**Nd-SOLID VARIETIES**

**Klaus Denecke**

*Universität Potsdam, Fachbereich Mathematik*
*Postfach 601553, 14415 Potsdam, Germany*
**e-mail:** kdenecke@rz.uni-potsdam.de

AND

**Prisana Glubudom**

*Chiangmai University, Department of Mathematics*
*Chiangmai, Thailand 50200*
**e-mail:** puprisana@yahoo.com

To the memory of Professor Kazimierz Glazek

**Abstract**

A non-deterministic hypersubstitution maps any operation symbol of a tree language of type $\tau$ to a set of trees of the same type, i.e. to a tree language. Non-deterministic hypersubstitutions can be extended to mappings which map tree languages to tree languages preserving the arities. We define the application of a non-deterministic hypersubstitution to an algebra of type $\tau$ and obtain a class of derived algebras. Non-deterministic hypersubstitutions can also be applied to equations of type $\tau$. Formally, we obtain two closure operators which turn out to form a conjugate pair of completely additive closure operators. This allows us to use the theory of conjugate pairs of additive closure operators for a characterization of $M$-solid non-deterministic varieties of algebras. As an application we consider $M$-solid non-deterministic varieties of semigroups.

**Keywords:** Non-deterministic hypersubstitution, conjugate pair of additive closure operators, $M$-solid non-deterministic variety.

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1. Introduction

Let \((f_i)_{i \in I}\) be an indexed set of operation symbols where \(f_i\) is \(n_i\)-ary, let \(X := \{x_1, \ldots, x_n, \ldots\}\) be a countably infinite set of variables and for each \(n \geq 1\) let \(X_n := \{x_1, \ldots, x_n\}\) be a finite set of variables. We denote by \(W_\tau(X)\) and \(W_\tau(X_n)\), respectively the sets of all terms of a finite type \(\tau = (n_i)_{i \in I}\) and of all \(n\)-ary terms of type \(\tau\). We use the well-known Galois connection \(\text{Id-Mod}\) between sets of identities and classes of algebras of a given type. For any set \(\Sigma\) of identities we denote by \(\text{Mod}_\Sigma\) the model class of all algebras of type \(\tau\) which satisfy all identities of \(\Sigma\); and for any class \(K\) of algebras of the same type we denote by \(\text{Id}_K\) the set of all identities satisfied by all algebras in \(K\). Classes of the form \(\text{Mod}_\Sigma\) are called varieties of algebras of type \(\tau\). If \(A\) satisfies the equation \(s \approx t\) as an identity, we write \(A \models s \approx t\) and if the class \(K\) of algebras of type \(\tau\) satisfies \(s \approx t\), we write \(K \models s \approx t\). If \(\Sigma \subseteq W_\tau(X)^2\) is a set of equations, then \(K \models s \approx t\) means that every equation from \(\Sigma\) is satisfied by every algebra from \(K\). Any subset of \(W_\tau(X)\), i.e. any element of the power set \(\mathcal{P}(W_\tau(X))\) or of \(\mathcal{P}(W_\tau(X_n))\) is called a tree language. Our restriction to a finite type is motivated by applications of tree languages in computer science. For tree languages one may define the following superposition operations

\[
\hat{S}_n^m : \mathcal{P}(W_\tau(X)) \times \mathcal{P}(W_\tau(X_n))^n \to \mathcal{P}(W_\tau(X_m))
\]

inductively by the following steps:

**Definition 1.1.** Let \(m, n \in \mathbb{N}^+ (:= \mathbb{N} \setminus \{0\})\) and let \(B \in \mathcal{P}(W_\tau(X_n))\) and \(B_1, \ldots, B_n \in \mathcal{P}(W_\tau(X_m))\) such that \(B, B_1, \ldots, B_n\) are non-empty.

(i) If \(B = \{x_j\}\) for \(1 \leq j \leq n\), then \(\hat{S}_n^m(\{x_j\}, B_1, \ldots, B_n) := B_j\).

(ii) If \(B = \{f_i(t_1, \ldots, t_{n_i})\}\), and if we assume that \(\hat{S}_n^m(\{t_j\}, B_1, \ldots, B_n)\) for \(1 \leq j \leq n\); are already defined, then \(\hat{S}_n^m(\{f_i(t_1, \ldots, t_{n_i})\}, B_1, \ldots, B_n) := \{f_i(r_1, \ldots, r_{n_i}) \mid r_j \in \hat{S}_n^m(\{t_j\}, B_1, \ldots, B_n)\text{ for }1 \leq j \leq n_i\}\).

(iii) If \(B\) is an arbitrary subset of \(W_\tau(X_n)\), we define

\[
\hat{S}_n^m(B, B_1, \ldots, B_n) := \bigcup_{b \in B} \hat{S}_n^m(\{b\}, B_1, \ldots, B_n).
\]
If one of the sets $B, B_1, \ldots, B_n$ is empty, we define $\hat{S}_m^n(B, B_1, \ldots, B_n) := \emptyset$. Then we may consider the heterogeneous algebra

$$\mathcal{P} - \text{clone } \tau := ((\mathcal{P}(W_\tau(X_n)))_{n \in \mathbb{N}^+}; (\hat{S}_m^n)_{m,n \in \mathbb{N}^+}, (\{x_i\})_{i \leq n, n \in \mathbb{N}^+})$$

which is called the power clone of $\tau$ ([?]). We mention that $\mathcal{P} - \text{clone } \tau$ satisfies the well-known clone axioms (C1), (C2), (C3) (see e.g. [?], [?]). If $\mathcal{P}_{\text{fin}}(W_\tau(X_n))$ is the set of all finite subsets of $W_\tau(X_n)$, then

$$\mathcal{P}_{\text{fin}} - \text{clone } \tau := ((\mathcal{P}_{\text{fin}}(W_\tau(X_n)))_{n \in \mathbb{N}^+}; (\hat{S}_m^n)_{m,n \in \mathbb{N}^+}, (\{x_i\})_{i \leq n, n \in \mathbb{N}^+})$$

is a subalgebra of $\mathcal{P} - \text{clone } \tau$ ([?]).

We mention also that there is a one-based version of $\mathcal{P} - \text{clone } \tau$, the algebra $\mathcal{P}_n - \text{clone } \tau_n := (\mathcal{P}(W_{\tau_n}(X_n)); \hat{S}_n, \{x_1\}, \ldots, \{x_n\})$ where $\tau_n$ is a finite type consisting of $n$-ary operation symbols only and where $\hat{S}_n := \hat{S}_n^1$. $\mathcal{P}_n - \text{clone } \tau_n$ is an example of a unitary Menger algebra of rank $n$ (see e.g [?]).

