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LATTICES OF RELATIVE COLOUR-FAMILIES AND ANTIVARIETIES

Aleksandr Kravchenko

Sobolev Institute of Mathematics SB RAS
Novosibirsk, Russia

e-mail: tclab@math.nsc.ru

Abstract

We consider general properties of lattices of relative colour-families and antivarieties. Several results generalise the corresponding assertions about colour-families of undirected loopless graphs, see [1]. Conditions are indicated under which relative colour-families form a lattice. We prove that such a lattice is distributive. In the class of lattices of antivarieties of relation structures of finite signature, we distinguish the most complicated (universal) objects. Meet decompositions in lattices of colour-families are considered. A criterion is found for existence of irredundant meet decompositions. A connection is found between meet decompositions and bases for anti-identities.

Keywords: colour-family, antivariety, lattice of antivarieties, meet decomposition, basis for anti-identities.

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1. Preliminary facts

Throughout the article, by a *structure* we mean a relation structure of a fixed signature $\sigma = (r_j)_{j \in J}$. A structure is said to be *finite* if its universe is a finite set. A *homomorphism* from a structure $\mathcal{A}$ into a structure $\mathcal{B}$ is a map $\varphi : A \to B$ such that $(\varphi(a_1), \ldots, \varphi(a_n)) \in r_j^{\mathcal{B}}$ for all $j \in J$ and

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$a_1, \dots, a_n \in A$ with $(a_1, \dots, a_n) \in r_j^{\mathcal{A}}$. If there exists a homomorphism from $\mathcal{A}$ into $\mathcal{B}$ then we write $\mathcal{A} \to \mathcal{B}$; otherwise, we write $\mathcal{A} \nrightarrow \mathcal{B}$.

For every class $\mathbf{K}$, let $\mathbf{K}_\mathrm{f}$ denote the set of isomorphism types of finite structures in $\mathbf{K}$. On $\mathbf{K}_\mathrm{f}$, define an equivalence relation $\equiv$ as follows: $\mathcal{A} \equiv \mathcal{B}$ if and only if $\mathcal{A} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{A}$. The relation $\to$ induces a partial order $\leq$ on the quotient set $\mathbf{K}_\mathrm{f}/_\equiv$. Let $\mathrm{Core}(\mathbf{K})$ denote the resulting partially ordered set.

In the sequel, it is convenient to consider an isomorphic partially ordered set whose universe is the set of cores of finite structures in $\mathbf{K}$. Recall [2, Section 2] that a finite structure is a *core* if all its endomorphisms are automorphisms. A structure $\mathcal{A}$ is a *core of a structure* $\mathcal{B}$ if $\mathcal{A}$ is a minimal retract of $\mathcal{B}$ (with respect to set inclusion). Simple properties of cores can be found, for example, in [2, Proposition 2.1]. It is easy to see that, in every coset $\mathcal{G}/_\equiv$, there exists a unique (up to isomorphism) core. We denote this core by $\mathrm{Core}(\mathcal{G})$. The map defined by the rule $\mathcal{G}/_\equiv \mapsto \mathrm{Core}(\mathcal{G})$ is an isomorphism.

Let $\mathbf{K}$ be a class of structures. For every $\mathcal{A} \in \mathbf{K}$, let

$$[\mathbf{K} \to \mathcal{A}] = \{\mathcal{B} \in \mathbf{K} : \mathcal{B} \to \mathcal{A}\}.$$

If there is no ambiguity or $\mathbf{K}$ is the class of all structures of a given signature then we write $[\to \mathcal{A}]$ instead of $[\mathbf{K} \to \mathcal{A}]$. For every set $\mathbf{A} \subseteq \mathbf{K}$, let $[\mathbf{K} \to \mathbf{A}] = \bigcup_{\mathcal{A} \in \mathbf{A}} [\mathbf{K} \to \mathcal{A}]$. If $\mathbf{A}$ is a finite set of finite structures then $[\mathbf{K} \to \mathbf{A}]$ is called a $\mathbf{K}$-*colour-family*. If $|\mathbf{A}| = 1$ then the $\mathbf{K}$-colour-family $[\mathbf{K} \to \mathbf{A}]$ is said to be *principal*. Let $\mathrm{L}_0(\mathbf{K})$ denote the partially ordered set of all $\mathbf{K}$-colour-families with respect to set inclusion. (If $\mathrm{L}_0(\mathbf{K})$ has no greatest element then by $\mathrm{L}_0(\mathbf{K})$ we mean the set of all $\mathbf{K}$-colour-families with a new greatest element $1_\mathbf{K}$).

We recall the definition of operations with structures from [3]. Let $\mathcal{A}$ and $\mathcal{B}$ be structures and let $(\mathcal{A}_i)_{i \in I}$ be a family of structures.

On the disjoint union of the universes of $\mathcal{A}$ and $\mathcal{B}$, define a structure $\mathcal{A} + \mathcal{B}$ of signature σ as follows: $(a_1, \dots, a_n) \in r^{\mathcal{A}+\mathcal{B}}$ if and only if either $(a_1, \dots, a_n) \in r^{\mathcal{A}}$ or $(a_1, \dots, a_n) \in r^{\mathcal{B}}$. The resulting structure is called the *sum* of $\mathcal{A}$ and $\mathcal{B}$. We have

$$\mathcal{A} + \mathcal{B} \to \mathcal{C} \iff \mathcal{A} \to \mathcal{C} \text{ and } \mathcal{B} \to \mathcal{C} \tag{1}$$

for every structure $\mathcal{C}$. A structure $\mathcal{A}$ is said to be *connected* if it cannot be represented in the form $\mathcal{A}_1 + \mathcal{A}_2$, where $\mathcal{A} \nrightarrow \mathcal{A}_i$, $i = 1, 2$.

On the Cartesian product of the universes of $\mathcal{A}_i$, $i \in I$, define a structure $\prod_{i\in I} \mathcal{A}_i$ of signature σ as follows: $(a_1, \dots, a_n) \in r^{\prod_{i\in I} \mathcal{A}_i}$ if and only if $(a_1(i), \dots, a_n(i)) \in r^{\mathcal{A}_i}$ for every $i \in I$. The resulting structure is called the *product* of the family $(\mathcal{A}_i)_{i\in I}$. We have

$$\mathcal{C} \to \prod_{i\in I} \mathcal{A}_i \iff \mathcal{C} \to \mathcal{A}_i \text{ for all } i \in I \tag{2}$$

for every structure $\mathcal{C}$.

On the set A^B of all functions from B into A, define a structure $\mathcal{A}^{\mathcal{B}}$ of signature σ as follows: $(f_1, \dots, f_n) \in r^{\mathcal{A}^{\mathcal{B}}}$ if and only if $(f_1(b_1), \dots, f_n(b_n)) \in r^{\mathcal{A}}$ for all $b_1, \dots, b_n \in B$ with $(b_1, \dots, b_n) \in r^{\mathcal{B}}$. The resulting structure is called the *exponent* of $\mathcal{A}$ by $\mathcal{B}$. We have

$$\mathcal{C} \to \mathcal{A}^{\mathcal{B}} \iff \mathcal{B} \times \mathcal{C} \to \mathcal{A} \tag{3}$$

for every structure $\mathcal{C}$.

