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COMMUTATIVE DIRECTOIDS WITH SECTIONAL INVOLUTIONS*

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Abstract

The concept of a commutative directoid was introduced by J. Ježek and R. Quackenbush in 1990. We complete this algebra with involutions in its sections and show that it can be converted into a certain implication algebra. Asking several additional conditions, we show whether this directoid is sectionally complemented or whether the section is an NMV-algebra.

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The concept of a commutative directoid was introduced by J. Ježek and R. Quackenbush [2] as follows: a groupoid $\mathcal{A} = (A; \sqcup)$ is a *commutative directoid* if it satisfies the following identities

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(i) $x \sqcup x = x$;

(ii) $x \sqcup y = y \sqcup x$;

(iii) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

When introducing a binary relation $\leq$ by setting

(S) $$x \leq y \quad \text{if and only if} \quad x \sqcup y = y,$$

then $\leq$ is the induced order of $\mathcal{A}$ and two conditions are satisfied:

(a) $x, y \leq x \sqcup y$;

(b) if $x \leq y$ then $x \sqcup y = y$.

Also conversely, if $(A; \leq)$ is an ordered set which is upward directed and for any couple of elements $x, y \in A$ we choose an element $x \sqcup y \in A$ such that (a), (b) are satisfied then $\mathcal{A} = (A; \sqcup)$ becomes a commutative directoid.

If a commutative directoid has a greatest element, it will be denoted by 1. Now, let $\mathcal{A} = (A; \sqcup)$ be a commutative directoid with 1. For $a \in A$ we call the interval $[a, 1]$ a *section*. We say that a mapping $f_a : [a, 1] \to [a, 1]$ is a *sectional involution* if

(c) $f_a(f_a(x)) = x$ for each $x \in [a, 1]$;

(d) $f_a(a) = 1$ and $f_a(1) = a$.

In the sake to avoid huge notation, the sectional mapping f_a will be denoted simply by a superscript, i.e. $f_a(x)$ will be denoted as x^a.

A commutative directoid with 1 will be called *with sectional antitone involutions* if there is a sectional involution on $[a, 1]$ for each $a \in A$. For each $x, y \in A$ clearly $y \leq x \sqcup y$ thus $x \sqcup y \in [y, 1]$ and we can define the so-called *derived operation* "$\circ$" as follows

$$x \circ y = (x \sqcup y)^y.$$

Theorem 1. *Let $\mathcal{A} = (A; \sqcup)$ be a commutative directoid with sectional involutions. The derived operation satisfies the following identities:*

(D1) $1 \circ x = x$, $x \circ x = 1$, $x \circ 1 = 1$;

(D2) $(x \circ y) \circ y = (y \circ x) \circ x$;

(D3) $x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1$;

(D4) $((x \circ y) \circ y) \circ y = x \circ y$.

Proof. We compute

$$1 \circ x = (1 \sqcup x)^x = 1^x = x;$$

$$x \circ x = (x \sqcup x)^x = x^x = 1;$$

$$x \circ 1 = (x \sqcup 1)^1 = 1^1 = 1.$$

Further, since $x \sqcup y \in [y, 1]$ also $(x \sqcup y)^y \geq y$ thus

$$(x \circ y) \circ y = ((x \sqcup y)^y \sqcup y)^y = (x \sqcup y)^{yy} = x \sqcup y$$

whence

$$(y \circ x) \circ x = y \sqcup x = x \sqcup y = (x \circ y) \circ y.$$

Analogously we compute using the previous identity

$$x \circ ((((x \circ y) \circ y) \circ z) \circ z) = x \circ ((x \sqcup y) \sqcup z)$$

$$= (x \sqcup ((x \sqcup y) \sqcup z))^{(x \sqcup y) \sqcup z} = ((x \sqcup y) \sqcup z)^{(x \sqcup y) \sqcup z} = 1.$$

Finally,

$$((x \circ y) \circ y) \circ y = (x \sqcup y) \circ y = ((x \sqcup y) \sqcup y)^y = (x \sqcup y)^y = x \circ y.$$

■

Let $\mathcal{P} = (P; \circ, 1)$ be an algebra of type $(2, 0)$ satisfying the identities (D1)–(D4) of Theorem 1. Then $\mathcal{P}$ will be called a *d-implication algebra*.

