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COMPLETION OF PARTIALLY ORDERED SETS*

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Abstract

The paper considers a generalization of the standard completion of a partially ordered set through the collection of all its lower sets.

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1. Introduction

Let **Pos** be the category of partially ordered sets (posets) and order-preserving maps and let **JCPos** be its subcategory consisting of complete lattices and join-preserving maps. It is known that the category **JCPos** is reflective in **Pos** (see, e.g., [1]). The completion of a poset goes through the collection of all its lower-sets.

Given a quantale Q one can consider the category Q-**Mod** of modules over Q (see, e.g., [5]). Since the categories **2-Mod** and **JCPos** are isomorphic one could ask about the generalization of the aforesaid result for an arbitrary quantale Q. We answer the question in two ways using the generalization of the category **Pos** in the latter one.

All results from Category Theory used in the paper can be found in [1].

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2. Definition of the category Q-**Mod**

In this section we recall basic facts about the category Q-**Mod** motivated by the category of modules over a ring [3, 4]. Start by recalling the definition of quantale (see, e.g., [5]).

Definition 2.1. A *quantale* is a triple $(Q, \leqslant, \cdot)$ such that

(i) $(Q, \leqslant)$ is a complete lattice;

(ii) $(Q, \cdot)$ is a semigroup;

(iii) $q \cdot (\bigvee S) = \bigvee_{s \in S} (q \cdot s)$ and $(\bigvee S) \cdot q = \bigvee_{s \in S} (s \cdot q)$ for every $q \in Q$ and every $S \subseteq Q$.

Given a quantale Q, denote its top (bottom) element by $\top$ ($\bot$) respectively.

Definition 2.2. A quantale Q is called *unital* provided that there exists an element $e \in Q$ such that $(Q, \cdot, e)$ is a monoid.

From now on without further references all quantales are supposed to be unital. The following are examples of quantales:

(i) $(\mathbf{2}, \leqslant, \wedge, 1)$ where $\mathbf{2} = \{0, 1\}$;

(ii) $([0,1], \leqslant, \wedge, 1)$ where $[0,1]$ is the unit interval;

(iii) $([0,1], \leqslant, \cdot, 1)$ where $\cdot$ is the usual multiplication;

(iv) the chain $\mathbf{3}$ with the usual order and the map $\mathbf{3} \times \mathbf{3} \xrightarrow{\cdot} \mathbf{3}$ given by the table:

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

Notice that $\top \neq e$.

As in the last example we *do not* assume that $\top = e$ for every quantale Q we use.

Now define the category Q-**Mod** of quantale modules or in other words the category of enriched complete lattices over the category **JCPos**.

Definition 2.3. Given a quantale Q, define the category Q-**Mod** as follows:

(i) The objects are triples $(A, \leqslant, *)$ where $(A, \leqslant)$ is a complete lattice and $Q \times A \xrightarrow{*} A$ is a map such that

(a) $q * (\bigvee S) = \bigvee_{s \in S} (q * s)$ for every $q \in Q$, $S \subseteq A$;

(b) $(\bigvee S) * a = \bigvee_{s \in S} (s * a)$ for every $a \in A$, $S \subseteq Q$;

(c) $q_1 * (q_2 * a) = (q_1 \cdot q_2) * a$ for every $q_1, q_2 \in Q$, $a \in A$;

(d) $e * a = a$ for every $a \in A$.

(ii) The morphisms are maps $(A, \leqslant, *) \xrightarrow{f} (B, \leqslant, *)$ such that

(a) $f(\bigvee S) = \bigvee f(S)$ for every $S \subseteq A$;

(b) $f(q * a) = q * f(a)$ for every $a \in A$, $q \in Q$.

We consider the category Q-**Mod** as a concrete category over **Set** in the following way.

Definition 2.4. Define the forgetful functor Q-**Mod** $\xrightarrow{U}$ **Set** as follows:

$$U((A, \leqslant, *) \xrightarrow{f} (B, \leqslant, *)) = A \xrightarrow{f} B.$$

Definitions 2.3 and 2.4 yield a construct $(Q\text{-}\mathbf{Mod}, U)$. One can easily see that **2-Mod** is concretely isomorphic to **JCPos** (compare with integers Z in case of the category R-**Mod** of modules over the ring R). Thus while considering the category Q-**Mod** we study the category **JCPos** as well.

Remark 2.1. For $Q = \mathbf{1}$ it follows that $A \in \mathcal{O}b(Q\text{-}\mathbf{Mod})$ iff $A \cong \mathbf{1}$, i.e., **1-Mod** is equivalent to the terminal category.

The following theorem states a useful property of the category Q-**Mod**.

Theorem 2.1. *The category Q-**Mod** is a monadic construct.*

Corollary 2.2. *The category Q-**Mod** is complete, cocomplete, wellpowered, extremally co-wellpowered, and has regular factorizations.*

3. Completion of partially ordered sets

In this section we generalize the standard method of completion of posets given in the following proposition (see [1]).