Similar structures can be obtained if one defines a superposition for sets of operations. Let $O^{(n)}(A)$ be the set of all $n$-ary operations $(n \geq 1)$ defined on the set $A$ and let $O(A) := \bigcup_{n \geq 1} O^{(n)}(A)$ be the set of all operations defined on $A$. Let $e_i^{n,A}$ be an $n$-ary projection defined on $A$, i.e., $e_i^{n,A}(a_1, \ldots, a_n) := a_i$ for $1 \leq i \leq n$, and let $\mathcal{P}(O^{(n)}(A))$ be the power set of $O^{(n)}(A)$.

**Definition 1.2.** Let $m, n \in \mathbb{N}^+$ and $B \in \mathcal{P}(O^{(n)}(A)), B_1, \ldots, B_n \in \mathcal{P}(O^{(m)}(A))$ such that $B, B_1, \ldots, B_n$ are non-empty.

(i) If $B = \{e_j^{n,A}\}$ for $1 \leq j \leq n$, then $\hat{S}_m^{n,A}(\{e_j^{n,A}\}, B_1, \ldots, B_n) := B_j$.

(ii) If $B = \{f_j^A(t_1^A, \ldots, t_n^A)\}$ with $f_j^A \in O^{(n_i)}(A), t_j^A \in O^{(n)}(A)$ and assume that $\hat{S}_m^{n,A}(t_j^A, B_1, \ldots, B_n)$ for $1 \leq j \leq n_i$ are already defined, then

$$\hat{S}_m^{n,A}(\{f_j^A(t_1^A, \ldots, t_n^A)\}, B_1, \ldots, B_n) := \{f_j^A(r_1^A, \ldots, r_n^A) \mid r_j^A \in \hat{S}_m^{n,A}(\{t_j^A\}, B_1, \ldots, B_n), 1 \leq j \leq n_i\}.$$ 

(iii) If $B \in \mathcal{P}(O^{(n)}(A))$ is arbitrary, then we define

$$\hat{S}_m^{n,A}(B, B_1, \ldots, B_n) := \bigcup_{b \in B} \hat{S}_m^{n,A}(\{b\}, B_1, \ldots, B_n).$$
If one of the sets \( B, B_1, \ldots, B_n \) is empty, then we define \( \hat{S}_m^{n,A}(B, B_1, \ldots, B_n) := \emptyset \). In this case we consider the heterogeneous algebra

\[
\mathcal{P}_A - \text{clone} := ((\mathcal{P}(O^{(n)}(A)))_{n \in \mathbb{N}^+}; (\hat{S}_m^{n,A})_{m,n \in \mathbb{N}^+}, (\{e_i^{n,A}\})_{i \leq n, n \in \mathbb{N}^+}).
\]

Let \( A = (A; (f_i^A)_{i \in I}) \) be an algebra of type \( \tau \). Then we may consider the subclone \( \mathcal{P}_A - \text{clone}A \) of \( \mathcal{P}_A - \text{clone} \) which is defined as follows.

**Definition 1.3.** Let \( n \in \mathbb{N}^+ \) and \( B \in \mathcal{P}(W_\tau(X_n)) \). Then we define the set \( B^A \) of term operations induced on the algebra \( A = (A; (f_i^A)_{i \in I}) \) as follows:

(i) If \( B = \{x_j\} \) for \( 1 \leq j \leq n \), then \( B^A := \{e_j^{n,A}\} \).

(ii) If \( B = \{f_i(t_1, \ldots, t_n)\} \) then \( B^A = \{f_i^A(t_1^A, \ldots, t_n^A)\} \) where \( f_i^A \) is the fundamental operation of \( A \) corresponding to the operation symbol \( f_i \) and where \( t_j^A \) are term operations on \( A \) which are induced in the usual way by the \( t_j \)'s.

(iii) If \( B \) is an arbitrary non-empty subset of \( W_\tau(X_n) \), then we define \( B^A := \bigcup_{b \in B} \{b\}^A \). If the set \( B \) is empty, then we define \( B^A := \emptyset \).

Let \( \mathcal{P}(W_\tau(X_n))^A \) be the collection of all sets of \( n \)-ary term operations induced by sets of \( n \)-ary terms of type \( \tau \) on the algebra \( A = (A; (f_i^A)_{i \in I}) \).

From these definitions we obtain the following

**Lemma 1.4.** Let \( B \in \mathcal{P}(W_\tau(X_n)) \) and let \( B_1, \ldots, B_n \in \mathcal{P}(W_\tau(X_m)) \). Then

\[
[\hat{S}_m^n(B, B_1, \ldots, B_n)]^A = \hat{S}_m^{n,A}(B^A, B_1^A, \ldots, B_n^A).
\]

**Proof.** If one of the sets \( B, B_1, \ldots, B_n \) is empty, then one of the sets \( B^A, B_1^A, \ldots, B_n^A \) is also empty. Thus

\[
[\hat{S}_m^n(B, B_1, \ldots, B_n)]^A = \emptyset^A = \emptyset = \hat{S}_m^{n,A}(B^A, B_1^A, \ldots, B_n^A).
\]

Assume now that all of \( B, B_1, \ldots, B_n \) are different from the empty set. At first we show by induction on the complexity of the term \( t \) that for one-element sets \( B = \{t\} \) our equation is satisfied.
For $t = x_i$ with $1 \leq i \leq n$, we have $B^A = \{x_i\}^A = \{e_i^{n,A}\}$ and

$$[\hat{S}_m^n(B, B_1, \ldots, B_n)]^A = [\hat{S}_m^n(\{x_i\}, B_1, \ldots, B_n)]^A$$

$$= B_i^A$$

$$= \hat{S}_m^n(\{e_i^{n,A}\}, B_1^A, \ldots, B_n^A)$$

$$= \hat{S}_m^n(\{x_i\}^A, B_1^A, \ldots, B_n^A)$$

$$= \hat{S}_m^n(B, B_1^A, \ldots, B_n^A).$$

Let now $t = f_i(t_1, \ldots, t_{n_i})$ and assume that for all $1 \leq k \leq n_i$,

$$[\hat{S}_m^n(\{t_k\}, B_1, \ldots, B_n)]^A = \hat{S}_m^n(\{t_k\}^A, B_1^A, \ldots, B_n^A).$$