2. LATTICES OF COLOUR-FAMILIES

In this section, we show that partially ordered sets of **K**-colour-families are usually lattices and study their lattice-theoretical properties.

Lemma 1. *If* **K** *is a class satisfying the condition*

$$\mathcal{A} \times \mathcal{B} \in \mathbf{K} \textit{ for all finite } \mathcal{A}, \mathcal{B} \in \mathbf{K} \tag{4}$$

then $\mathrm{L}_0(\mathbf{K})$ *is a distributive lattice with respect to the set-theoretical operations.*

Proof. Let $\mathbf{K}_1 = [\mathbf{K} \to (\mathcal{A}_i)_{i<n}]$, $\mathbf{K}_2 = [\mathbf{K} \to (\mathcal{B}_j)_{j<m}]$. It is clear that $\mathbf{K}_1 \cup \mathbf{K}_2$ is a **K**-colour-family, i.e., $\mathbf{K}_1 \vee \mathbf{K}_2 = \mathbf{K}_1 \cup \mathbf{K}_2$.

Let $\mathcal{A} \in \mathbf{K}_1 \cap \mathbf{K}_2$. Then $\mathcal{A} \in \mathbf{K}$ and there exist $i < n$ and $j < m$ such that $\mathcal{A} \to \mathcal{A}_i$ and $\mathcal{A} \to \mathcal{B}_j$. Hence, $\mathcal{A} \to \mathcal{A}_i \times \mathcal{B}_j$, cf. (2). By (4), we have $\mathcal{A}_i \times \mathcal{B}_j \in \mathbf{K}$. Conversely, if $\mathcal{A} \to \mathcal{A}_i \times \mathcal{B}_j$, where $\mathcal{A} \in \mathbf{K}$, $i < n$, and $j < m$, then we have $\mathcal{A} \to \mathcal{A}_i$ and $\mathcal{A} \to \mathcal{B}_j$. Hence, $\mathbf{K}_1 \cap \mathbf{K}_2 = \left[\mathbf{K} \to (\mathcal{A}_i \times \mathcal{B}_j)_{\substack{i<n,\\ j<m}}\right]$. In view of (4), we have $\mathcal{A}_i \times \mathcal{B}_j \in \mathbf{K}$ for all $i < n$ and $j < m$. ∎

Remark 2. Another lattice of (principal) colour-families was considered in [4, 5]. In fact, the universe of that lattice is the set of cores, the meet operation coincides with the meet operation in $\mathrm{L}_0(\mathbf{K})$, while the join operation corresponds to the sum of relation structures. That lattice is distributive too.

Let L be a lattice and let $a, b \in L$. By a *relative pseudocomplement* of a with respect to b we mean an element $a * b$ such that

$$a \wedge x \leqslant b \iff x \leqslant a * b$$

for all $x \in L$. If a relative pseudocomplement exists for every pair of elements of L then L is said to be a *relatively pseudocomplemented lattice.*

Lemma 3. *If* $\mathbf{K}$ *is a class satisfying* (4) *and the condition*

$$\mathcal{A}^{\mathcal{B}} \in \mathbf{K} \text{ for all finite } \mathcal{A}, \mathcal{B} \in \mathbf{K} \text{ with } \mathcal{B} \nrightarrow \mathcal{A} \tag{5}$$

then $\mathrm{L}_0(\mathbf{K})$ *is a relatively pseudocomplemented lattice.*

Proof. Let $\mathbf{K}_1 = [\mathbf{K} \to (\mathcal{A}_i)_{i<n}]$ and let $\mathbf{K}_2 = [\mathbf{K} \to (\mathcal{B}_j)_{j<m}]$. We introduce the notation $\mathbf{K}^i = \left[\mathbf{K} \to (\mathcal{B}_j^{\mathcal{A}_i})_{j<m}\right]$. By (5), we have $\mathcal{B}_j^{\mathcal{A}_i} \in \mathbf{K}$ provided $\mathcal{A}_i \nrightarrow \mathcal{B}_j$. If $\mathcal{A}_i \to \mathcal{B}_j$ then $\mathcal{A}_i \times \mathcal{C} \to \mathcal{B}_j$ for every $\mathcal{C} \in \mathbf{K}$. By (3), we obtain $\mathcal{C} \to \mathcal{B}_j^{\mathcal{A}_i}$, i.e., $\left[\mathbf{K} \to \mathcal{B}_j^{\mathcal{A}_i}\right] = \mathbf{K}$ is the greatest element of $\mathrm{L}_0(\mathbf{K})$. Therefore, $\left[\mathbf{K} \to \mathcal{B}_j^{\mathcal{A}_i}\right] \in \mathrm{L}_0(\mathbf{K})$ for all $i < n$ and $j < m$.

By Lemma 1, we have $[\mathbf{K} \to \mathcal{A}_i] \cap \mathbf{K}^i = \left[\mathbf{K} \to (\mathcal{B}_j^{\mathcal{A}_i} \times \mathcal{A}_i)_{j<m}\right]$. It is immediate from (3) that $\mathcal{B}_j^{\mathcal{A}_i} \times \mathcal{A}_i \to \mathcal{B}_j$. Hence, $[\mathbf{K} \to \mathcal{A}_i] \cap \mathbf{K}^i \subseteq \mathbf{K}_2$. Let $\mathbf{K}_3 = [\mathbf{K} \to (\mathcal{C}_k)_{k<l}]$ be a $\mathbf{K}$-colour-family such that $[\mathbf{K} \to \mathcal{A}_i] \cap \mathbf{K}_3 \subseteq \mathbf{K}_2$. By Lemma 1, we have $[\mathbf{K} \to \mathcal{A}_i] \cap \mathbf{K}_3 = [\mathbf{K} \to (\mathcal{A}_i \times \mathcal{C}_k)_{k<l}]$. Therefore, for every $k < l$, there exists a $j < m$ such that $\mathcal{A}_i \times \mathcal{C}_k \to \mathcal{B}_j$. By definition, $\mathcal{C}_k \to \mathcal{B}_j^{\mathcal{A}_i}$. Thus, $\mathbf{K}_3 \subseteq \mathbf{K}^i$.

We have proven that $\mathbf{K}^i$ is a pseudocomplement of $[\mathbf{K} \to \mathcal{A}_i]$ with respect to $\mathbf{K}_2$. For every distributive lattice L and elements $a, b, c \in L$, if $a * c$ and $b * c$ exist then so does $(a \vee b) * c$ and $(a \vee b) * c = (a * c) \wedge (b * c)$, cf. [6, Theorem 9.2.3]. Hence, $\mathbf{K}_1 * \mathbf{K}_2 = \bigcap_{i<n} \mathbf{K}^i$. ■

3. Lattices of antivarieties

Recall [2] that a $\mathbf{K}$-*antivariety* is a class defined in $\mathbf{K}$ by some (possibly, empty) set of *anti-identities*, i.e., sentences of the form

$$\forall x_1 \ldots \forall x_n(\neg R_1(\overline{x}) \vee \cdots \vee \neg R_m(\overline{x})),$$

where each $R_i(\overline{x})$ is an atomic formula. By [2, Theorem 1.2], for every universal Horn class $\mathbf{K}$, a subclass $\mathbf{K}'$ is a $\mathbf{K}$-antivariety if and only if $\mathbf{K}' = \mathbf{K} \cap \mathbf{H^{-1}SP^*_u}(\mathbf{K}')$, where $\mathbf{H^{-1}}$, $\mathbf{S}$, and $\mathbf{P^*_u}$ are operators for taking homomorphic pre-images, substructures, and nontrivial ultraproducts. In particular, each $\mathbf{K}$-colour-family is a $\mathbf{K}$-antivariety. Let $\mathrm{L}(\mathbf{K})$ denote the partially ordered set of all $\mathbf{K}$-antivarieties with respect to set inclusion.