Proposition 3.1. *The category **JCPos** is reflective in **Pos**.*

Proof. Given a poset A one has the complete lattice B_A of all lower-sets of A and the embedding $A \hookrightarrow B_A : a \mapsto \downarrow a$ which is the reflection arrow for A. ∎

We are going to generalize the result for the category Q-**Mod**. The first approach is as follows.

Proposition 3.2. *Let Q be a quantale. Then the category Q-**Mod** is reflective in **Pos**.*

Proof. Given a poset A, the reflection arrow can be constructed as follows. Define $B_A = \{h \in Q^A \mid a \leqslant b \text{ implies } h(b) \leqslant h(a)\}$ and let $A \xrightarrow{r} B_A : a \mapsto \downarrow a$ where

$$\downarrow a : A \longrightarrow Q : b \mapsto \begin{cases} e, & b \leqslant a \\ \bot, & \text{otherwise.} \end{cases}$$

Given a Q-**Mod**-object B and a **Pos**-morphism $A \xrightarrow{f} B$, there exists a unique Q-**Mod**-morphism $B_A \xrightarrow{\overline{f}} B$ such that $\overline{f} \circ r = f$, i.e., $\overline{f} : B_A \longrightarrow B : h \mapsto \bigvee_{a \in A} h(a) * f(a)$. ∎

Another approach is more sophisticated. Start with the following definition.

Definition 3.1. Given a quantale Q, define the category Q-**Pos** as follows:

(i) The objects are triples $(A, \leqslant, *)$ where $(A, \leqslant)$ is a poset and $Q \times A \xrightarrow{*} A$ is a map such that

(a) the map $A \xrightarrow{q*\_} A$ is order-preserving for every $q \in Q$;

(b) the map $Q \xrightarrow{\_*a} A$ is order-preserving for every $a \in A$;

(c) $q_1 * (q_2 * a) = (q_1 \cdot q_2) * a$ for every q_1, $q_2 \in Q$, $a \in A$;

(d) $e * a = a$ for every $a \in A$.

(ii) The morphisms are maps $(A, \leqslant, *) \xrightarrow{f} (B, \leqslant, *)$ such that

(a) f is order-preserving;

(b) $q * f(a) \leqslant f(q * a)$ for every $a \in A$, $q \in Q$ (notice that we use a rather non-standard definition of morphisms since one would expect ”=” instead of ”$\leqslant$”).

The objects of the category Q-**Pos** will be referred to as Q-posets. One can consider the category Q-**Pos** as a concrete category over **Set** in the following way.

Definition 3.2. Define the forgetful functor Q-**Pos** $\xrightarrow{U}$ **Set** as follows:

$$U((A, \leqslant, *) \xrightarrow{f} (B, \leqslant, *)) = A \xrightarrow{f} B.$$

Definitions 3.1 and 3.2 give a construct (Q-**Pos**, U). One can easily see that **2**-**Pos** is concretely isomorphic to **Pos** (for a poset A let $* : \mathbf{2} \times A \longrightarrow A : (q, a) \mapsto a$). Thus while considering the category Q-**Pos** we study the category **Pos** as well.

Every Q-**Pos**-object A has the following map

$$\_ \to \_ : A \times A \longrightarrow Q : (a, b) \mapsto \bigvee \{q \in Q \mid q * a \leqslant b\}.$$

Consider a property of the aforesaid map. Start by recalling some preliminary notions (cf. Chapter 0–3 in [2]).

Definition 3.3. Let **C** be an ordered category (i.e., *hom*-sets are partially ordered and composition on both sides is order-preserving). A pair of **C**-morphisms $A \underset{d}{\overset{g}{\rightleftarrows}} B$ is called *an adjunction* between A and B provided that $id_B \leqslant g \circ d$ and $d \circ g \leqslant id_A$.

Notice that for every quantale Q one has the ordered category Q-**Pos**.

Definition 3.4. Let A be a Q-poset and let $q \in Q$. Define $A \xrightarrow{q_A} A : a \mapsto q * a$.

Given a Q-**Pos**-morphism $A \xrightarrow{f} B$, it follows that $q_B \circ f \leqslant f \circ q_A$ for every $q \in Q$. Moreover, Definition 3.4 gives the following characterization of adjunctions in Q-**Pos**.

Lemma 3.3. *Let* $A \underset{d}{\overset{g}{\rightleftarrows}} B$ *be* Q-**Pos**-*morphisms. The following are equivalent:*

(i) (g, d) *is an adjunction between* A *and* B*;*

(ii) $q_B \leqslant g \circ q_A \circ d$ *and* $d \circ q_B \circ g \leqslant q_A$ *for every* $q \in Q$*.*

Proof. (i)⇒(ii) If $q \in Q$, then $q_B \leqslant q_B \circ (g \circ d) \leqslant g \circ q_A \circ d$ and $d \circ q_B \circ g \leqslant (d \circ g) \circ q_A \leqslant q_A$.