This name is motivated by the fact that $\mathcal{P}$ satisfies several axioms of the propositional connective "implication" thus $x \circ y$ can be read as $x \Rightarrow y$ in a certain non-classical logic. Moreover, if $\mathcal{A} = (A; \sqcup)$ is a commutative directoid with sectional involutions and $\circ$ its derived operation then the $d-$implication algebra $(A; \circ)$ will be said *induced by* $\mathcal{A}$ and denoted by $\mathcal{P}(A) = (A; \circ, 1)$.

Lemma 1. *Let $\mathcal{P} = (P; \circ, 1)$ be a d-implication algebra. Define a binary relation $\leq$ on P as follows*

$$\text{(R)} \qquad x \leq y \ \textit{ if and only if } \ x \circ y = 1.$$

Then $\leq$ is an order on P and $x \leq 1$ for each $x \in P$. If $\mathcal{A} = (A; \sqcup)$ is a commutative directoid with sectional involutions then the order induced on $\mathcal{P}(A)$ (by the rule (R)*) coincides with the induced order of $\mathcal{A}$.*

Proof. By (D1), $\leq$ is reflexive and $x \leq 1$ for each $x \in P$. If $x \leq y$ and $y \leq x$ then, by (D2) and (D1),

$$x = 1 \circ x = (y \circ x) \circ x = (x \circ y) \circ y = 1 \circ y = y.$$

If $x \leq y$ and $y \leq z$ then $x \circ y = 1$, $y \circ z = 1$ and applying (D3), we have

$$x \circ z = x \circ (1 \circ z) = x \circ ((y \circ z) \circ z) = x \circ (((1 \circ y) \circ z) \circ z)$$
$$= x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1.$$

Hence $\leq$ is an order on P.

Now, suppose $x \sqcup y = y$ in $\mathcal{A}$. Then in $\mathcal{P}(A)$ we have $x \circ y = (x \sqcup y)^y = y^y = 1$. Conversely, if $x \circ y = 1$ in $\mathcal{P}(A)$ then, as shown in the proof of Theorem 1,

$$x \sqcup y = (x \circ y) \circ y = 1 \circ y = y.$$

Hence, the order defined by (R) on $\mathcal{P}(A)$ coincides with that of $\mathcal{A}$ defined by (S). ∎

Lemma 2. *Let $\mathcal{P} = (P; \circ, 1)$ be a d-implication algebra. Then $\mathcal{P}$ satisfies the identity $x \circ (y \circ x) = 1$.*

Proof. Using of (D4), (D2) and (D1), we derive

$$x \circ (y \circ x) = ((x \circ (y \circ x))(y \circ x)) \circ (y \circ x)$$

$$= (((y \circ x) \circ x) \circ x) \circ (y \circ x) = (y \circ x) \circ (y \circ x) = 1.$$

■

We are ready to prove the converse of Theorem 1.

Theorem 2. *Let $\mathcal{P} = (P; \circ, 1)$ be a d-implication algebra. Define $x \sqcup y = (x \circ y) \circ y$ and for $x \in [y, 1]$ we put $x^y = x \circ y$. Then $\mathcal{A}(P) = (P; \sqcup)$ be a commutative directoid with sectional involutions.*

Proof. We compute easily

(i) $x \sqcup x = (x \circ x) \circ x = 1 \circ x = x,$

(ii) $x \sqcup y = (x \circ y) \circ y = (y \circ x) \circ x = y \sqcup x,$

(iii) $x \sqcup ((x \sqcup y) \sqcup z) = (x \circ ((((x \circ y) \circ y) \circ z) \circ z)) \circ ((((x \circ y) \circ y) \circ z) \circ z)$

$= 1 \circ ((((x \circ y) \circ y) \circ z) \circ z) = (x \sqcup y) \sqcup z,$

thus $(P; \sqcup)$ is a commutative directoid. Moreover, $x \sqcup 1 = (x \circ 1) \circ 1 = 1$ thus 1 is the greatest element of $(P; \sqcup)$.

Suppose $x \in [y, 1]$. Then $y \leq x$ and $x^{yy} = (x \circ y) \circ y = x \sqcup y = x$. Further, by Lemma 2, $x \circ y \geq y$ thus $x^y \in [y, 1]$, i.e. the mapping $x \mapsto x^y$ is an involution on $[y, 1]$ with $y^y = y \circ y = 1$ and $1^y = 1 \circ y = y$ thus $\mathcal{A}(P)$ has sectional involutions. ■

The directoid $\mathcal{A}(P)$ will be called *induced* by a d-implication algebra $\mathcal{P} = (P; \circ, 1)$. We show that these derived algebras are in one-to-one correspondence.

