

THE DIMENSION OF A VARIETY

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Abstract

Derived varieties were invented by P. Cohn in [4]. *Derived varieties of a given type* were invented by the authors in [10]. In the paper we deal with the derived variety V_σ of a given variety, by a fixed hypersubstitution σ . We introduce the notion of the *dimension of a variety* as the cardinality κ of the set of all proper derived varieties of V included in V .

We examine dimensions of some varieties in the lattice of all varieties of a given type τ . Dimensions of varieties of lattices and all subvarieties of regular bands are determined.

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1. NOTATIONS

By τ we denote a fixed type $\tau : I \rightarrow N$, where I is an index set and N is the set of all natural numbers. In the paper we deal only with finite types, i.e., $\text{card}(I)$ is finite. We use the definition of an n -ary *term* of type τ from [4, p. 6].

$T(\tau)$ denotes the set of all term symbols of type τ . For a given variety V of type τ , two terms p and q of type τ are called *equivalent* (in V) if the identity $p \approx q$ holds in V .

Definition 1.1. For a given type τ , F denotes the set of all fundamental operations $F = \{f_i : i \in I\}$ of type τ , i.e., $\tau(i)$ is the arity of the operation symbol f_i , for $i \in I$. Let $\sigma = (t_i : i \in I)$ be a fixed choice of terms of type τ with $\tau(t_i) = \tau(f_i)$, for every $i \in I$.

Recall from [10] (cf. [4, p. 13]), that for a given σ , the extension of σ to the map $\bar{\sigma}$ from the set $T(\tau)$ to $T(\tau)$, leaving all the variables unchanged and acting on composed terms as:

$$\bar{\sigma}(f_i(p_0, \dots, p_{n-1})) = \sigma(f_i)(\bar{\sigma}(p_0), \dots, \bar{\sigma}(p_{n-1}))$$

is called a *hypersubstitution* of type τ .

In the sequel, we shall use σ instead of $\bar{\sigma}$ for a hypersubstitution.

A hypersubstitution σ will be called *trivial*, if it is the identity mapping.

The set of all hypersubstitutions of type τ will be denoted by $H(\tau)$.

For any algebra $\mathbf{A} = (A, \Omega) = (A, (f_i^{\mathbf{A}} : i \in I)) \in V$, of type τ , the algebra $\mathbf{A}_\sigma = (A, (t_i^{\mathbf{A}} : i \in I))$ or shortly $\mathbf{A}_\sigma = (A, \Omega_\sigma)$, for $\Omega_\sigma = (t_i : i \in I)$ is called a *derived algebra* (of a given type τ) of \mathbf{A} , corresponding to σ , for any $\sigma \in H(\tau)$ (cf. [10, 17]).

Definition 1.2. The variety generated by the class of all derived algebras \mathbf{A}_σ , of algebras $\mathbf{A} \in V$ will be called the *derived variety* of V using σ and it will be denoted by V_σ , for any fixed $\sigma \in H(\tau)$.

For a class K of algebras of a given type τ , $D(K)$ denotes the class of all derived algebras of K for all possible choices of σ of type τ , i.e.:

$$D(K) = \bigcup \{K_\sigma : \sigma \in H(\tau)\}.$$

D is a class operator examined in [10] (cf. [16, 17]).

Let us note, that $V_\sigma = HSP(\sigma(V))$, for a given variety V and σ , where $\sigma(V)$ denotes the class of all derived algebras \mathbf{A}_σ , for $\mathbf{A} \in V$.

Recall from [12]:

Definition 1.3. For a given set Σ of identities of type τ , $E(\Sigma)$ denotes the set of all consequences of Σ by the rules (1)–(5) of inferences of G. Birkhoff (cf. [1, 12]).

$Mod(\Sigma)$ denotes the variety of algebras determined by Σ .

A variety V is *trivial* if all algebras in V are *trivial* (i.e., one-element). Trivial varieties will be denoted by T . A subclass W of a variety V which is also a variety is called *subvariety* of V .

V is a *minimal* (or *equationally complete*) variety if V is not trivial but the only subvariety of V , which is not equal to V is trivial.