Lemma 4. *For every universal Horn class* $\mathbf{K}$*, the partially ordered set* $\mathrm{L}(\mathbf{K})$ *is a distributive lattice with respect to the set-theoretical operations.*

Proof. It is clear that $\mathbf{K}_1 \wedge \mathbf{K}_2 = \mathbf{K}_1 \cap \mathbf{K}_2$ and $\mathbf{K}_1 \cup \mathbf{K}_2 \subseteq \mathbf{K}_1 \vee \mathbf{K}_2$ for all $\mathbf{K}_1, \mathbf{K}_2 \in \mathrm{L}(\mathbf{K})$. Since $\mathbf{K}_1$ and $\mathbf{K}_2$ are elementary classes, the class $\mathbf{K}_1 \cup \mathbf{K}_2$ is elementary too. In particular, $\mathbf{P^*_u}(\mathbf{K}_1 \cup \mathbf{K}_2) \subseteq \mathbf{K}_1 \cup \mathbf{K}_2$. By [2, Theorem 1.2], we have $\mathbf{K}_1 \vee \mathbf{K}_2 = \mathbf{K} \cap \mathbf{H^{-1}SP^*_u}(\mathbf{K}_1 \cup \mathbf{K}_2)$. Hence, for every $\mathcal{A} \in \mathbf{K}_1 \vee \mathbf{K}_2$, there exists a $\mathcal{B} \in \mathbf{K}_1 \cup \mathbf{K}_2$ such that $\mathcal{A} \to \mathcal{B}$. Since $\mathbf{K}_1$ and $\mathbf{K}_2$ are closed under $\mathbf{H^{-1}S}$ in $\mathbf{K}$, we obtain $\mathcal{A} \in \mathbf{K}_1 \cup \mathbf{K}_2$ and, consequently, $\mathbf{K}_1 \vee \mathbf{K}_2 = \mathbf{K}_1 \cup \mathbf{K}_2$. ∎

The proof of the following lemma is similar to that of [1, Lemma 3.4]

Lemma 5. *For every universal Horn class* $\mathbf{K}$ *and* $\mathbf{K}$*-antivariety* $\mathbf{X}$*, we have* $\mathbf{X} = \mathbf{H^{-1}SP^*_u}(\mathrm{Core}(\mathbf{X})) \cap \mathbf{K}$.

If σ is a finite signature then the partially ordered set $\mathrm{Core}(\mathbf{K})$ and the lattices $\mathrm{L}_0(\mathbf{K})$ and $\mathrm{L}(\mathbf{K})$ are related as follows.

Lemma 6. *For every universal Horn class* $\mathbf{K}$ *of finite signature, the lattice* $\mathrm{L}(\mathbf{K})$ *is isomorphic to the ideal lattice* $\mathrm{I}(\mathrm{L}_0(\mathbf{K}))$ *of the lattice* $\mathrm{L}_0(\mathbf{K})$ *and to the lattice* $\mathrm{I}_\mathrm{o}(\mathrm{Core}(\mathbf{K}))$ *of order ideals of the partially ordered set* $\mathrm{Core}(\mathbf{K})$.

This lemma generalises [1, Theorem 3.6] to the case of arbitrary relation structures of finite signature. The proof follows the lines of the proof in [1].

Proof. Put $\varphi(\mathbf{K}') = \{\mathbf{A} \in \mathrm{L}_0(\mathbf{K}) : \mathbf{A} \subseteq \mathbf{K}'\}$ for every $\mathbf{K}' \in \mathrm{L}(\mathbf{K})$ and put $\psi(J) = \bigvee\{\mathbf{A} \in \mathrm{L}_0(\mathbf{K}) : \mathbf{A} \in J\}$ for every ideal J of $\mathrm{L}_0(\mathbf{K})$. It is easy to see that φ is a map from $\mathrm{L}(\mathbf{K})$ into $\mathrm{I}(\mathrm{L}_0(\mathbf{K}))$ and ψ is a map from $\mathrm{I}(\mathrm{L}_0(\mathbf{K}))$ into $\mathrm{L}(\mathbf{K})$; moreover, both maps are monotone. We prove that $\varphi = \psi^{-1}$, which implies that φ and ψ are isomorphisms.

It is clear that $\varphi\psi(J) \supseteq J$ and $\psi\varphi(\mathbf{K}') \subseteq \mathbf{K}'$ for all $J \in \mathrm{I}(\mathrm{L}_0(\mathbf{K}))$ and $\mathbf{K}' \in \mathrm{L}(\mathbf{K})$.

Let $\mathcal{A} \in \mathbf{K}'$ be a finite structure. Then $[\mathbf{K} \to \mathcal{A}] \subseteq \mathbf{K}'$ because $\mathbf{K}'$ is a $\mathbf{K}$-antivariety. We have $\mathcal{A} \in [\mathbf{K} \to \mathcal{A}] \subseteq \bigcup\{\mathbf{A} \in \mathrm{L}_0(\mathbf{K}) : \mathbf{A} \subseteq \mathbf{K}'\}$. Since the $\mathbf{K}$-antivariety $\psi\varphi(\mathbf{K}')$ is generated by its finite structures, we obtain $\psi\varphi(\mathbf{K}') = \mathbf{K}'$.

Let $\mathbf{A} \in \varphi\psi(J)$, i.e., $\mathbf{A} \in \mathrm{L}_0(\mathbf{K})$ and $\mathbf{A} \subseteq \bigvee_{\mathbf{B}\in J} \mathbf{B}$. Then $\mathbf{A} = [\mathbf{K} \to (\mathcal{A}_i)_{i<n}]$, where each finite structure $\mathcal{A}_i$ belongs to $\bigvee_{\mathbf{B}\in J} \mathbf{B}$. By [2, Theorem 1.2], for every $i < n$, there exist a family $(\mathcal{B}_{ij})_{j\in J_i} \subseteq \bigcup_{\mathbf{B}\in J} \mathbf{B}$ and an ultrafilter U_i over J_i such that $\mathcal{A}_i \to \prod_{j\in J_i} \mathcal{B}_{ij}$. Since $\mathcal{A}_i$ is a finite structure of finite signature, from [1, Lemma 3.2] it follows that there exists a $j(i) \in J_i$ such that $\mathcal{A}_i \to \mathcal{B}_{j(i)}$. Hence, $\mathbf{A} \subseteq \left[\mathbf{K} \to (\mathcal{B}_{j(i)})_{i<n}\right]$. Since $\mathcal{B}_{j(i)} \in \bigcup_{\mathbf{B}\in J} \mathbf{B}$, we obtain $\left[\mathbf{K} \to \mathcal{B}_{j(i)}\right] \in J$. Since J is an ideal, we conclude that $\bigvee_{i<n} \left[\mathbf{K} \to \mathcal{B}_{j(i)}\right] \in J$. Thus, $\mathbf{A} \in J$ and, consequently, $\varphi\psi(J) = J$.