Theorem 3. *Let $\mathcal{A} = (A; \sqcup)$ be a commutative directoid with sectional involutions and $\mathcal{P} = (P; \circ, 1)$ be a d-implication algebra. Then $\mathcal{A}(\mathcal{P}(A)) = \mathcal{A}$ and $\mathcal{P}(\mathcal{A}(P)) = \mathcal{P}$.*

Proof. Denote by $\cup$ the operation in $\mathcal{A}(\mathcal{P}(A))$ and by f_y the sectional mapping in $\mathcal{A}(\mathcal{P}(A))$. Then for $x, y \in A$ we have

$$x \cup y = (x \circ y) \circ y = ((x \sqcup y)^y \sqcup y)^y = (x \sqcup y)^{yy} = x \sqcup y$$

and for $x \in [y, 1]$ we have

$$f_y(x) = x \circ y = (x \sqcup y)^y = x^y$$

thus $\mathcal{A}(\mathcal{P}(A)) = \mathcal{A}$.
Denote by $\odot$ the operation of $\mathcal{P}(\mathcal{A}(P))$. Then

$$x \odot y = (x \sqcup y)^y = ((x \circ y) \circ y) \circ y = x \circ y \quad \text{for } x, y \in P$$

thus also $\mathcal{P}(\mathcal{A}(P)) = \mathcal{P}$. ■

Example 1. Consider a commutative directoid $\mathcal{A} = (A; \sqcup)$ where $A = \{a, b, c, d, 1\}$ and the operation is given by the table

$\sqcup$	a	b	c	d	1
a	a	c	c	d	1
b	c	b	c	d	1
c	c	c	c	1	1
d	d	d	1	d	1
1	1	1	1	1	1

Its induced order is visualized in Figure 1

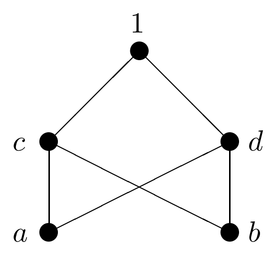

Figure 1

Pick up the following sectional involutions:

in $[a, 1]$ we have $a^a = 1$, $c^a = c$, $d^a = d$, $1^a = a$

in $[b, 1]$ we have $b^b = 1$, $c^b = d$, $d^b = c$, $1^b = b$

and trivially in other at most two-element sections.

Then the induced d-implication algebra $\mathcal{P}(A)$ is determined by the following table of the derived operation

$\circ$	a	b	c	d	1
a	1	d	1	1	1
b	c	1	1	1	1
c	c	d	1	d	1
d	d	c	c	1	1
1	a	b	c	d	1

Example shows that in the section $[b, 1]$ we have $x^b \sqcup x = 1$ for each $x \in [b, 1]$ but in $[a, 1]$ it is not valid since e.g. $c^a \sqcup c = c \sqcup c = c \neq 1$. This motivates us to introduce the following concept.

Let $\mathcal{A} = (A; \sqcup)$ be a commutative directoid with sectional involutions. $\mathcal{A}$ is called ***sectionally complemented*** if $x \sqcup x^a = 1$ for any $a \in A$ and each $x \in [a, 1]$.

Remark. Let $\mathcal{A} = (A; \sqcup)$ be a sectionally complemented commutative directoid. For each $a \in A$ define a new binary operation $\sqcap_a$ on $[a, 1]$ as follows

$$x \sqcap_a y = (x^a \sqcup y^a)^a.$$

Then clearly $x^a \sqcap_a x = (x^{aa} \sqcup x^a)^a = (x \sqcup x^a)^a = 1^a = a$ thus x^a is really a complement of x in the section $[a, 1]$.

We can characterize sectionally complemented commutative directoids by a simple property of the induced d-implication algebra.