We accept the following definition from [17]:

Definition 1.4. A derived variety V_σ is *proper* if V_σ is not equal to V , i.e., $V_\sigma \neq V$.

Note, that V_σ may be not proper only for nontrivial σ .

Recall from [10]:

Definition 1.5. A variety V of type τ is *solid* if V contains all derived varieties V_σ for every choice of σ of type τ , i.e., $D(V) \subseteq V$.

Definition 1.6. A variety V of type τ is *fluid* if the variety V contains no proper derived varieties V_σ for every choice of σ of type τ .

Fluid varieties appear naturally in many well known examples (cf. [11]). Derived varieties are an important tool for describing the lattice of all subvarieties of a given variety and therefore we expect some practical applications of the invented notion.

Note, that our definition of a *fluid variety* does not coincide with that of [17].

2. THE DIMENSION

Definition 2.1. If V is a variety of type τ , then the dimension of V is the cardinality κ of the set of all proper derived varieties V_σ of V included in V , for $\sigma \in H(\tau)$. We write then that $\kappa = \dim(V)$.

From the definitions above it follows that the trivial variety T of a given type is of dimension 0.

Theorem 2.1. *Minimal varieties are of dimension 0. Fluid varieties are of dimension 0.*

Later on we shall use the well-known *conjugate property* of [3] (cf. [9, p. 35] and [11]) and quote as:

Theorem 2.2. *Let \mathbf{A} be an algebra and σ be a hypersubstitution of type τ . Then an identity $p \approx q$ of type τ is satisfied in the derived algebra \mathbf{A}_σ if and only if the derived identity $\sigma(p) \approx \sigma(q)$ holds in \mathbf{A} .*

From the theorem above, it immediately follows:

Theorem 2.3. *Let V be a variety and two hypersubstitutions σ_1 and σ_2 of type τ be given. If $\sigma_1(f_i) \approx \sigma_2(f_i)$, is an identity of V for every $i \in I$, then the derived varieties V_{σ_1} and V_{σ_2} are equal.*

Proof. The proof follows by induction on the complexity of terms of type τ . ■

In the proof we use the relation \sim_V on sets of hypersubstitutions which was introduced by J. Płonka in [15] and used in [3] to determine the notion of *V-equivalent hypersubstitutions* in order to simplify the procedure of checking whether an identity is satisfied in a variety V as a hyperidentity.

Recall from [13, p. 221], that an algebra \mathbf{A} is locally finite iff every finitely generated subalgebra of \mathbf{A} is finite. A class of algebras is *locally finite* iff each of its members is a locally finite algebra.

Theorem 2.4. *Assume that a variety V (of a finite type) is locally finite. Then V is of a finite dimension.*

Proof. As V is locally finite, therefore every finitely generated free algebra in V is finite and therefore for every $n \in N$ there is only a finite number of non-equivalent n -ary terms in V . Moreover, in V there are only finitely many fundamental operations (by the assumption). Therefore in V there is only a finite number of non-equivalent hypersubstitutions of type τ . In consequence there are only finitely many derived varieties of V and $\dim(V)$ is finite. ■

3. DIMENSIONS OF VARIETIES OF LATTICES

We present some examples in lattice varieties as an answer to a problem posed by Brian Davey (La Trobe University, Australia) during the Conference on Universal Algebra and Lattice Theory (July 2005) at Szeged University, Szeged (Hungary).

Let $\mathbf{L} = (L, \vee, \wedge)$ be a lattice. A variety L_σ derived from a variety L of lattices must not be a variety of lattices.

This follows from the fact, that there are only four non-equivalent binary terms in lattices, namely x , y , $x \vee y$ and $x \wedge y$. Given a hypersubstitution σ of type (2,2). If σ is trivial, then the derived algebra \mathbf{L}_σ is \mathbf{L} itself. If one takes σ generated by $\sigma(x \vee y) = x \wedge y$ and $\sigma(x \wedge y) = x \vee y$, then \mathbf{L}_σ is the dual lattice $\mathbf{L}^d = (L, \wedge, \vee)$. Otherwise the derived algebra \mathbf{L}_σ is not a lattice at all, as some lattice axioms will be failed, unless \mathbf{L} is trivial (i.e., one-element lattice).