(ii)⇒(i) Set $q = e$. ∎

Theorem 3.4. *Let* $A \underset{d}{\overset{g}{\rightleftarrows}} B$ *be maps between* Q*-posets. The following are equivalent:*

(i) (g, d) *is an adjunction between* A *and* B*;*

(ii) (a) d *is a* Q-**Pos**-*morphism;*

(b) $g(a) = \max d^{-1}[\downarrow a]$ *for every* $a \in A$*;*

(c) $d \circ q_B \circ g \leqslant q_A$ *for every* $q \in Q$*.*

Proof. (i)⇒(ii) See the proof of Theorem 0–3.2 in [2] and use Lemma 3.3.

(ii)⇒(i) By Theorem 0–3.2 in [2] (g, d) is an adjunction in **Pos**. By $q_B \circ g \leqslant (g \circ d) \circ q_B \circ g = g \circ (d \circ q_B \circ g) \leqslant g \circ q_A$, g is a Q-**Pos**-morphism. ■

Now the promised property.

Corollary 3.5. *Let A be a Q-poset and let $a \in A$. The following are equivalent:*

(i) *$(a \to \_, \_ * a)$ is an adjunction between A and Q (and thus $a \to (\bigwedge S) = \bigwedge_{s \in S} (a \to s)$ for every $S \subseteq A$ such that $\bigwedge S$ exists in A);*

(ii) *$(a \to b) * a \leqslant b$ for every $b \in A$.*

Corollary 3.5 gives rise to the following definition which will be useful for us later.

Definition 3.5. Given a Q-poset A, say that it satisfies condition $(\mathfrak{A})$ provided that $(a \to b) * a \leqslant b$ for every $a, b \in A$.

Every Q-module satisfies condition $(\mathfrak{A})$. The situation with Q-posets, however, is different as shows the following example. Let $\mathbf{2}$ be ordered by equality. Define $Q \times \mathbf{2} \xrightarrow{*} \mathbf{2} : (q, a) \mapsto a$. Then $\mathbf{2}$ is a Q-poset which does not satisfy $(\mathfrak{A})$.

Return to the completion of posets.

Lemma 3.6. *Let A be a Q-**Pos**-object. Then $(b \to c) \cdot (a \to b) \leqslant (a \to c)$ for every a, b, $c \in A$.*

Proof. Straightforward computations. ■

Now the main proposition.

Proposition 3.7. *Let Q be a quantale. Then the category Q-**Mod** is reflective in Q-**Pos**.*

Proof. Given a Q-poset A, the reflection arrow can be constructed as follows. Define $B_A = \{h \in Q^A \,|\, h(b) \cdot (a \to b) \leqslant h(a)$ for every $a, b \in A\}$. Then B_A is a submodule of Q^A (closed under arbitrary meets) and $h \in B_A$ iff $h(a) = \bigvee_{b \in A} h(b) \cdot (a \to b)$ for every $a \in A$. Let $A \xrightarrow{r} B_A : a \mapsto \_ \to a$. By Lemma 3.6 the map r is correct. For the rest see Proposition 3.2. ■

Below are some properties of the reflection arrow $A \xrightarrow{r} B_A$.

Lemma 3.8. *Let Q be completely distributive. Then r preserves all existing meets.*

Proof. Let $S \subseteq A$ be such that $\bigwedge S$ exists in A and let $a \in A$. Show that $a \to (\bigwedge S) = \bigwedge_{s \in S}(a \to s)$. Set $a \to (\bigwedge S) = \bigvee T$ and $\bigwedge_{s \in S}(a \to s) = \bigwedge_{s \in S} \bigvee T_s$. It will be enough to show that $\bigwedge_{s \in S} \bigvee T_s \leqslant \bigvee T$. By the assumption, $\bigwedge_{s \in S} \bigvee T_s = \bigvee_{f \in F} \bigwedge_{s \in S} f(s)$ where F is the set of choice functions defined on S. Since $\bigwedge_{s \in S} f(s) \in T$ for every $f \in F$, the result follows. ■

Lemma 3.9. *Let A satisfy condition $(\mathfrak{A})$. Then r is injective and preserves all existing meets.*

Proof. Since the second statement follows from Corollary 3.5 we show that r is injective. Let $a, b \in A$ with $r(a) = r(b)$. Then $e \leqslant a \to a = (r(a))(a) = (r(b))(a) = a \to b$ implies $a = e * a \leqslant (a \to b) * a \leqslant b$. Similarly $b \leqslant a$. ■

In conclusion let us note that it would be interesting to consider other generalizations of completions, e.g., of the Dedekind-MacNeille completion (see, e.g., [6]).

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