A}^m \overset{*}{\leq} \mathbf{B}^m \;\Rightarrow\; \mathbf{AB} = \mathbf{BA}. \tag{2.8}$$

The above result is valid for Hermitian matrices only when m is odd [6], Theorem 4.1.

Theorem 2.9. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\mathsf{H}}$. Then, for any positive integer k,*

$$\mathbf{A} \overset{*}{\leq} \mathbf{B} \;\Leftrightarrow\; \mathbf{A}^{2k+1} \overset{*}{\leq} \mathbf{B}^{2k+1} \;\Rightarrow\; \mathbf{AB} = \mathbf{BA}.$$

A version of Theorem 2.5 for powers of matrices presented below is valid only for nonnegative definite matrices [6], Theorem 4.2.

Theorem 2.10. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\geq}$. Then, for any integer $m \geq 2$,*

$$\mathbf{A} \overset{-}{\leq} \mathbf{B} \;\text{ and }\; \mathbf{A}^m \leq_{\mathsf{L}} \mathbf{B}^m \;\Leftrightarrow\; \mathbf{A} \overset{*}{\leq} \mathbf{B}. \tag{2.9}$$

When $\mathbf{A}$ and $\mathbf{B}$ are not necessarily nonnegative definite, the assertion (2.9) is no longer valid for every $m \geq 2$, see counterexample - matrices (4.5) in [6]. An extension of the validity of (2.9) to all Hermitian matrices appears possible under the additional assumption that m is even.

Theorem 2.11. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\mathsf{H}}$ and let k be any positive integer. Then*

$$\mathbf{A} \overset{-}{\leq} \mathbf{B} \;\text{ and }\; \mathbf{A}^{2k} \leq_{\mathsf{L}} \mathbf{B}^{2k} \;\Leftrightarrow\; \mathbf{A} \overset{*}{\leq} \mathbf{B}.$$

3. Inheritance of orderings - new results

Let us define a matrix-valued polynomial

$$W_k(\mathbf{A}) = c_k\mathbf{A}^k + c_{k-1}\mathbf{A}^{k-1} + \ldots + c_0\mathbf{A}^0,$$

where $\mathbf{A}^0 = \mathbf{AA}^+$, $c_0, c_1, \ldots, c_k$ are **fixed nonnegative real scalars**, and $c_k > 0$.

Next we present some generalizations of results discussed in Section 2 using the above defined polynomials instead of powers of the matrices considered.

Theorem 3.1. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\geq}$ *and* $\mathbf{AB} = \mathbf{BA}$. *Then*

$$\mathbf{A} \leq_{\mathsf{L}} \mathbf{B} \;\Leftrightarrow\; W_k(\mathbf{A}) \leq_{\mathsf{L}} W_k\mathbf{B}). \tag{3.1}$$

Proof. The commutativity of $\mathbf{A}$ and $\mathbf{B}$ implies their simultaneous diagonalization by a unitary similarity transformation:

$$\mathbf{A} = \mathbf{U}^*\mathbf{X}\mathbf{U} \quad \text{and} \quad \mathbf{B} = \mathbf{U}^*\mathbf{Y}\mathbf{U}, \tag{3.2}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are diagonal nonnegative definite matrices. Let us observe that the right part of (3.1) is equivalent to:

$$c_k\left(y_i^k - x_i^k\right) + c_{k-1}\left(y_i^{k-1} - x_i^{k-1}\right) + \ldots + c_0\left(y_i^0 - x_i^0\right) \geq 0 \tag{3.3}$$

for $i = 1, \ldots, n$, which, in turn, leads equivalently to

$$y_i - x_i \geq 0 \tag{3.4}$$

for x_i and y_i being i-th diagonal elements of $\mathbf{X}$ and $\mathbf{Y}$, respectively, $i = 1, \ldots, n$. (3.4) in fact, represents the left part of (3.1) and the proof is complete. ∎

Theorem 3.2. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_n^{\geq}$ *and* $\mathbf{AB} = \mathbf{BA}$. *Then*

$$\mathbf{A} \overset{*}{\leq} \mathbf{B} \;\Leftrightarrow\; W_k(\mathbf{A}) \overset{*}{\leq} W_k(\mathbf{B}). \tag{3.5}$$

Proof. Using decomposition similar as in (3.3) we obtain that the right part of (3.5) is equivalent to

$$\Sigma_{s=0}^{k}\Sigma_{t=0}^{k}c_s c_t x_i^s(y_i^t - x_i^t) = 0$$

for $i = 1, \ldots, n$, which, in turn, leads equivalently to

$$x_i(y_i - x_i) = 0$$

which completes the proof. ■

Further generalizations of results presented in Section 2 are rather complicated and need adding some restrictions on coefficients c_i of polynomials $W_k(\mathbf{A})$ and $W_k(\mathbf{B})$.

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Received 14 December 2007