We got immediately:

Example 3.1. Let V be a nontrivial variety of lattices. Then a derived variety V_σ is the dual variety of lattices V^d or a variety which is not a variety of lattices.

Example 3.2. The variety L of all lattices in type (2,2) is fluid and not solid.

The variety L is fluid as it is selfdual, i.e., $L = L^d$. It is not solid, as the commutativity laws for \vee and \wedge are not satisfied as hyperidentities in lattices, for example.

Theorem 3.1. *Every variety of lattices is fluid.*

Proof. Let V be a variety of lattices. Consider the dual variety of V , i.e., the variety V^d of all dual lattices of V . Then there are only two possibilities:

(i) $V^d \subseteq V$ and consequently $V = V^d$

or

(ii) V and V^d are incomparable in the lattice of all varieties of lattices.

Therefore we conclude, that either V is selfdual or V and V^d are incomparable. In consequence V is fluid and $\dim(V) = 0$. ■

4. DIMENSIONS OF SUBVARIETIES OF REGULAR BANDS

In this section we concentrate on the lattice of all subvarieties of regular bands, described in [6, 7] and [8].

Definition 4.1. *Bands* is the variety B of algebras of type (2), defined by: associativity and idempotency (i.e., a *band* is an idempotent semigroup).

Following [5, p. 11], let us note, that the variety of bands has only six non-equivalent binary terms, therefore only six hypersubstitutions of type (2) in the variety of bands should be checked, namely: $\sigma_1 - \sigma_6$ defined as follows: $\sigma_1(xy) = x$, $\sigma_2(xy) = y$, $\sigma_3(xy) = xy$, $\sigma_4(x, y) = yx$, $\sigma_5(xy) = xyx$ and $\sigma_6(xy) = yxy$ to be considered in order to determine all derived varieties of a given subvariety of regular bands.

Recall Proposition 3.1.5(i) from [4, p. 11, 77]:

Definition 4.2. A variety V of type (2) is called *hyperassociative* if the associativity law is satisfied in V as a hyperidentity.

Proposition 4.1. *A variety of bands is hyperassociative if and only if it is contained in the variety $\text{Reg}B$ of regular bands.*

The propositions above may be considered as a motivation of our interest in the lattice of all subvarieties of the variety of regular bands.

In order to determine the dimension of all subvarieties of $\text{Reg}B$, we shall use the following two theorems of [11]:

Theorem 4.1. *The variety of B of all bands constitutes a not fluid and not solid variety of type (2).*

Theorem 4.2. *A variety V of bands is fluid if and only if it is minimal.*

Remark 4.1. Note, that a nontrivial variety V is of dimension 0 if and only if it is fluid.

Definition 4.3. An identity e of the form $p \approx q$ is called *leftmost* (*rightmost*) if and only if it has the same first (last) variable on each side. An identity which meets both of these conditions is called *outermost*.

First we express three technical lemmas:

Lemma 4.1. *Let Σ be a set of identities of type τ which are leftmost (or rightmost). Then the set $E(\Sigma)$ consists only of leftmost (rightmost) identities.*

Proof. The proof follows from the observation that all rules of inference (1)–(5) preserve the property of being the *leftmost* (or *rightmost*) identity. Therefore the closure of the set of left(right)most identities consists of left(right)most identities. ■

From [6, 7] and [8] it follows that every subvariety of the variety B of all bands is defined by one additional identity added to two axioms of bands (i.e., associativity and idempotency).

Lemma 4.2. *Assume that V and W are varieties of regular bands, W is defined by a single identity $p \approx q$, i.e., $W = \text{Mod}(p \approx q)$ (in the variety of regular bands). Then:*

$$V_\sigma \subseteq W, \text{ for a given } \sigma \in H(\tau),$$

if and only if the derived identity $\sigma(p) \approx \sigma(q)$ is satisfied in V , i.e., $V \models \sigma(p) \approx \sigma(q)$.