Then

$$[\hat{S}_m^n(\{f_i(t_1, \ldots, t_{n_i})\}, B_1, \ldots, B_n)]^A$$

$$= \{f_i(r_1, \ldots, r_{n_i}) \mid r_k \in \hat{S}_m^n(\{t_k\}, B_1, \ldots, B_n), 1 \leq k \leq n_i\}^A$$

$$= \{f_i^A(r_1^A, \ldots, r_{n_i}^A) \mid r_k \in \hat{S}_m^n(\{t_k\}, B_1, \ldots, B_n), 1 \leq k \leq n_i\}$$

$$= \{f_i^A(r_1^A, \ldots, r_{n_i}^A) \mid r_k^A \in \hat{S}_m^n(\{t_k\}^A, B_1^A, \ldots, B_n^A), 1 \leq k \leq n_i\}$$

$$= \hat{S}_m^n(\{f_i^A(t_1^A, \ldots, t_{n_i}^A)\}, B_1^A, \ldots, B_n^A)$$

$$= \hat{S}_m^n(\{f_i(t_1, \ldots, t_{n_i})\}^A, B_1^A, \ldots, B_n^A).$$

If $B$ is a set of terms consisting of more than one element, then we have
\[\hat{S}_n^m(B, B_1, \ldots, B_n)^A = \left[ \hat{S}_m^\bigcup b \in B \{b\}, B_1, \ldots, B_n \right]^A\]

\[= \left[ \bigcup_{b \in B} \hat{S}_m^\{b\}, B_1, \ldots, B_n \right]^A\]

\[= \bigcup_{b \in B} \hat{S}_m^A(\{b\}, B_1^A, \ldots, B_n^A)\]

\[= \hat{S}_m^A \left( \bigcup_{b \in B} \{b\}^A, B_1^A, \ldots, B_n^A \right)\]

\[= \hat{S}_m^A(B^A, B_1^A, \ldots, B_n^A).\]

**Proposition 1.5.**

\[P_A - \text{clone } A = ( (P(W_\tau(X_n))^A)_{n \in \mathbb{N}^+}; (\hat{S}_m^A)_{m, n \in \mathbb{N}^+}; ( \{ e_i^{n, A} \})_{i \leq n, n \in \mathbb{N}^+} )\]

is a subalgebra of \( P_A - \text{clone}. \)

**Proof.** Let \( B^A \in P(W_\tau(X_n))^A \) and let \( B_1^A, \ldots, B_n^A \in P(W_\tau(X_m))^A \), then \( B \in P(W_\tau(X_n)) \) and \( B_1, \ldots, B_n \in P(W_\tau(X_m)) \).

From Lemma 1.4 we have that

\[\hat{S}_m^A(B^A, B_1^A, \ldots, B_n^A) = [\hat{S}_m^A(B, B_1, \ldots, B_n)]^A \in P(W_\tau(X_m))^A.\]

If \( T^{(n)}(A) \) is the set of all derived \( n \)-ary operations of the algebra \( A = (A; (f_i^A)_{i \in I}) \), then we can also consider the algebra \( P(T(A)) := ((P(T^{(n)}(A)))_{n \in \mathbb{N}^+}; (\hat{S}_m^A)_{n, m \in \mathbb{N}^+}; ( \{ e_i^{n, A} \})_{i \leq n, n \in \mathbb{N}^+}) \). It is not difficult to prove that \( P_A - \text{clone } A = P(T(A)) \).

Any mapping \( \sigma : \{ f_i \mid i \in I \} \rightarrow P(W_\tau(X)) \) with \( \sigma(f_i) \subseteq W_\tau(X_{n_i}), \) for \( i \in I \), is called a non-deterministic hypersubstitution (for short \( nd \)-hypersubstitution) of type \( \tau \). We denote by \( Hyp^{nd}(\tau) \) the set of all non-deterministic hypersubstitutions of type \( \tau \). Every \( nd \)-hypersubstitution can
be extended in the following inductive way to a mapping $\hat{\sigma} : \mathcal{P}(W_\tau(X)) \to \mathcal{P}(W_\tau(X))$.

(i) $\hat{\sigma}[\emptyset] := \emptyset$.

(ii) $\hat{\sigma}[\{x_i\}] := \{x_i\}$ for every variable $x_i \in X$.

(iii) $\hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}] := \hat{S}_n^{n_i}(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}])$ if we inductively assume that $\hat{\sigma}[\{t_k\}], 1 \leq k \leq n_i$, are already defined.

(iv) $\hat{\sigma}[B] := \bigcup\{\hat{\sigma}[\{b\}] \mid b \in B\}$ for $B \subseteq W_\tau(X)$.

In the sequel instead of $\hat{\sigma}[\{t\}]$ for a term $t \in W_\tau(X)$ we will simply write $\hat{\sigma}[t]$.

In [?] was proved that for every nd-hypersubstitution $\sigma$ the mapping $\hat{\sigma}$ is an endomorphism of $\mathcal{P}_-\text{clone } \tau$. We recall also that the set $Hyp_{nd}(\tau)$ forms a monoid with respect to the operation $\circ_{nd}$ defined by $\sigma_1 \circ_{nd} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and the identity element $\sigma_{pid} : f_i \mapsto \{f_i(x_1, \ldots, x_{n_i})\}$ for every $i \in I$.

In the next section we apply nd-hypersubstitutions to equations and to algebras.

2. The Conjugate Pair $(\chi^A_{nd}, \chi^E_{nd})$

If $A = (A; (f^A_i)_{i \in I})$ is an algebra of type $\tau$ and if $\sigma \in Hyp_{nd}(\tau)$ is an nd-hypersubstitution, then we define

$$\sigma(A) := \{(A; (l^A_i)_{i \in I}) \mid l_i \in \sigma(f_i)\}.$$ 

The set $\sigma(A)$ is called the set of derived algebras. Since for every sequence $(l_i)_{i \in I}$ of terms there is a hypersubstitution mapping $f_i$ to $l_i$ we can write $\sigma(A)$ also in the form $\sigma(A) = \{\rho(A) \mid \rho \in Hyp(\tau) \text{ with } \rho(f_i) \in \sigma(f_i) \text{ for } i \in I\}$. For a class $K$ of algebras of type $\tau$ we define

$$\sigma(K) := \bigcup_{A \in K} \sigma(A).$$

If $M \subseteq Hyp_{nd}(\tau)$ is the universe of a submonoid of $Hyp_{nd}(\tau)$, then we define $\chi^A_{M-nd}[K] := \bigcup_{\sigma \in M} \sigma(K)$. For $M = Hyp_{nd}(\tau)$ we will simply write $\chi^A_{nd}$. We notice that $\chi^A_{M-nd}[K]$ consists of algebras of the same type. For a set $K \in \mathcal{P}(\mathcal{P}(Alg(\tau)))$ of sets of algebras of type $\tau$ and a monoid $M$
algebras as follows:

\[ \sigma \text{an algebra of type } n \]