For proving the fact that $\mathrm{L}(\mathbf{K})$ and $\mathrm{I}_\mathrm{o}(\mathrm{Core}(\mathbf{K}))$ are isomorphic put

$$\varphi_0(\mathbf{K}') = \{\mathrm{Core}(\mathcal{A}) : \mathcal{A} \in \mathbf{K}'\}, \quad \psi_0(J) = \mathbf{H^{-1}SP^*_u}(J) \cap \mathbf{K}.$$

By Lemma 5 and [1, Lemma 3.2], we have $\varphi_0^{-1} = \psi_0$. It is clear that φ_0 and ψ_0 are monotone. Therefore, the lattices $\mathrm{L}(\mathbf{K})$ and $\mathrm{I}_\mathrm{o}(\mathrm{Core}(\mathbf{K}))$ are isomorphic. ∎

Corollary 7. *For every universal Horn class* $\mathbf{K}$ *of finite signature, the lattice* $\mathrm{L}(\mathbf{K})$ *is relatively pseudocomplemented. For all* $\mathbf{K}_1, \mathbf{K}_2 \in \mathrm{L}(\mathbf{K})$*, the following equality holds*: $\mathbf{K}_1 * \mathbf{K}_2 = \mathbf{H^{-1}SP^*_u}\{\mathcal{A} \in \mathbf{K}_\mathrm{f} : \mathcal{A} \times \mathcal{B} \in \mathbf{K}_2$ *for all* $\mathcal{B} \in (\mathbf{K}_1)_\mathrm{f}\} \cap \mathbf{K}$.

Proof. By [7, Corollary II.1.4], we have $I * J = \{a \in L : a \wedge i \in J$ for all $i \in I\}$ for every distributive lattice L and ideals I and J of L. This equality, together with Lemma 6, yields the required assertion. ∎

4. Complexity of lattices of antivarieties

In this section, we introduce the notion of a universal (the most complicated) lattice among the lattices of antivarieties of relation structures of finite signature and give examples of universal lattices.

Let $\mathbf{K}$ be a class of structures. By the category $\mathbf{K}$ we mean the category whose objects are structures in $\mathbf{K}$ and morphisms are homomorphisms. A one-to-one functor Φ from a category $\mathbf{K}_1$ into a category $\mathbf{K}_2$ is called a *full embedding* if, for every morphism $\alpha : \Phi(\mathcal{A}) \to \Phi(\mathcal{B})$ in $\mathbf{K}_2$, there exists a morphism $\beta : \mathcal{A} \to \mathcal{B}$ in $\mathbf{K}_1$ such that $\Phi(\beta) = \alpha$. For more information about categories, the reader is referred to [8]. By $\mathbf{G}$ we denote the class (and the category) of undirected loopless graphs.

Lemma 8. *Let $\mathbf{K}$ be a class of structures and let there exist a full embedding $\Phi : \mathbf{G} \to \mathbf{K}$ such that, for every finite graph $\mathcal{G}$, the structure $\Phi(\mathcal{G})$ if finite. Then there exists an embedding $\varphi : \mathrm{Core}(\mathbf{G}) \to \mathrm{Core}(\mathbf{K})$.*

Proof. Put $\varphi(\mathcal{G}) = \mathrm{Core}(\Phi(\mathcal{G}))$ for every $\mathcal{G} \in \mathrm{Core}(\mathbf{G})$. It is clear that φ is a map from $\mathrm{Core}(\mathbf{G})$ into $\mathrm{Core}(\mathbf{K})$. We show that φ is an embedding.

Let $\mathcal{G} \le \mathcal{H}$, where $\mathcal{G}, \mathcal{H} \in \mathrm{Core}(\mathbf{G})$, and let ψ be the corresponding homomorphism. Then the composition

$$\varphi(\mathcal{G}) = \mathrm{Core}(\Phi(\mathcal{G})) \xrightarrow{e} \Phi(\mathcal{G}) \xrightarrow{\Phi(\psi)} \Phi(\mathcal{H}) \xrightarrow{r} \mathrm{Core}(\Phi(\mathcal{H})) = \varphi(\mathcal{H})$$

is a homomorphism from $\varphi(\mathcal{G})$ into $\varphi(\mathcal{H})$. Hence, $\varphi(\mathcal{G}) \le \varphi(\mathcal{H})$.

Let $\varphi(\mathcal{G}) \le \varphi(\mathcal{H})$ for some $\mathcal{G}, \mathcal{H} \in \mathrm{Core}(\mathbf{G})$ and let ψ be the corresponding homomorphism. Then the composition

$$\Phi(\mathcal{G}) \xrightarrow{r} \mathrm{Core}(\Phi(\mathcal{G})) = \varphi(\mathcal{G}) \xrightarrow{\psi} \varphi(\mathcal{H}) = \mathrm{Core}(\Phi(\mathcal{H})) \xrightarrow{e} \Phi(\mathcal{H})$$

is a homomorphism from $\Phi(\mathcal{G})$ into $\Phi(\mathcal{H})$. Denote this homomorphism by α. Since Φ is a full embedding, we have $\alpha = \Phi(\beta)$ for some homomorphism $\beta : \mathcal{G} \to \mathcal{H}$. Hence, $\mathcal{G} \le \mathcal{H}$.

It remains to show that φ is a one-to-one map. If $\varphi(\mathcal{G}) = \varphi(\mathcal{H})$ then $\varphi(\mathcal{G}) \le \varphi(\mathcal{H})$ and $\varphi(\mathcal{H}) \le \varphi(\mathcal{G})$. By the above, $\mathcal{G} \le \mathcal{H}$ and $\mathcal{H} \le \mathcal{G}$. Hence, $\mathcal{G} = \mathcal{H}$. ∎

A category $\mathbf{K}$ satisfying the conditions of Lemma 8 is said to be *finite-to-finite universal*. As is known [9] (see also [10, Theorem 2.10]), the partially ordered set $\mathrm{Core}(\mathbf{G})$ is ω-universal, i.e., each countable partially ordered set is embeddable into Core($\mathbf{G}$). By Lemma 8, for every finite-to-finite universal category $\mathbf{K}$, the partially ordered set Core($\mathbf{K}$) is ω-universal.

Recall that a lattice L is called a *factor* of a lattice K if L is a homomorphic image of a suitable sublattice of K. We say that L($\mathbf{K}$) is a *universal* lattice if, for every universal Horn class $\mathbf{K}'$ of relation structures of finite signature, the lattice L($\mathbf{K}'$) is a factor of the lattice L($\mathbf{K}$).

Theorem 9. *Let* $\mathbf{K}$ *be a universal Horn class of relation structures of finite signature. If* $\mathbf{K}$ *is a finite-to-finite universal category then* L($\mathbf{K}$) *is a universal lattice.*

Proof. By Lemma 8, for every universal Horn class $\mathbf{K}'$ of relation structures of finite signature, there exists an embedding $\varphi : \mathrm{Core}(\mathbf{K}') \to \mathrm{Core}(\mathbf{K})$.