Proof. $V_\sigma = \text{HSP}(\sigma(V)) \subseteq W$ if and only if $\sigma(V) \models p \approx q$. By Theorem 2.2 we conclude that $\mathbf{A}_\sigma \models p \approx q$, for every algebra $\mathbf{A}_\sigma \in \sigma(V)$, if and only if $\mathbf{A} \models \sigma(p) \approx \sigma(q)$, for every algebra $\mathbf{A} \in V$, i.e., $V \models \sigma(p) \approx \sigma(q)$. ■

A simple generalization of the above lemma is the following:

Lemma 4.3. *Assume that V and W are varieties of type τ , W is defined by a set Σ of identities of type τ , i.e., $W = \text{Mod}(\Sigma)$. Then:*

$$V_\sigma \subseteq W, \text{ for a given } \sigma \in H(\tau),$$

if and only if the derived identity $\sigma(p) \approx \sigma(q)$ is satisfied in V , i.e.,

$$V \models \sigma(p) \approx \sigma(q), \text{ for every identity } p \approx q \in \Sigma.$$

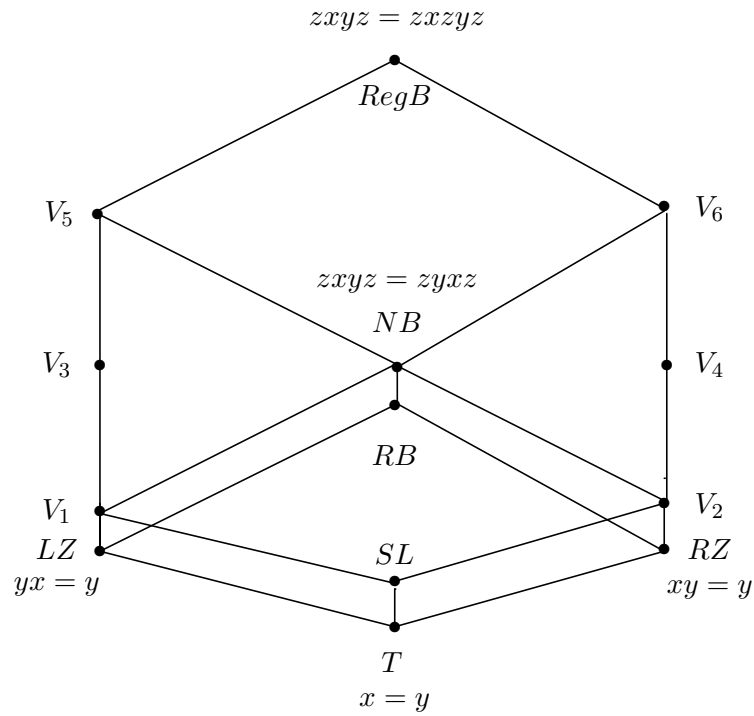
Proof. The proof is similar as that of Lemma 4.2, where $p \approx q$ is any identity of the given axiomatic Σ . ■

The next three propositions show some regularities in the dimensions of all subvarieties of regular bands described in [7, p. 244] and [8]:

Definition 4.4. The variety RB in the variety B bands is defined by the identity: $y \approx yxy$. It is called the variety of rectangular bands.

The fact that the variety RB is solid was proved in [5, p. 96].

We expressed the situation of theorems above on the diagram, which describes the bottom part of the lattice of all identities of bands, see [10] and [12, p. 244] Proposition 3.1.5 of [4]:



Theorem 4.3. *The variety RB is of dimension 2.*

Proof. The variety of RB of rectangular bands have only two nontrivial subvarieties, namely the variety LZ defined by the identities: $yx \approx y$ (called the variety of left-zero semigroups) and the variety RZ defined by $xy \approx y$

(called the variety of right-zero semigroups), respectively. Both of them are derived varieties of RB by the first and the second projection, respectively. To prove that, let $\mathbf{A}_{\sigma_1} \in (RB)_{\sigma_1}$, for $\mathbf{A} \in RB$. Then the identity $yx \approx y$ is satisfied in \mathbf{A}_{σ_1} , as: $\sigma_1(yx) \approx y \approx y \approx \sigma_1(y)$ is satisfied in \mathbf{A} and consequently in $(RB)_{\sigma_1}$. Similarly for σ_2 . We conclude that $\dim(RB) = 2$. ■

Theorem 4.4. *The varieties V_1 and V_2 of bands defined by the identities:*

$$(1) \quad zxy \approx zyx$$

and

$$(2) \quad yxz \approx xyz, \quad \text{respectively,}$$

are mutually derived by σ_4 . Moreover, $\dim(V_1) = \dim(V_2) = 1$.