For the set \( \Sigma \) and for a set \( B \) we define the set \( \sigma \) of \( n \)-ary terms we define the set \( \{ \sigma \} \) of induced term operations induced by the set \( \sigma \) of derived algebras and for a set \( B \in \mathcal{P}(W_{\tau}(X)) \) of \( n \)-ary terms we define the set \( B^{\sigma(A)} \) of term operations induced by the set \( \sigma(A) \) of derived algebras as follows:

**Definition 2.1.** Let \( n \in \mathbb{N}^+ \) and \( B \in \mathcal{P}(W_{\tau}(X_n)) \), let \( A = (A; (f_i^A))_{i \in I} \) be an algebra of type \( \tau \), let \( \sigma \in Hyp^{nd}(\tau) \) be an \( nd \)-hypersubstitution and let \( \sigma(A) = \{(A; (t_i^A))_{i \in I} \mid i \in \sigma(f_i)\} \) be the set of derived algebras. Then we define the set \( B^{\sigma(A)} \) of term operations induced by the set \( \sigma(A) \) of derived algebras as follows:

(i) If \( B := \{x_j\} \) for \( 1 \leq j \leq n \), then \( B^{\sigma(A)} := \{e_j^{n,\rho(A)} \mid \rho(A) \in \sigma(A)\} = \{e_j^{n,\rho(A)}\} \).

(ii) If \( B = \{f_i(t_1, \ldots, t_{n_i})\} \) then

\[
B^{\sigma(A)} := \{\tilde{S}^{\rho(A)}(\{f_i^{\rho(A)} \mid \rho(A) \in \sigma(A)\}, \{t_1\}^{\sigma(A)}, \ldots, \{t_{n_i}\}^{\sigma(A)})
\]

\[
= \bigcup_{\rho(A) \in \sigma(A)} \{\tilde{S}^{\rho(A)}(\{f_i^{\rho(A)}\}, \{t_1\}^{\sigma(A)}, \ldots, \{t_{n_i}\}^{\sigma(A)})\}
\]

\[
= \bigcup_{\rho(A) \in \sigma(A)} \{f_i^{\rho(A)}(r_1, \ldots, r_{n_i}) \mid r_k \in \{t_k\}^{\sigma(A)}, \text{ for } 1 \leq k \leq n_i\}.
\]
where \( f_i^{\rho(A)} \) denotes the fundamental operation of the algebra \( \rho(A) \) belonging to the operation symbol \( f_i \) and assume that \( \{ t_k \}^{\sigma(A)}_{\leq k \leq n_i} \) are already defined.

(iii) If \( B \) is an arbitrary non-empty subset of \( W_{\tau}(X_n) \), then we define
\[
B^{\sigma(A)} := \bigcup_{b \in B} \{ b \}^{\sigma(A)}.
\]
If the set \( B \) is empty, then we define \( B^{\sigma(A)} := \emptyset \).

For any term \( t \in W_{\tau}(X_n) \) and a class \( G \) of algebras of type \( \tau \) we define
\[
t^{G} := \{ t \}^{G} := \{ t^A \mid A \in G \}.
\]

**Definition 2.2.** Let \( A \) be an algebra of type \( \tau \) and let \( K \subseteq \chi_{M-nd}[A] \) and let \( n \geq 1 \) be an integer. Then we define

(i) If \( B = \{ x_j \} \) for \( 1 \leq j \leq n \), then \( B^K = \{ e_j^{n,A} \} \subseteq T^{(n)}(A) \).

(ii) If \( B = \{ f_i(t_1, \ldots, t_n) \} \) and let \( B_j = t_j^K \subseteq T^{(n)}(A) \) for \( 1 \leq j \leq n_i \) are already known, then
\[
B^K := \{ \hat{S}^{n,A}_{n_i}(S, B_1, \ldots, B_{n_i}) \mid S = \{ \rho(f_i)^A \mid \rho \in Hyp(\tau), \rho(A) \in K \} \subseteq T^{(n)}(A) \}.
\]

Finally for an arbitrary nonempty set \( B \in \mathcal{P}(W_{\tau}(X)) \) we set \( B^K := \bigcup_{b \in B} \{ b \}^{K} \) and for the empty set \( B \) we let \( B^K := \emptyset \).

Definition 2.2 contains Definition 2.1 as a special case since for every \( \sigma \in Hyp^{nd}(\tau) \) we have \( \sigma(A) \subseteq \chi_{M-nd}[A] \). We have also \( \{ A \} \subseteq \chi_{M-nd}[A] \) and \( \{ \rho(A) \} \subseteq \chi_{M-nd}[A] \) for a hypersubstitution \( \rho \in Hyp(\tau) \) and it is easy to see that for a single term \( s \in W_{\tau}(X_n) \) we have \( \{ \hat{\rho}[s] \}^{A} = \{ \hat{\rho}[s]^A = s^{\rho(A)} = \{ s \}^{\rho(A)} \} \).

Now we prove:

**Lemma 2.3.** Let \( B \in \mathcal{P}(W_{\tau}(X_n)) \) be an arbitrary set of \( n \)-ary terms of type \( \tau \), let \( A = (A_i^{f_i})_{i \in I} \) be an algebra of type \( \tau \) and let \( \sigma \) be an \( nd \)-hypersubstitution of type \( \tau \). Then \( \hat{\sigma}[B]^A = B^{\sigma(A)} \).