By Lemma 6, we may consider the lattice of order ideals of the partially ordered set of cores instead of the lattice of antivarieties. For every $I \in \mathrm{I_o}(\mathrm{Core}(\mathbf{K}'))$, put

$$\psi(I) = \{\mathcal{H} \in \mathrm{Core}(\mathbf{K}) : \mathcal{H} \leq \varphi(\mathcal{G}) \text{ for some } \mathcal{G} \in I\}.$$

It is easy to verify that, for every order ideal I of Core($\mathbf{K}'$), the set $\psi(I)$ is an order ideal of Core($\mathbf{K}$).

We prove that ψ is one-to-one. Let $\psi(I) = \psi(J)$. For every $\mathcal{H} \in I$, we have $\varphi(\mathcal{H}) \in \psi(I) = \psi(J)$, i.e., there exists an element $\mathcal{G} \in J$ such that $\varphi(\mathcal{H}) \leq \varphi(\mathcal{G})$. Since φ is an embedding, we have $\mathcal{H} \leq \mathcal{G}$, i.e., $\mathcal{H} \in J$. We have proven that $I \subseteq J$. The proof of the converse inclusion is similar.

We prove that ψ is a join homomorphism. Consequently, the join semilattice of $\mathrm{I_o}(\mathrm{Core}(\mathbf{K}'))$ is embeddable into the join semilattice of $\mathrm{I_o}(\mathrm{Core}(\mathbf{K}))$. The inclusion $\psi(I) \vee \psi(J) \subseteq \psi(I \vee J)$ is obvious. Conversely, let $\mathcal{H} \in \psi(I \vee J)$. Then there exists a $\mathcal{G} \in I \vee J$ such that $\mathcal{H} \leq \varphi(\mathcal{G})$. Since $I \vee J = I \cup J$, we obtain $\mathcal{H} \in \psi(I) \cup \psi(J) = \psi(I) \vee \psi(J)$.

Let L be the sublattice of $\mathrm{I_o}(\mathbf{K})$ generated by the set $\{\psi(I) : I \in \mathrm{I_o}(\mathrm{Core}(\mathbf{K}'))\}$. Then, for every $X \in L$, there exist a lattice term $t(v_0, \ldots, v_{n-1})$ and order ideals $J_0, \ldots, J_{n-1}$ of Core($\mathbf{K}'$) such that $X = t(\psi(J_0), \ldots, \psi(J_{n-1}))$.

We prove that, for every lattice term $t(v_0, \ldots, v_{n-1})$ and order ideals $J_0, \ldots, J_{n-1}$ of Core($\mathbf{K}'$), the equality

$$t(\psi(J_0), \ldots, \psi(J_{n-1})) \cap \varphi(\mathrm{Core}(\mathbf{K}')) = \varphi(t(J_0, \ldots, J_{n-1})) \tag{6}$$

holds, where $\varphi(M) = \{\varphi(m) : m \in M\}$ for every set M.

We use induction on the length of the term. Let $t(v_0,\dots,v_{n-1}) = v_i$. Then the right-hand side of (6) is $\varphi(J_i)$ and the left-hand side of (6) is $\psi(J_i) \cap \varphi(\mathrm{Core}(\mathbf{K}'))$. It is clear that $\varphi(J_i) \subseteq \psi(J_i) \cap \varphi(\mathrm{Core}(\mathbf{K}'))$. Conversely, let $\mathcal{H} \in \psi(J_i) \cap \varphi(\mathrm{Core}(\mathbf{K}'))$. Then $\mathcal{H} = \varphi(\mathcal{G})$ for some $\mathcal{G} \in \mathrm{Core}(\mathbf{K}')$. Let J be the least ideal of $\mathrm{Core}(\mathbf{K}')$ containing $J_i \cup \{\mathcal{G}\}$. We have $\psi(J) = \psi(J_i) \cup \{\mathcal{A} \in \mathrm{Core}(\mathbf{K}') : \mathcal{A} \le \varphi(\mathcal{G})\}$. Since $\varphi(\mathcal{G}) = \mathcal{H} \in \psi(J_i)$, we obtain $\psi(J) = \psi(J_i)$. Since ψ is a one-to-one map, we have $J = J_i$, i.e., $\mathcal{G} \in J_i$. Therefore, $\mathcal{H} \in \varphi(J_i) \subseteq \psi(J_i)$.

Assume that $t = t_1 \wedge t_2$ or $t = t_1 \vee t_2$ for some terms t_1 and t_2. We introduce the notation

$$Y_i = t_i(\psi(J_0),\dots,\psi(J_{n-1})), \quad X_i = t_i(J_0,\dots,J_{n-1}),$$

where $i = 1, 2$. By induction, $Y_i \cap \mathrm{Core}(\mathbf{K}') = \varphi(X_i)$, $i = 1, 2$.

If $t = t_1 \wedge t_2$ then $t(\psi(J_0),\dots,\psi(J_{n-1})) = Y_1 \cap Y_2$, $t(J_0,\dots,J_{n-1}) = X_1 \cap X_2$. By induction, $Y_1 \cap Y_2 \cap \varphi(\mathrm{Core}(\mathbf{K}')) = \varphi(X_1) \cap \varphi(X_2) \supseteq \varphi(X_1 \cap X_2)$. For every $\mathcal{A} \in \varphi(X_1) \cap \varphi(X_2)$, there exist $\mathcal{A}_i \in X_i$, $i = 1, 2$, such that $\mathcal{A} = \varphi(\mathcal{A}_1) = \varphi(\mathcal{A}_2)$. Since φ is a one-to-one map, we obtain $\mathcal{A}_1 = \mathcal{A}_2 \in X_1 \cap X_2$.

It $t = t_1 \vee t_2$ then $t(\psi(J_0),\dots,\psi(J_{n-1})) = Y_1 \cup Y_2$, $t(J_0,\dots,J_{n-1}) = X_1 \cup X_2$. By induction, $(Y_1 \cup Y_2) \cap \varphi(\mathrm{Core}(\mathbf{K}')) = (Y_1 \cap \varphi(\mathrm{Core}(\mathbf{K}'))) \cup (Y_2 \cap \varphi(\mathrm{Core}(\mathbf{K}'))) = \varphi(X_1) \cup \varphi(X_2) \subseteq \varphi(X_1 \vee X_2)$. The converse inclusion is an easy consequence of the equality $X_1 \vee X_2 = X_1 \cup X_2$.

Since the operations of the lattice of order ideals are the set-theoretical operations, the union and the intersection, from (6) we obtain

$$\varphi^{-1}(t(\psi(J_0),\dots,\psi(J_{n-1})) \cap \varphi(\mathrm{Core}(\mathbf{K}'))) = t(J_0,\dots,J_{n-1}). \tag{7}$$

Let $X = t(\psi(J_0),\dots,\psi(J_{n-1})) \in L$. Put $\alpha(X) = \varphi^{-1}(X \cap \varphi(\mathrm{Core}(\mathbf{K}')))$. It is immediate from (7) that α is a map from L onto $\mathrm{I_o}(\mathrm{Core}(\mathbf{K}'))$. By (6), α is a homomorphism. ■

We present an example showing that the converse to Theorem 9 is not true. Namely, we indicate a quasivariety $\mathbf{K}$ of loopless digraphs such that $\mathbf{K}$ is not a finite-to-finite universal category but $\mathrm{L}(\mathbf{K})$ is a universal lattice.