Proof. Note, that the varieties V_1 and V_2 has only two proper nontrivial subvarieties, namely: the variety of left (right) zero-semigroups (respectively) and the variety SL of semilattices. The variety of semilattices, defined (in the variety of bands) by the commutativity law: $xy \approx yx$ is not a derived variety of V_1 , neither of V_2 . This follows from the fact, that if the variety SL of semilattices would be a derived variety of V_1 , then $SL = (V_1)_{\sigma_5}$ or $SL = (V_1)_{\sigma_6}$. This is impossible, via Theorem 1.3, as the derived identity of $xy \approx yx$ by the hypersubstitutions σ_5 (or σ_6), i.e., $\sigma_5(xy) \approx \sigma_5(yx)$ (or $\sigma_6(xy) \approx \sigma_6(yx)$) is of the form $xyx \approx yxy$ is neither leftmost nor rightmost and therefore, by Lemma 4.1 is not satisfied in V_1 as every identity satisfied in V_1 is leftmost. Similarly for V_2 . The proof follows from the fact that the only proper derived variety of V_1 included in V_1 is the variety LZ of left zero semigroups. Similarly, one can show that the only proper derived subvariety of V_2 by the second projection σ_2 is the variety RZ of right zero semigroups. Finally we conclude that $\dim(V_1) = \dim(V_2) = 1$. ■

Definition 4.5. Varieties of dimension 1 will be called *prefluid*.

Theorem 4.5. *The varieties V_3 and V_4 of bands defined by the identities:*

$$(3) \quad yx \approx yxy$$

and

$$(4) \quad xy \approx yxy, \quad \text{respectively,}$$

are mutually derived by σ_4 . Moreover, $\dim(V_3) = \dim(V_4) = 1$.

Proof. The variety $(V_3)_{\sigma_1}$ is the variety LZ of bands defined by $yx \approx y$. We obtain that: $(V_3)_{\sigma_1}$ is different from V_3 , therefore $(V_3)_{\sigma_1}$ is proper and $(V_3)_{\sigma_1} \subseteq V_3$. Note, that the derived variety $(V_3)_{\sigma_2}$ is proper and is the variety RZ of right-zero semigroups but is not included in V_3 . Similarly as in the previous theorem we conclude, that the variety SL of semilattices is not a derived variety of V_3 , as all the identities of V_3 are left-most. In order to exclude that, the variety V_1 defined by the identity (1) $zxy \approx zyx$ is the derived variety of V_3 by σ_5 consider the derived identity $\sigma_5(zxy) \approx \sigma_5(zyx)$ of (1) by σ_5 , i.e., the identity $zxyxz \approx zyxyz$. If this identity would be satisfied in V_3 , then the identity $zxyz \approx zyxz$ would be satisfied in V_3 , which is not true due to the results of [6]–[8]. Dually, the derived variety of V_3 by σ_6 is not the variety V_1 . Therefore we conclude, that $\dim(V_3) = 1$. Similarly one can prove that $\dim(V_4) = 1$. ■

Theorem 4.6. *The varieties V_5 and V_6 of bands defined by the identities:*

$$(5) \quad zxy \approx zxzy$$

and

$$(6) \quad yxz \approx yzxz, \quad \text{respectively,}$$

are mutually derived by σ_4 . Moreover, $\dim(V_5) = \dim(V_6) = 3$.