**Proof.** If \( B \) is empty, then all is clear. If \( B \) is nonempty we will give a proof by induction on the complexity of the terms from the set \( B \).
If $B = \{x_j\}$ for $1 \leq j \leq n$, then $\hat{\sigma}[B]^A = \{x_j\}^A = \{e_j^n.A\}$ by the definition of $\sigma$ and by Definition 1.3. Further, by Definition 2.1 we have

$$B^\sigma(A) = \{x_j\}^{\sigma(A)} = \{ e_j^n.\rho(A) \mid \rho(A) \in \sigma(A) \} = \{ e_j^n.A \}$$

since all algebras $\rho(A)$ have the same universe. Therefore $\hat{\sigma}[B]^A = B^\sigma(A)$ for $B = \{x_j\}$ for $1 \leq j \leq n$

Now let $B = \{f_i(t_1, \ldots, t_{n_i})\}$ and assume that $\hat{\sigma}([t_k]^A) = \{t_k\}^{\sigma(A)}$ for $1 \leq k \leq n_i$. Then

$$\hat{\sigma}[B]^A = \hat{\sigma}([f_i(t_1, \ldots, t_{n_i})]^A)$$

$$= \left[ \hat{S}_n^A(\sigma(f_i), \hat{\sigma}([t_1]), \ldots, \hat{\sigma}([t_{n_i}])) \right]^A$$

$$= \hat{S}_n^A(\sigma(f_i)^A, \hat{\sigma}([t_1]^A), \ldots, \hat{\sigma}([t_{n_i}]^A))$$

$$= \hat{S}_n^A(\{l_i \mid l_i \in \sigma(f_i)\}^A, \hat{\sigma}([t_1]^A), \ldots, \hat{\sigma}([t_{n_i}]^A))$$

$$= \bigcup_{l_i \in \sigma(f_i)} \hat{S}_n^A(\{l_i^A, \hat{\sigma}([t_1]^A), \ldots, \hat{\sigma}([t_{n_i}]^A))$$

$$= \bigcup_{l_i \in \sigma(f_i)} \hat{S}_n^A(\{l_i^A, \{t_1\}^{\sigma(A)}, \ldots, \{t_{n_i}\}^{\sigma(A)})$$

$$= \hat{S}_n^A(\{f_i^A \mid \rho(A) \in \sigma(A)\}, \{t_1\}^{\sigma(A)}, \ldots, \{t_{n_i}\}^{\sigma(A)})$$

$$= \{f_i(t_1, \ldots, t_{n_i})\}^{\sigma(A)}$$

$$= B^\sigma(A).$$
If $B$ is a set of terms consisting of more than one element, then we have

$$\hat{\sigma}[B]^A = \left\{ \bigcup_{{b \in B}} \hat{\sigma}[\{b\}] \right\}^A = \bigcup_{{b \in B}} \hat{\sigma}[\{b\}]^A = \bigcup_{{b \in B}} \{b \}^{\sigma(A)} = B^{\sigma(A)}.$$ 

From Lemma 2.3 we obtain the "conjugate pair property" for the pair $(x^A_{M\text{-nd}}, x^A_{M\text{-nd}})$ of operators. We use the notation $A \models s \approx t$ if the algebra $A$ of type $\tau$ satisfies the equation $s \approx t$ of type $\tau$ as an identity and $K \models s \approx t$ if the class $K$ satisfies $s \approx t$. Moreover, we define

**Definition 2.4.** Let $B_1, B_2 \subseteq W_\tau(X)$ be sets of terms of type $\tau$ and assume that $A$ is an algebra of type $\tau$ and that $K \subseteq \chi^A_{M\text{-nd}}[A]$ for a monoid $M \subseteq \mathcal{H}(yp^{nd}(\tau))$ of non-deterministic hypersubstitution. Then

$$K \models B_1 \approx B_2 \iff B_1^K = B_2^K.$$ 

Especially we have $\sigma[A] \models B_1 \approx B_2 \iff B_1^{\sigma[A]} = B_2^{\sigma[A]}$ and $\{A\} \models B_1 \approx B_2 \iff B_1^{\{A\}} = B_2^{\{A\}}$ and this means $A \models B_1 \approx B_2$ if $B_1^A = B_2^A$.

From Lemma 2.3 we obtain the following conjugate property.

**Theorem 2.5.** Let $A$ be an algebra of type $\tau$, and let $B_1 \approx B_2 \in \mathcal{P}(W_\tau(X)) \times \mathcal{P}(W_\tau(X))$ and assume that $\sigma \in Hyp^{nd}(\tau)$ be a non-deterministic hypersubstitution of type $\tau$. Then

$$\sigma(A) \models B_1 \approx B_2 \iff A \models \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2].$$

**Proof.**

$$\sigma(A) \models B_1 \approx B_2 \iff B_1^{\sigma(A)} = B_2^{\sigma(A)}$$

$$\iff \hat{\sigma}[B_1]^A = \hat{\sigma}[B_2]^A$$

$$\iff A \models \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2].$$
Let now $M \subseteq Hyp^{nd}(\tau)$ be a monoid of non-deterministic hypersubstitutions. Then we form the set $\bigcup \{ \mathcal{P}(\chi^A_{M-nd}[A]) \mid A \in Alg(\tau) \}$ and consider $
abla \subseteq \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ and $\mathcal{K} \subseteq \bigcup \{ \mathcal{P}(\chi^A_{M-nd}[A]) \mid A \in Alg(\tau) \}$. Definition 2.4 defines a relation between both sets. In the usual way we obtain a Galois connection $(\mathcal{P}Mod; \mathcal{P}Id)$ of non-deterministic models and non-deterministic identities defined by

$$\mathcal{P}Mod\Sigma := \{ K \mid K \subseteq \chi^A_{M-nd}[A] \text{ for some algebra } A \in Alg(\tau) \}
$$

and $\forall B_1 \approx B_2 \in \Sigma(K \models B_1 \approx B_2)$

$$\mathcal{P}Id\mathcal{K} := \{ B_1 \approx B_2 \mid B_1 \approx B_2 \in \mathcal{P}(W_\tau(X))^2 \text{ and } \forall K \in \mathcal{K}(K \models B_1 \approx B_2) \}
$$

By definition, the operators $\chi^A_{M-nd} : \mathcal{P}(\mathcal{P}(Alg(\tau))) \rightarrow \mathcal{P}(\mathcal{P}(Alg(\tau)))$ and $\chi^E_{M-nd} : \mathcal{P}((\mathcal{P}(W_\tau(X)))^2) \rightarrow \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ are completely additive. This means, for classes $\mathcal{K} \subseteq \mathcal{P}(\mathcal{P}(Alg(\tau)))$ the result of the application of $\chi^A_{M-nd}$ to $\mathcal{K}$ is the union of the results obtained by application of $\chi^A_{M-nd}$ to the single classes $K \subseteq Alg(\tau) : \chi^A_{M-nd}[\mathcal{K}] = \bigcup_{\sigma \in M} \bigcup_{K \in \mathcal{K}} \sigma[K]$. In a corresponding way for a set $\Sigma \subseteq \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ and a submonoid $M \subseteq Hyp^{nd}(\tau)$ we have $\chi^E_{M-nd}[\Sigma] = \bigcup_{\sigma \in M} \bigcup_{K \approx B_1 \approx B_2 \in \Sigma} \sigma[B_1] \approx \sigma[B_2]$. Therefore, both operators are monotone, i.e.