Example 10. Let σ consist of one binary relation symbol r. Denote by $\mathbf{K}$ the quasivariety of structures of the signature σ defined by the quasi-identities

$$\forall x \forall y \big(r(x,x) \to x \approx y \big),$$

$$\forall x \forall y \forall z \big(r(x,y) \,\&\, r(x,z) \to y \approx z \big),$$

$$\forall x \forall y \forall z \big(r(y,x) \,\&\, r(z,x) \to y \approx z \big).$$

Let $\mathcal{C}_n$, $n \geqslant 2$, denote the *cycle* of length n, i.e., the structure whose universe is $C_n = \{0, 1, \ldots, n-1\}$ and $(i,j) \in r^{\mathcal{C}_n}$ if and only if $i + 1 \equiv j \pmod{n}$. It is easy to see that, for every $n \geqslant 2$, we have $\mathcal{C}_n \in \mathbf{K}$.

Let $\mathbb{P}$ denote the set of prime numbers. Denote by a_p, $p \in \mathbb{P}$, the $\mathbf{K}$-colour-family $[\mathbf{K} \to \mathcal{C}_p]$. Let L be the sublattice of $\mathrm{L}_0(\mathbf{K})$ generated by the elements $(a_p)_{p \in \mathbb{P}}$.

We show that the distributive lattice L is freely generated by the set $(a_p)_{p \in \mathbb{P}}$. Since $|\mathbb{P}| = \omega$, this means that the free distributive lattice $\mathrm{F}_{\mathbf{D}}(\omega)$ of countable rank is embedded into $\mathrm{L}_0(\mathbf{K})$. We use [7, Theorem II.2.3]. It suffices to verify that, for all finite nonempty subsets $I, J \subseteq \mathbb{P}$, from $\bigwedge_{i \in I} a_i \leqslant \bigvee_{j \in J} a_j$ it follows that $I \cap J \neq \varnothing$.

Let I and J be finite and nonempty. Assume that $\bigwedge_{i \in I} a_i \leqslant \bigvee_{j \in J} a_j$. By Lemma 1, we have $\bigwedge_{i \in I} a_i = \big[\mathbf{K} \to \prod_{i \in I} \mathcal{C}_i\big]$ and $\bigvee_{j \in J} a_j = [\mathbf{K} \to (\mathcal{C}_j)_{j \in J}]$. Let $k = \prod_{i \in I} i$. It is easy to see that $\prod_{i \in I} \mathcal{C}_i \simeq \mathcal{C}_k$ (cf., for example, [11]). Since $\bigwedge_{i \in I} a_i \leqslant \bigvee_{j \in J} a_j$, there exists a prime $j \in J$ with $\mathcal{C}_k \in [\mathbf{K} \to \mathcal{C}_j]$. We have $\mathcal{C}_k \to \mathcal{C}_j$ if and only if j divides k (cf., for example, [12]). Since j is prime and k is a product of distinct primes, we conclude that $j \in I$. Thus, $I \cap J \neq \varnothing$.

We show that the ideal lattice $\mathrm{I}(\mathrm{F}_{\mathbf{D}}(\omega))$ of the free distributive lattice of countable rank is embeddable into $\mathrm{L}(\mathbf{K})$. Let L and K be distributive lattices and let $\varphi : L \to K$ be an embedding. Define a map $\psi : \mathrm{I}(L) \to \mathrm{I}(K)$ by the following rule: *$\psi(I)$ is the ideal of K generated by $\varphi(I)$*. Using the definition of an ideal generated by a set, we easily find that ψ is an embedding. In particular, $\mathrm{I}(\mathrm{F}_{\mathbf{D}}(\omega))$ is embeddable into $\mathrm{I}(\mathrm{L}_0(\mathbf{K}))$. The latter lattice is isomorphic to $\mathrm{L}(\mathbf{K})$ in view of Lemma 6.

We show that the lattice $\mathrm{L}(\mathbf{G})$ of antivarieties of undirected loopless graphs is a homomorphic image of the lattice $\mathrm{I}(\mathrm{F}_{\mathbf{D}}(\omega))$. Since $\mathrm{L}_0(\mathbf{G})$ is a countable distributive lattice, there exists a homomorphism from $\mathrm{F}_{\mathbf{D}}(\omega)$ onto $\mathrm{L}_0(\mathbf{G})$. As above, this homomorphism induces a homomorphism between the corresponding ideal lattices. It remains to use Lemma 6.

Therefore, L(**G**) is a factor of L(**K**). We conclude that L(**K**) is a universal lattice. The class of rigid objects in the category **K** consists of trivial structures and finite directed chains only (cf. [8, Exercise IV.1.6]). Therefore, the category **K** is not universal and, consequently, is not finite-to-finite universal.

5. IRREDUNDANT MEET DECOMPOSITIONS IN LATTICES OF COLOUR-FAMILIES

Recall that **G** denotes the universal Horn class and the category of undirected loopless graphs. The study of the lattice $\mathrm{L}_0(\mathbf{G})$ was initiated in [1]. It was proven that this lattice possesses neither completely join irreducible nor completely meet irreducible nonzero elements. A simple description for join irreducible colour-families was found. The question on meet irreducible elements turned to be closely connected with a well-known problem in the graph theory, Hedetniemi's conjecture [13].

Here, we consider meet decompositions of **K**-colour-families with the help of Lemma 3, which says that the lattice of **K**-colour-families is relatively pseudocomplemented. A similar approach was first used in [4].

We present necessary definitions. By a *meet decomposition* of an element $x \in L$, where L is an arbitrary lattice, we mean a representation

$$x = \bigwedge_{i \in I} m_i, \tag{8}$$

where $(m_i)_{i \in I}$ is a family of meet irreducible elements, i.e., for each $i \in I$, we have $m_i \neq 1$ and from $m_i = a \wedge b$ it follows that either $m_i = a$ or $m_i = b$. A meet decomposition (8) is *irredundant* if $x < \bigwedge_{i \in J} m_i$ for every proper subset $J \subsetneq I$. For distributive relatively pseudocomplemented lattices, the following criterion for meet irreducibility of elements is known [14].

Proposition 11. *Let L be a distributive relatively pseudocomplemented lattice and let $m \in L$. The element m is meet irreducible if and only if $x * m = m$ for every $x \in L$ with $x \nleq m$.*

Throughout this section, we assume that L is an arbitrary distributive relatively pseudocomplemented lattice. Let $\vee$, $\wedge$, and $*$ denote the operations of L. For every $x \in L$, let $\mathrm{Reg}(x) = \{y * x : y \in L\}$, i.e., let $\mathrm{Reg}(x)$ denote the set of regular elements of the principal filter $[x)$ of L. For all $u, v \in \mathrm{Reg}(x)$, put

$$u + v = ((u \vee v) * x) * x, \quad u \cdot v = u \wedge v, \quad 0 = x, \quad 1 = 1_L, \quad u' = u * x.$$

The set $\mathrm{Reg}(x)$ with the operations $+$, $\cdot$, and $'$ and constants 1 and 0 is a Boolean algebra; moreover, the map r from L to $\mathrm{Reg}(x)$ defined by the rule $r(y) = (y * x) * x$ is a homomorphism between Heyting algebras [6, Theorem 8.4.3].