Theorem 4. *Let $\mathcal{A} = (A; \sqcup)$ be a commutative directoid with sectional involutions. Then $\mathcal{A}$ is sectionally complemented if and only if the induced d-implication algebra $\mathcal{P}(A)$ satisfies the identity*

(C) $((x \circ y) \circ y) \circ (x \circ y) = x \circ y.$

Proof. Let $\mathcal{P}(A)$ satisfy (C) and $x \in [y, 1]$. Then $y \leq x$ thus $x \sqcup y = x$ and $x^y = x \circ y$ and we have by (C)

$$x \sqcup x^y = (((x \circ y) \circ y) \circ (x \circ y)) \circ (x \circ y) = (x \circ y) \circ (x \circ y) = 1.$$

Conversely, let $\mathcal{A}$ be a sectionally complemented commutative directoid. Since $x \sqcup y \in [y, 1]$ we have

$$(x \sqcup y) \sqcup (x \sqcup y)^y = 1.$$

This yields

$$(((x \circ y) \circ y) \circ (x \circ y)) \circ (x \circ y) = ((x \circ y) \circ y) \sqcup (x \circ y)$$

$$((x \circ y) \circ y) \sqcup (((x \circ y) \circ y) \circ y) = (x \sqcup y) \sqcup (x \sqcup y)^y = 1.$$

Multiplying this identity by $x \circ y$ in both sides and using of (D4) we obtain (C). ∎

Example 2. Consider the commutative directoid of Example 1 and change only the sectional involution in $[a, 1]$ as follows: $a^a = 1$, $c^a = d$, $d^a = c$, $1^a = a$. Then the corresponding table of the derived operation will be

$\circ$	a	b	c	d	1
a	1	d	1	1	1
b	d	1	1	1	1
c	d	d	1	d	1
d	c	c	c	1	1
1	a	b	c	d	1

One can easily check the identity (C) and hence this directoid is sectionally complemented.

On the contrary, that of Example 1 is not sectionally complemented since e.g. $((c \circ a) \circ a) \circ (c \circ a) = c \circ c = 1 \neq c = c \circ a$.

A certain non-associative generalization of C. C. Chang's MV-algebra was introduced in [2] under the name NMV-algebra. We recall that an algebra $\mathcal{M} = (M; \oplus, \neg, 0)$ is an *NMV-algebra* if it satisfies the following seven axioms where $1 = \neg 0$:

(N1) $x \oplus y = y \oplus x$;

(N2) $x \oplus 0 = x$;

(N3) $\neg\neg x = x$;

(N4) $x \oplus 1 = 1$;

(N5) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;

(N6) $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$;

(N7) $\neg x \oplus (x \oplus y) = 1$.

We can show that, under an additional condition, a section of a commutative directoid with sectional involutions can become an NMV-algebra.

Theorem 5. *Let $\mathcal{A} = (A; \circ, 1)$ be a d-implication algebra. If for some $b \in A$ it holds*

$$x \circ (y \circ b) = y \circ (x \circ b) \text{ for all } x, y \in [b, 1]$$

then for $x \oplus y = (x \circ b) \circ x$, $\neg x = x \circ b$ we have that $\mathcal{B} = ([b, 1]; \oplus, \neg, b)$ is an NMV-algebra.

Proof. Of course, $\neg b = b \circ b = 1$ and $(x \circ b) \circ b = x \sqcup b = x$ in the induced directoid thus (N2), (N3) are evident. Further, $x \oplus y = (x \circ b) \circ y = (x \circ b) \circ ((y \circ b) \circ b) = (y \circ b) \circ ((x \circ b) \circ b) = (y \circ b) \circ x = y \oplus x$ proving (N1). For (N4) we compute $x \oplus 1 = (x \circ b) \circ 1 = 1$.

Further, we have $\neg x \oplus y = ((x \circ b) \circ b) \circ y = x \circ y$ thus $\neg(\neg x \oplus y) \oplus y = (x \circ y) \circ y = (y \circ x) \circ x = \neg(\neg y \oplus x) \oplus x$ proving (N5).

The next condition (N6) is an easy consequence of (D3):

$$\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1.$$

It remains to prove (N7). Applying Lemma 2 we conclude

$$\neg x \oplus (x \oplus y) = x \circ (x \oplus y) = x \circ (y \oplus x) = x \circ ((x \circ b) \circ x) = 1. \qquad \blacksquare$$

Example 3. One can easily see that the section $[b, 1]$ of $\mathcal{A}$ in Example 1 can be converted into NMV-algebra as shown by Theorem 5 but the section $[a, 1]$ not because

$$c \circ (d \circ a) = c \circ d = d \neq c = d \circ c = d \circ (c \circ a).$$

On the contrary, in Example 2 both the non-trivial sections $[a, 1]$ and $[b, 1]$ can be converted into NMV-algebras.

References

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