$$\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \chi^A_{M-nd}[\mathcal{K}_1] \subseteq \chi^A_{M-nd}[\mathcal{K}_2]
$$

and

$$\Sigma_1 \subseteq \Sigma_2 \Rightarrow \chi^E_{M-nd}[\Sigma_1] \subseteq \chi^E_{M-nd}[\Sigma_2].$$

Since $\sigma_{pid} \in M$ and $\sigma_{pid}(K) = \{ K \}$, the operator $\chi^A_{M-nd}$ is extensive, i.e. $\mathcal{K} \subseteq \chi^A_{M-nd}[\mathcal{K}]$ for every class $\mathcal{K} \subseteq \mathcal{P}(\mathcal{P}(Alg(\tau)))$. Since $\sigma_{pid}(\{ B \}) = \{ B \}$ for every $B \in \mathcal{P}(W_\tau(X))$, the operator $\chi^E_{M-nd}$ is also extensive. It turns out that both operators, $\chi^A_{M-nd}$ and $\chi^E_{M-nd}$ are closure operators. Altogether, we have

**Theorem 2.6.** The pair $(\chi^A_{M-nd}, \chi^E_{M-nd})$ is a conjugate pair of additive closure operators.

**Proof.** From Theorem 2.5, there follows $\chi^A_{M-nd}[K] \models B_1 \approx B_2 \iff K \models \chi^E_{M-nd}[B_1 \approx B_2]$. By the previous remarks it is left to show that the operators $\chi^A_{M-nd}$ and $\chi^E_{M-nd}$ are idempotent. Extensivity of $\chi^A_{M-nd}$ and $\chi^E_{M-nd}$,
implies $\chi_{M-\text{nd}}^A[\mathcal{K}] \subseteq \chi_{M-\text{nd}}^A[\chi_{M-\text{nd}}^A[\mathcal{K}]]$ and $\chi_{M-\text{nd}}^E[\Sigma] \subseteq \chi_{M-\text{nd}}^E[\chi_{M-\text{nd}}^E[\Sigma]]$ for $\mathcal{K} \subseteq \{\{\mathcal{P}(\chi_{M-\text{nd}}^A[A]) \mid A \in \text{Alg}(\tau)\} \mid W \in \mathcal{P}(\mathcal{P}(\mathcal{W}(\tau(\mathcal{X})))^2)\}$. We write $\mathcal{K} \models W$ iff $K \models A \approx B$ for all $K \in \mathcal{K}$ and all $B_1 \approx B_2 \in W$. We have to show that the opposite inclusions are satisfied. Let $B \in \chi_{M-\text{nd}}^A[\chi_{M-\text{nd}}^A[\mathcal{K}]]$. Then there are $A$-hypersubstitutions $\sigma_1, \sigma_2 \in M$ and an algebra $A \in \mathcal{K}$ such that

$$B \in \sigma_1[\sigma_2(A)] = \sigma_1[\{(A; (l_i^A)_{i \in I}) \mid l_i \in \sigma_2(f_i)\}]$$

$$= \{(A; (l_i^A)_{i \in I}) \mid l_i \in \sigma_2(f_i)\}$$

$$= \{(A; (h_i^A)_{i \in I}) \mid h_i \in \hat{\sigma}_1[l_i] \text{ and } l_i \in \sigma_2(f_i)\}$$

$$= \{(A; (h_i^A)_{i \in I}) \mid h_i \in \hat{\sigma}_1[\sigma_2(f_i)]\}$$

$$= \{(A; (h_i^A)_{i \in I}) \mid h_i \in (\sigma_1 \circ_{\text{nd}} \sigma_2)(f_i)\}$$

$$= (\sigma_1 \circ_{\text{nd}} \sigma_2)(A) \in \chi_{M-\text{nd}}^A[\mathcal{K}].$$

This shows $\chi_{M-\text{nd}}^A[\chi_{M-\text{nd}}^A[\mathcal{K}]] = \chi_{M-\text{nd}}^A[\mathcal{K}]$. Now let $B_1 \approx B_2 \in \chi_{M-\text{nd}}^E[\chi_{M-\text{nd}}^E[\Sigma]]$. Then there is an equation $U \approx V$ in $\Sigma$ and an $nd$-hypersubstitution $\sigma_1, \sigma_2 \in M$ such that $B_1 \approx B_2 \in \hat{\sigma}_1[\sigma_2[U]] \approx \hat{\sigma}_1[\sigma_2[V]]$, i.e. $B_1 \approx B_2 \in (\sigma_1 \circ_{\text{nd}} \sigma_2)^{-1}[U] \approx (\sigma_1 \circ_{\text{nd}} \sigma_2)^{-1}[V] \in \chi_{M-\text{nd}}^E[U \approx V] \subseteq \chi_{M-\text{nd}}^E[\Sigma].$

### 3. $M$-Nd-Solid Varieties

A solid variety $V$ admits every mapping $\sigma : \{f_i \mid i \in I\} \to W(\tau)$ which maps $n_i - ary$ operation symbols $f_i$ to $n_i - ary$ terms in the sense that every derived algebra $\sigma(A) = (A; (\sigma(f_i^A))_{i \in I})$ belongs to $V$. Equivalently if $s \approx t$ is an identity in a solid variety $V$, then $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are also satisfied as identities in $V$ for every hypersubstitution $\sigma$. We generalize the definition of a solid variety to $M$-solid non-deterministic varieties.
Definition 3.1. Let $M \subseteq \mathcal{Hyp}^{nd}(\tau)$ be a monoid of non-deterministic hypersubstitutions of type $\tau$. A variety $V$ of type $\tau$ is said to be an $M$-solid non-deterministic variety, for short an $M - nd$-solid variety, if \( \{ \{ A \} \mid A \in V \} \models \{ \hat{\sigma}(s) \approx \hat{\sigma}(t) \} \mid s \approx t \in IdV, \sigma \in M \} \). In the case that $M = \mathcal{Hyp}^{nd}(\tau)$ we will speak of a solid non-deterministic variety, for short of an $nd$-solid variety.