We mention the following relationship between meet irreducible elements of L and dual atoms of $\mathrm{Reg}(x)$ [4, Theorem 6].

Proposition 12. *Let $x, y \in L$ and let $x < y$. The element y is a dual atom of the Boolean algebra* $\mathrm{Reg}(x)$ *if and only if $y * x > x$ and y is meet irreducible in L.*

Recall [15] that a Boolean algebra A is *atomic* if, for every nonzero element $a \in A$, there exists an atom b such that $b \leqslant a$. An element a is said to be *atomless* if $a \neq 0$ and there is no atom b such that $b \leqslant a$.

Theorem 13. *Let L be a distributive pseudocomplemented lattice and let $a \in L$. The element a admits an irredundant meet decomposition in L if and only if the Boolean algebra* $\mathrm{Reg}(a)$ *is atomic.*

Proof. Let $a = \bigwedge_{i \in I} m_i$ be an irredundant meet decomposition. We prove that $m_i * a > a$ for every $i \in I$. In view of Proposition 12, this means that each m_i, $i \in I$, is a dual atom of $\mathrm{Reg}(a)$. Since $a = \bigwedge_{i \in I} m_i$, we have $m_i * a = m_i * \left(\bigwedge_{i \in I} m_i\right) = \bigwedge_{m_j \nleqslant m_i} m_j = \bigwedge_{j \neq i} m_j$ (cf. [16, IV.7.2 (8)]). Since the meet decomposition is irredundant, we have $m_i * a = \bigwedge_{j \neq i} m_j > a$.

Assume that there exists an atomless element $b \in \mathrm{Reg}(a)$. Since the complement of a dual atom is an atom, we conclude that $m_i * a \nleqslant b$ for all $i \in I$. Hence, $b \wedge (m_i * a) = a$ for all $i \in I$. By the definition of a relative pseudocomplement, we have $b \leqslant (m_i * a) * a = m_i$ for all $i \in I$. Therefore, $b \leqslant \bigwedge_{i \in I} m_i = a$. This proves that $\mathrm{Reg}(a)$ possesses no atomless element, i.e., $\mathrm{Reg}(a)$ is an atomic Boolean algebra.

Conversely, assume that $\mathrm{Reg}(a)$ is an atomic Boolean algebra. Let $(a_i)_{i \in I}$ be the set of atoms. If $|\mathrm{Reg}(a)| = 2$ then the element a is meet irreducible in view of Proposition 11. In the sequel, we assume that $|\mathrm{Reg}(a)| > 2$. We denote $m_i = a_i * a$, $i \in I$. For every $i \in I$, the element m_i is a dual atom of $\mathrm{Reg}(a)$; moreover, each dual atom is of the form m_i, $i \in I$.

By Proposition 12, we have $m_i * a > a$ and m_i is meet irreducible for every $i \in I$. It remains to show that $a = \bigwedge_{i \in I} m_i$ (in the lattice L). It is clear that a is a lower bound for $(m_i)_{i \in I}$. If a is not the greatest lower bound then there exists a lower bound $b_0 \in L$ for $(m_i)_{i \in I}$ such that $b_0 \nleqslant a$. Consider the element $(b_0 * a) * a \in \mathrm{Reg}(a)$. Since $b_0 \leqslant m_i$, we conclude that $(b_0 * a) * a \leqslant (m_i * a) * a = m_i$, $i \in I$. Hence, $b = (b_0 * a) * a \in \mathrm{Reg}(a)$ is a lower bound for $(m_i)_{i \in I}$. Since $b_0 \nleqslant a$ and $b_0 \leqslant b$, we find that $b \neq a$, i.e., $b > a$. Since $\mathrm{Reg}(a)$ is an atomic Boolean algebra, there exists an atom a_j such that $a_j \leqslant b$. Passing to the complements, we find that $b' \leqslant m_j$. Since $b \leqslant m_i$ for all $i \in I$, we obtain $b + b' = 1 \leqslant m_j + m_j = m_j < 1$, a contradiction. ■

Similar questions for undirected loopless graphs were considered in [17], where the notion of the *level of nonmultiplicativity* of a graph was introduced. In our terminology, the level of nonmultiplicativity of a graph $\mathcal{G}$ is the number of dual atoms of the Boolean algebra $\mathrm{Reg}([\mathbf{G} \to \mathcal{G}])$. In [17], the following conjecture is stated: *The level of nonmultiplicativity of each finite graph is finite.* In connection with Theorem 13, we formulate the following

Problem 14. Let $\mathbf{K}$ be a universal Horn class of relation structures of finite signature. Is this true that, for every $\mathbf{K}$-colour-family $\mathbf{A}$, the following conditions are equivalent:

(1) there exists an irredundant meet decomposition $\mathbf{A} = \bigwedge_{i \in I} \mathbf{M}_i$,
(2) there exists a finite meet decomposition $\mathbf{A} = \bigwedge_{i < n} \mathbf{M}_i$, $n < \omega$,
(3) the Boolean algebra $\mathrm{Reg}(\mathbf{A})$ is finite?

In the next section, we find a connection between this problem and existence of independent bases for anti-identities.

6. Anti-identities of finite structures

Recall that a set Σ of anti-identities is a *basis for anti-identities* of a class $\mathbf{K}$ if $\mathbf{K}$ is the class of structures in which all anti-identities of Σ are valid, i.e., $\mathbf{K} = \mathrm{Mod}(\Sigma)$. By a basis for anti-identities of a structure $\mathcal{A}$ we mean a basis for anti-identities of the antivariety generated by $\mathcal{A}$. A basis Σ is said to be *independent* if, for every $\varphi \in \Sigma$, the proper inclusion $\mathrm{Mod}(\Sigma) \subsetneq \mathrm{Mod}(\Sigma \setminus \{\varphi\})$ holds.

A structure $\mathcal{A}$ is said to be *weakly atomic compact* if every locally consistent in $\mathcal{A}$ set of atomic formulas is consistent in $\mathcal{A}$.

We reduce the question on existence of an independent basis for anti-identities of a finite relation structure of finite signature to Problem 14.

Let $\mathcal{A}$ be a finite relation structure of finite signature and let $\Sigma = (\varphi_i)_{i\in I}$ be an independent basis for anti-identities of $\mathcal{A}$. With each anti-identity φ_i, $i \in I$, we associate a finitely presented structure $\mathcal{B}_i$ as follows:

Let $\varphi_i \leftrightharpoons \forall\overline{x}(\neg\psi_1(\overline{x}) \vee \cdots \vee \neg\psi_n(\overline{x}))$; then $\mathcal{B}_i$ is the structure defined by generators $\overline{x}$ and relations $\psi_1(\overline{x}), \ldots, \psi_n(\overline{x})$.

It is easy to see that the antivariety defined by Σ coincides with the class

$$\bigcap_{i\in I} [\mathcal{B}_i \not\to] = \{\mathcal{B} : \mathcal{B}_i \not\to \mathcal{B} \text{ for all } i \in I\}.$$

Since Σ is an independent basis, we have $\mathcal{B}_i \to \mathcal{B}_j$ if and only if $i = j$.