Proof. The proof that $(V_5)_{\sigma_1}$ ($(V_5)_{\sigma_2}$) is the variety $LZ(RZ)$ of left (right) zero semigroups follows from the proof of previous observations. Obviously: $\sigma_4(V_5) = V_6$, as the derived identity $\sigma_4(yxz) \approx \sigma_4(yzxz)$ of (6) by σ_4 gives rise to the identity (5) $zxy \approx zxzy$ and vice versa. Therefore V_6 and V_5 are mutually derived by σ_4 . We will show that the derived variety of V_5 by the hypersubstitution σ_5 is the variety V_3 , i.e., $\sigma_5(V_5) = V_3$. To show this consider the derived identity of (3) by σ_5 , i.e., the identity $\sigma_5(yx) \approx \sigma_5(yxy)$. This gives rise to the identity $yxy \approx yxyxy$, which is obviously satisfied in V_5 . Moreover, note that the derived identity of (1) by σ_5 , i.e., $\sigma_5(zxy) \approx \sigma_5(zyx)$ gives rise to the identity $zxyxz \approx zyxyz$, which can not be satisfied in V_5 , as otherwise the identity $zxyz \approx zyxz$ would be satisfied in V_3 , which is impossible by the results of [6]–[8] and it has been shown already in the proof of Theorem 4.5. Similarly one can show, that the derived variety of V_5 by σ_6 is the variety V_4 . We conclude that $\dim(V_5) = 3$. Similarly, $\dim(V_6) = 3$. ■

Definition 4.6. The variety NB of normal bands is defined by the identity:

$$(7) \quad zxyz \approx zyxz.$$

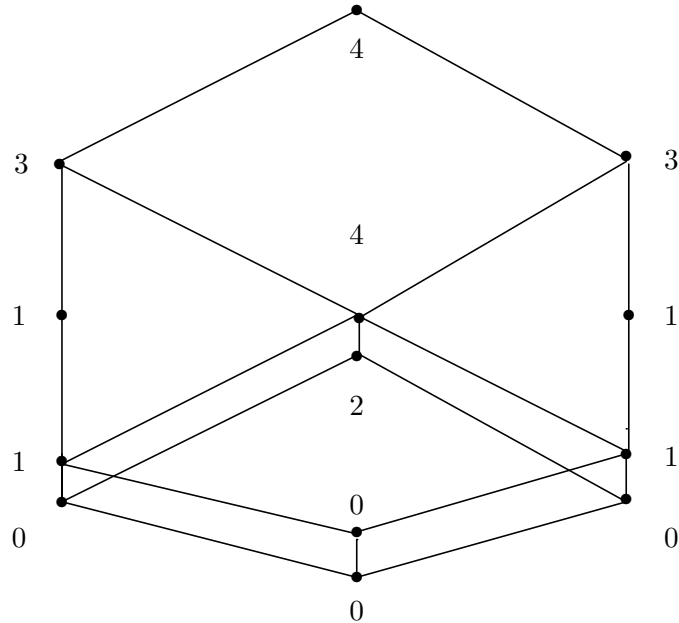
Theorem 4.7. $\dim(NB) = 4$.

Proof. For solidity of the variety NB confront [5, p. 96]. It follows, that all derived varieties of the variety NB are included in the variety of NB . Similarly as before we show that $(NB)_{\sigma_1}$ is the variety LZ of left-zero semigroups and $(NB)_{\sigma_2}$ is the variety RZ of right-zero semigroups. Both of them are proper subvarieties of NB . It is obvious that $(NB)_{\sigma_3} = (NB)_{\sigma_4} = NB$. We show only that $(NB)_{\sigma_5} = V_1$, as the derived identity of (1) $zxy \approx zyx$ by σ_5 , i.e., $\sigma_5(zxy) \approx \sigma_5(zyx)$ gives rise to the identity $zxyxz \approx zyxyz$ satisfied in NB . In order to exclude that the variety LZ of left zero semigroups, defined by the identity $yx \approx y$ equals to $(NB)_{\sigma_5}$, notice that the derived identity of $yx \approx y$ by σ_5 is the identity $xyy \approx y$, which is not satisfied in NB , as the variety of NB is defined by the set of regular identities (cf. [14]), which has only regular consequences. Similarly $(NB)_{\sigma_6} = V_2$, as the derived identity of (2) $yxz \approx xyz$ by σ_6 , i.e., $\sigma_6(yxz) \approx \sigma_6(xyz)$ gives rise to the identity $zxzyzxx \approx zyxzyz$ satisfied in NB and we conclude that $\dim(NB) = 4$. ■

Theorem 4.8. $\dim(RegB) = 4$.