Clearly, the class $Alg(\tau)$ of all algebras of type $\tau$ is $nd$-solid. The trivial variety (consisting only of one-element algebras of type $\tau$) is also $nd$-solid. The class of all $nd$-solid varieties of type $\tau$ is contained in the class of all solid varieties of this type.

Example 3.2. There is no nontrivial $nd$-solid variety of semigroups.

Let $V$ be a variety of semigroups. For a proof we consider the $nd$-hypersubstitutions $\sigma_1, \sigma_2 \in Hyp^{nd}(2)$ defined by $\sigma_1(f) = \{ x, xy \}$ and $\sigma_2(f) = \{ xy, yx \}$. If $V$ were an $nd$-solid variety of semigroups, then the application of $\sigma_1$ to the associative law gives identities which are satisfied in $V$. Let $V^* := \{ \{ A \} \mid A \in V \}$, then $V^* \models \{ \hat{\sigma_1}(f(x, f(y, z))) \approx \hat{\sigma_1}(f(f(x, y), z)) \} \approx \{ x, f(x, y), f(x, z), f(f(x, y), z) \}$. Since every $nd$-solid variety is solid, this gives especially $V^* \models \{ f(x, f(y, z)) \} \approx \{ f(x, z), f(z, x) \}$. Applying $\sigma_2$ to this identity gives $V^* \models \{ f(x, f(y, z)), f(x, f(z, y)), f(z, f(y, x)), f(y, f(z, x)) \} \approx \{ f(x, z), f(z, x) \}$. We use again the fact that every $nd$-solid variety is solid and the previous identity and obtain $V^* \models \{ f(x, z) \} \approx \{ f(z, x) \}$ or $V^* \models \{ f(x, y) \} \approx \{ f(x, z) \}$ or $V^* \models \{ f(x, z) \} \approx \{ f(y, x) \}$. If we use again the fact that every $nd$-solid variety must be solid in each of the cases we obtain that $V$ is trivial.

If an identity $s \approx t$ in a variety $V$ is satisfied for all $nd$-hypersubstitutions we speak of an $nd$-hyperidentity. More generally we define

Definition 3.3. Let $V$ be a variety of algebras of type $\tau$, let $s \approx t$ be an identity satisfied in $V$ and let $M \subseteq \mathcal{Hyp}^{nd}(\tau)$ be a monoid of non-deterministic hypersubstitutions. Then $s \approx t$ is an $M - nd$-hyperidentity in $V$ if $V^* \models \chi_{M - nd}^\mathcal{E}[s] \approx \{ t \}$ where $V^* = \{ \{ A \} \mid A \in V \}$. In this case we write $V \models_{M - \text{nd-hyp}} s \approx t$ and for $M = Hyp^{nd}(\tau)$ we will simply write $V \models_{\text{nd-hyp}} s \approx t$ and call $s \approx t$ an $nd$-hyperidentity in $V$.

The relation $K \models B_1 \approx B_2$ introduced in Definition 2.4 defines the Galois connection $\langle \mathcal{P}Mod, \mathcal{P}Id \rangle$ with the operations
\[ \mathcal{P} \text{Mod} : \mathcal{P}((\mathcal{P}(W_\tau(X)))^2) \to \mathcal{P} \left( \bigcup \{ \mathcal{P}(\chi^A_{M-nd}[A]) \mid A \in \text{Alg}(\tau) \} \right), \]

\[ \mathcal{P} \text{Id} : \mathcal{P} \left( \bigcup \{ \mathcal{P}(\chi^A_{M-nd}[A]) \mid A \in \text{Alg}(\tau) \} \right) \to \mathcal{P}((\mathcal{P}(W_\tau(X)))^2). \]

The relation \(|=_{M-nd-hyp}\) defines one more Galois connection

\[ (H_{M-nd} \mathcal{P} \text{Mod}, H_{M-nd} \mathcal{P} \text{Id}) \]

for sets \(\Sigma \subseteq \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)\) and classes \(\mathcal{K} \subseteq \bigcup\{ \mathcal{P}(\chi^A_{M-nd}[A]) \mid A \in \text{Alg}(\tau) \}\) as follows

\[ H_{M-nd} \mathcal{P} \text{Mod} : \mathcal{P}((\mathcal{P}(W_\tau(X)))^2) \to \mathcal{P} \left( \bigcup \{ \mathcal{P}(\chi^A_{M-nd}[A]) \mid A \in \text{Alg}(\tau) \} \right), \]

\[ H_{M-nd} \mathcal{P} \text{Id} : \mathcal{P} \left( \bigcup \{ \mathcal{P}(\chi^A_{M-nd}[A]) \mid A \in \text{Alg}(\tau) \} \right) \to \mathcal{P}((\mathcal{P}(W_\tau(X)))^2). \]

The products \(\mathcal{P} \text{Mod} \mathcal{P} \text{Id}, \mathcal{P} \text{Id} \mathcal{P} \text{Mod}, H_{M-nd} \mathcal{P} \text{Id} H_{M-nd} \mathcal{P} \text{Mod}, H_{M-nd} \mathcal{P} \text{Mod} H_{M-nd} \mathcal{P} \text{Id}\) are closure operators and their fixed points are complete lattices. The lattice of all \(M-nd\)-solid varieties arises if we restrict the operator \(H_{M-nd} \mathcal{P} \text{Mod} H_{M-nd} \mathcal{P} \text{Id}\) to classes of the form \(V^*\) where \(V\) is a variety of algebras of type \(\tau\). Moreover we have the conjugate pair \((\chi^A_{M-nd}, \chi^E_{M-nd})\) of additive closure operators. Their fixed points form two more complete lattices. Now we may apply the theory of conjugate pairs of additive closure operators (see e.g. [?]) and obtain the following propositions:

**Lemma 3.4.** Let \(K \subseteq \text{Alg}(\tau)\) be a class of algebras and let \(\Sigma \subseteq (\mathcal{P}W_\tau(X))^2\) be a set of equations. Then the following properties hold:

(i) \(H_{M-nd} \mathcal{P} \text{Id}(K^*) = \mathcal{P} \text{Id}\chi^A_{M-nd}[K^*],\)

(ii) \(H_{M-nd} \mathcal{P} \text{Id}(K^*) \subseteq \mathcal{P} \text{Id}(K^*),\)

(iii) \(\chi^E_{M-nd}[H_{M-nd} \mathcal{P} \text{Id}(K^*)] = H_{M-nd} \mathcal{P} \text{Id}(K^*),\)

(iv) \(\chi^A_{M-nd}[\mathcal{P} \text{Mod}(H_{M-nd} \mathcal{P} \text{Id}(K^*))] = \mathcal{P} \text{Mod}(H_{M-nd} \mathcal{P} \text{Id}(K^*)),\)

(v) \(H_{M-nd} \mathcal{P} \text{Id}(H_{M-nd} \mathcal{P} \text{Mod}(\Sigma)) = \mathcal{P} \text{Id}(\mathcal{P} \text{Mod}(\chi^E_{M-nd}[\Sigma]));\) and dually
Using these propositions one obtains the following characterization of $M$ – nd-solid varieties.