Lemma 15. *The following two conditions are equivalent*:

(1) $[\to \mathcal{A}] = \bigcap_{i\in I} [\mathcal{B}_i \not\to]$,

(2) *there exists a family of finite structures* $(\mathcal{A}_i)_{i\in I}$ *such that* $[\mathcal{B}_i \not\to] = [\to \mathcal{A}_i]$ *for all* $i \in I$ *and* $[\to \mathcal{A}] = [\to \prod_{i\in I} \mathcal{A}_i]$.

Proof. It is clear that (2) implies (1). Indeed, if such a family exists then, for every structure $\mathcal{C}$, we have

$$\begin{aligned} \mathcal{C} \in [\to \mathcal{A}] = \left[\to \prod_{i\in I} \mathcal{A}_i\right] &\iff \mathcal{C} \to \mathcal{A}_i \text{ for all } i \in I \iff \\ &\iff \mathcal{C} \in [\mathcal{B}_i \not\to] \text{ for all } i \in I \iff \\ &\iff \mathcal{C} \in \bigcap_{i\in I} [\mathcal{B}_i \not\to]. \end{aligned}$$

We prove that (1) implies (2).

Notice that each structure $\mathcal{B}_i$, $i \in I$, is connected. Assume the contrary, i.e., let there exist an element $i \in I$ such that $\mathcal{B}_i$ is not connected. Then $\mathcal{B}_i = \mathcal{B}_i^1 + \mathcal{B}_i^2$ for some structures $\mathcal{B}_i^k$ with $\mathcal{B}_i \not\to \mathcal{B}_i^k$, $k = 1, 2$. Since $\mathcal{B}_j \not\to \mathcal{B}_i$ provided $j \neq i$, we have $\mathcal{B}_i^k \in [\mathcal{B}_j \not\to]$ for all $j \neq i$ and $k = 1, 2$. Therefore, $\mathcal{B}_i^k \in \bigcap_{i\in I} [\mathcal{B}_i \not\to] = [\to \mathcal{A}]$, $k = 1, 2$. Thus, $\mathcal{B}_i \in [\to \mathcal{A}] \subseteq [\mathcal{B}_i \not\to]$, which is a contradiction.

For an arbitrary $i \in I$, consider the interval $[\mathcal{A}, \mathcal{A} + \mathcal{B}_i]$ of the partially ordered set of cores. If there exists a core $\mathcal{C}$ such that $\mathcal{A} \to \mathcal{C} \to \mathcal{A} + \mathcal{B}_i$ and $\mathcal{B}_i \not\to \mathcal{C} \not\to \mathcal{A}$ then, by (1), we obtain $\mathcal{C} \notin \bigcap_{i \in I} [\mathcal{B}_i \not\to]$. Hence, there exists a $j \in I$ such that $\mathcal{B}_j \to \mathcal{C}$. It is easy to see that $i \neq j$. Since $\mathcal{B}_j \to \mathcal{C} \to \mathcal{A} + \mathcal{B}_i$ and $\mathcal{B}_j$ is connected, we conclude that $\mathcal{B}_j \to \mathcal{B}_i$, where $i \neq j$. Since Σ is an independent basis, we arrive at a contradiction. Thus, $\mathcal{A} + \mathcal{B}_i$ covers $\mathcal{A}$ in the partially ordered set of cores (in symbols: $\mathcal{A} \prec \mathcal{A} + \mathcal{B}_i$). Since this is a distributive lattice (cf. Remark 2), we conclude that $\mathcal{A} \times \mathcal{B}_i \prec \mathcal{B}_i$. We denote $\mathcal{C}_i = \mathcal{A} \times \mathcal{B}_i$, $i \in I$. By (2) and (3), we obtain $\mathcal{C}_i \to \mathcal{A} \to \mathcal{C}_i^{\mathcal{B}_i}$ for all $i \in I$. By [5, Lemma 2.5], for every $i \in I$, the equality $\left[\to \mathcal{C}_i^{\mathcal{B}_i}\right] = [\mathcal{B}_i \not\to]$ holds. Put $\mathcal{A}_i = \mathcal{C}_i^{\mathcal{B}_i}$. We prove that $\bigcap_{i \in I} [\mathcal{B}_i \not\to] = \left[\to \prod_{i \in I} \mathcal{A}_i\right]$. We deduce

$$\mathcal{D} \in \bigcap_{i \in I} [\mathcal{B}_i \not\to] \iff \mathcal{D} \in [\mathcal{B}_i \not\to] \text{ for all } i \in I \iff$$

$$\iff \mathcal{D} \in [\to \mathcal{A}_i] \text{ for all } i \in I \iff$$

$$\iff \mathcal{D} \in \left[\to \prod_{i \in I} \mathcal{A}_i\right].$$

Since $\bigcap_{i \in I} [\mathcal{B}_i \not\to] = [\to \mathcal{A}]$, we obtain $[\to \mathcal{A}] = \left[\to \prod_{i \in I} \mathcal{A}_i\right]$. Moreover, if the structure $\mathcal{A}$ has no trivial substructure then the structure $\prod_{i \in I} \mathcal{A}_i$ has no trivial substructure either. ■

In the sequel, we assume that (equivalent) conditions (1) and (2) of Lemma 15 are satisfied. Without loss of generality, we may assume that $\mathcal{A}$ is a core. Since $\mathcal{A}$ is finite, the class $[\to \mathcal{A}]$ is elementary. By [2, Proposition 2.2], the structure $\prod_{i \in I} \mathcal{A}_i$ is weakly atomic compact and $\left[\to \prod_{i \in I} \mathcal{A}_i\right]$ is the antivariety generated by $\prod_{i \in I} \mathcal{A}_i$. In view of [2, Corollary 2.6], there exists a unique (up to isomorphism) core $\mathcal{A}^*$ of $\prod_{i \in I} \mathcal{A}_i$; moreover, the antivarieties generated by $\mathcal{A}^*$ and $\prod_{i \in I} \mathcal{A}_i$ coincide. Therefore, the antivarieties generated by $\mathcal{A}$ and $\mathcal{A}^*$ coincide. By [2, Corollary 2.5], the structures $\mathcal{A}$ and $\mathcal{A}^*$ are isomorphic.

Since $\mathcal{A}^*$ is a finite structure of finite signature, there exists a finite subset $F \subseteq I$ such that $\mathcal{A}^*$ is embeddable into $\prod_{i \in F} \mathcal{A}_i$.

We suggest the following

Conjecture 16. The equality $[\to \mathcal{A}] = \left[\to \prod_{i \in F'} \mathcal{A}_i\right]$ holds for some finite subset $F' \subseteq I$ with $F \subseteq F'$.

If this conjecture is true then Σ is a finite basis, which means that every finite relation structure of finite signature having no finite basis for its anti-identities possesses no independent basis for its anti-identities.

We return to Problem 14. Let $\mathbf{K}_i$ be the principal colour-family generated by $\mathcal{A}_i$, $i \in I$. Then

$$[\to \mathcal{A}] = \bigwedge_{i \in I} \mathbf{K}_i$$

is an irredundant meet decomposition of $[\to \mathcal{A}]$ (in the lattice of colour-families). Therefore, if the answer to the question in Problem 14 is positive then Conjecture 16 is true.

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