Proof. For solidity of the variety $RegB$ confront [5, p. 96]. Two derived subvarieties of $RegB$ are LZ and RZ , by σ_1 and σ_2 , respectively. The derived varieties of $RegB$ via σ_3 and σ_4 are equal to $RegB$. We show that $(RegB)_{\sigma_5} = V_3$. To prove that, consider the derived identity of the identity (3) $yx \approx yxy$ by σ_5 , i.e., the identity $\sigma_5(yx) \approx \sigma_5(yxy)$ which gives rise to the identity $xyy \approx yxyxy$ which is satisfied in $RegB$. In order to show that the derived variety of $RegB$ by σ_5 is not equal to the variety V_1 , note that the derived identity of (1) $zxy \approx zyx$ by σ_5 , i.e., the identity $\sigma_5(zxy) \approx \sigma_5(zyx)$ gives rise to the identity $zxyxz \approx zyxyz$ which is not satisfied in $RegB$, as it was shown in the proof of Theorem 4.6 that this identity is not satisfied in V_5 , which is a subvariety of $RegB$. Similarly, one can show, that the derived variety of $RegB$ by σ_6 is the variety V_4 . This finishes the proof that $\dim(RegB) = 4$. ■

We expressed the situation of theorems above on the diagram:



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REFERENCES

- [1] G. Birkhoff, *On the structure of abstract algebras*, J. Proc. Cambridge Phil. Soc. **31** (1935), 433–454.
- [2] P.M. Cohn, *Universal Algebra*, Reidel, 1981 Dordrecht.
- [3] K. Denecke and J. Koppitz, *M-solid varieties of algebras*, Advances in Mathematics, Vol. 10, Springer 2006.
- [4] K. Denecke and S.L. Wismath, *Hyperidentities and Clones*, Gordon & Breach, 2000, ISBN 90-5699-235-X. ISSN 1041-5394.
- [5] T. Evans, *The lattice of semigroups varieties*, Semigroup Forum **2** (1971), 1–43.
- [6] Ch.F. Fennemore, *All varieties of bands*, Ph.D. dissertation, Pennsylvania State University 1969.

- [7] Ch.F. Fennemore, *All varieties of bands I*, Mathematische Nachrichten **48** (1971), 237–252.
- [8] J.A. Gerhard, *The lattice of equational classes of idempotent semigroups*, J. of Algebra **15** (1970), 195–224.
- [9] E. Graczyńska, *Universal algebra via tree operads*, Opole 2000, ISSN 1429-6063, ISBN 83-88492-75-6.
- [10] E. Graczyńska and D. Schweigert, *Hyperidentities of a given type*, Algebra Universalis **27** (1990), 305–318.
- [11] E. Graczyńska and D. Schweigert, *Derived and fluid varieties*, in print.
- [12] G. Grätzer, *Universal Algebra*. 2nd ed., Springer, New York 1979.
- [13] R. McKenzie, G.F. McNulty and W. Taylor, *Algebras, Lattices, Varieties*, vol. I, 1987, ISBN 0-534-07651-3.
- [14] J. Płonka, *On equational classes of abstract algebras defined by regular equations*, Fund. Math. **64** (1969), 241–247.
- [15] J. Płonka, *Proper and inner hypersubstitutions of varieties*, pp. 106–116 in: "Proceedings of the International Conference Summer School on General Algebra and Ordered Sets", Olomouc 1994.
- [16] D. Schweigert, *Hyperidentities*, pp. 405–506 in: Algebras and Orders, I.G. Rosenberg and G. Sabidussi, Kluwer Academic Publishers, 1993, ISBN 0-7923-2143-X.
- [17] D. Schweigert, *On derived varieties*, Discussiones Mathematicae Algebra and Stochastic Methods **18** (1998), 17–26.

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