**Theorem 3.5.** Let $V$ be a variety of type $\tau$ and let $\Sigma$ be an equational theory of type $\tau$ (i.e. $\text{IdMod}(\Sigma) = \Sigma$). Further we assume that $M \subseteq \text{Hyp}^{\text{nd}}(\tau)$ is a monoid of non-deterministic hypersubstitutions of type $\tau$.

Then the following propositions are equivalent:

(i) $H_{M\text{-nd}\text{Mod}}(\Sigma) = \mathcal{P}\text{Mod}\chi_{M\text{-nd}}^E(\Sigma)$,

(ii) $H_{M\text{-nd}\text{Mod}}(\Sigma) \subseteq \mathcal{P}\text{Mod}(\Sigma)$,

(iii) $\chi_{M\text{-nd}}^A[H_{M\text{-nd}\text{Mod}}(\Sigma)] = H_{M\text{-nd}\text{Mod}}(\Sigma)$,

(iv) $\chi_{M\text{-nd}}^E[\mathcal{P}\text{Id}(H_{M\text{-nd}\text{Mod}}(\Sigma))] = \mathcal{P}\text{Id}(H_{M\text{-nd}\text{Mod}}(\Sigma))$,

(v) $H_{M\text{-nd}\text{Mod}}[H_{M\text{-nd}\text{Id}}(K^*)] = \mathcal{P}\text{Mod}(\mathcal{P}\text{Id}(\chi_{M\text{-nd}}^A[K^*]))$.

4. **$M$–Nd-Solid Varieties of Semigroups**

We consider some examples of $M$ – nd-solid varieties of semigroups and use the following notation for varieties of semigroups;

$B = \text{Mod}\{x(yz) \approx (xy)z, x^2 \approx x\}$ – the variety of bands,

$RB = \text{Mod}\{x(yz) \approx (xy)z \approx xz, x^2 \approx x\}$ – the variety of rectangular bands

$SL = \text{Mod}\{x(yz) \approx (xy)z, x^2 \approx x, xy \approx yx\}$ – the variety of semilattices, bands,

$LZ = \text{Mod}\{xy \approx x\}$ – the variety of left-zero bands.
Let $M = \{\sigma_{pid}, \sigma_1, \sigma_2\}$ with $\sigma_1(f) = \{x\}$ and $\sigma_2(f) = \{y\}$. Then $M$ forms a monoid and the multiplication $\circ_{nd}$ is given by the following table:

<table>
<thead>
<tr>
<th>$\circ_{nd}$</th>
<th>$\sigma_{pid}$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{pid}$</td>
<td>$\sigma_{pid}$</td>
<td>$\sigma_1$</td>
<td>$\sigma_2$</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>$\sigma_1$</td>
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<tr>
<td>$\sigma_2$</td>
<td>$\sigma_2$</td>
<td>$\sigma_1$</td>
<td>$\sigma_2$</td>
</tr>
</tbody>
</table>

We will prove the following proposition:

**Proposition 4.1.** Let $M = \{\sigma_{pid}, \sigma_1, \sigma_2\}$ as defined before. A non-trivial variety $V$ of semigroups is $M - nd$-solid iff $RB \subseteq V$.

**Proof.** It is well-known that $IdRB$ is the set of all outermost equations of type $\tau = (2)$, i.e. the set of all equations $s \approx t$ such that the first variables in $s$ and in $t$ and the last variables in $s$ and in $t$ agree. Therefore $RB \subseteq V$ means that all identities in $V$ are outermost and for any $s \approx t \in Id$ we have $\delta_1[s] = \{\text{first variable in } s\} = \{\text{first variable in } t\} = \delta_1[t]$ and $\delta_2[s] = \{\text{last variable in } s\} = \{\text{last variable in } t\} = \delta_2[t]$. Clearly $s \approx t$ is closed under $\sigma_{pid}$.

Conversely, let $V$ be a nontrivial $M - nd$-solid variety. Then $\sigma_1, \sigma_2 \in M$ requires $RB \subseteq V$. $\blacksquare$

Let $var(B)$ be the set of all variables occurring in the set $B$ of terms. Now let

$M' = \{\sigma \in Hyp^{nd}(\tau) \mid var(\sigma(f)) = \{x\}\}$.

Clearly $M' \cup \{\sigma_{pid}\}$ forms a submonoid of $Hyp^{nd}(\tau)$. Then we have

**Proposition 4.2.** A non-trivial variety $V$ of semigroups is $M' - nd$-solid iff $LZ \subseteq V \subseteq B$.

**Proof.** It is well-known that $IdLZ$ is the set of all equations $s \approx t$ of type $\tau = (2)$ such that the first variable in $s$ is equal to the first variable in $t$. Because of $var(\sigma(f)) = \{x\}$ the terms in $\delta[s]$ and the terms in $\delta[t]$ can be written as $x^r$ and as $x^l$ for some $r, l \in \mathbb{N}^+$. Since $V \subseteq B$ by the idempotent law all equations of the form $x^r \approx x^l$ are satisfied in $V$. This shows that $V$ is $M' - nd$-solid.
Conversely, let $V$ be a nontrivial $M' - nd$-solid variety of semigroups. If we apply $\sigma$ with $\sigma(f) = \{x, x^2\}$ to the identity $f(x, y) \approx f(x, y)$ we obtain $x \approx x^2$, i.e. $V \subseteq B$. If we apply $\sigma'$ with $\sigma'(f) = \{x\}$ we get $\text{leftmost}(s) \approx \text{leftmost}(t) \in IdV$ and this means $LZ \subseteq V$. Altogether, we have $LZ \subseteq V \subseteq B$. 

**References**


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