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JOINT ESTIMATION FOR NORMAL ORTHOGONAL MIXED MODELS

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Abstract

Commutative Jordan algebras are used to express the structure of mixed orthogonal models and to derive complete sufficient statistics. From these statistics, UMVUE, (Uniformly Minimum Variance Unbiased Estimators), are derived for the relevant parameters, first of single models then of several such models. These models may correspond to experiments designed separately so our results may be seen as a contribution to this meta-analysis.

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1. Introduction

In this paper we consider joint estimation for normal orthogonal mixed models.

Firstly we consider single models using commutative Jordan algebras to study their structure. This study, along with the assumption of normality, leads to the obtention of complete sufficient statistics. From these statistics UMVUE for the relevant parameters are derived.

Nextly the sufficient statistics for the individual models are pooled to give sufficient statistics for the joint model. Again UMVUE are derived from these statistics.

The joined experiments may have been designed separately so our results may be seen as a contribution to their meta-analysis.

2. Single models

A linear space $\mathcal{A}$ constituted by symmetrical matrices will be a Jordan algebra if it contains the square of it's matrices. If the matrices of an algebra commute, the Jordan algebra is said to be commutative. We will consider only commutative Jordan algebras.

Seely (1971), showed that any commutative Jordan algebra has one and only one basis constituted by pairwise orthogonal projection matrices. The basis will be the principal basis of the algebra.

We are interested in models

$$\boldsymbol{Y}^n = \sum_{i=1}^{w} \boldsymbol{X}_i \boldsymbol{\beta}_i^{c_i} + \boldsymbol{e}^n, \qquad (1)$$

where $\boldsymbol{\beta}_1^{c_1}, ..., \boldsymbol{\beta}_m^{c_m}$ are fixed vectors and the $\boldsymbol{\beta}_{m+1}^{c_{m+1}}, ..., \boldsymbol{\beta}_w^{c_w}$ and $\boldsymbol{e}^n$ are random independent vectors, with null mean vectors and covariance matrices $\sigma_i^2 \boldsymbol{I}_{c_i}, i = m+1, ..., w$, and $\sigma^2 \boldsymbol{I}_n$, respectively. Then $\boldsymbol{Y}^n$ has mean vector and covariance matrix

(2)
$$\begin{cases} \boldsymbol{\mu}^n = \sum_{i=1}^{m} \boldsymbol{X}_i \boldsymbol{\beta}_i^{c_i} \\ \\ \boldsymbol{V} = \sum_{i=m+1}^{w} \sigma_i^2 \boldsymbol{M}_i + \sigma^2 \boldsymbol{I}_n, \end{cases}$$

with $\boldsymbol{M}_i = \boldsymbol{X}_i \boldsymbol{X}_i^\top, i = 1, ..., w$. We will assume that $\boldsymbol{X}_1 = \mathbf{1}^n$ and that $\boldsymbol{\beta}_1 = \mu^n$.

This model will be associated to a commutative Jordan algebra $\mathcal{A}$ if the matrices $\boldsymbol{M}_i, i = 1, ..., w$, and $\boldsymbol{I}_n$ constitute a basis for $\mathcal{A}$. With $\{\boldsymbol{Q}_1, ..., \boldsymbol{Q}_{w+1}\}$ the principal basis of $\mathcal{A}$ we have

(3)
$$\begin{cases} \boldsymbol{M}_i = \sum_{j=1}^{w+1} b_{i,j} \boldsymbol{Q}_j, i = 1, ..., w \\ \\ \boldsymbol{I}_n = \sum_{j=1}^{w+1} \boldsymbol{Q}_j \end{cases}.$$

Putting $b_{w+1,j} = 1, j = 1, ..., w + 1$, the matrix $\boldsymbol{B} = [b_{i,j}]$ will be regular since it is a matrix for change of basis. We say that $\boldsymbol{B}$ is a transition matrix of $\mathcal{A}$.

So, $\boldsymbol{Y}^n$ will have normal distribution $\boldsymbol{Y}^n \sim \mathcal{N}(\boldsymbol{\mu}^n, \boldsymbol{V})$, with mean vector $\boldsymbol{\mu}^n$ and putting $\sigma^2 = \sigma_{w+1}^2$, we have the covariance matrix

(4)
$$\boldsymbol{V} = \sum_{i=m+1}^{w} \sigma_i^2 \left(\sum_{j=1}^{w+1} b_{i,j} \boldsymbol{Q}_j \right) + \sigma^2 \sum_{j=1}^{w+1} \boldsymbol{Q}_j = \sum_{j=1}^{w+1} \boldsymbol{\gamma}_j \boldsymbol{Q}_j,$$

where

$$\gamma_j = \sum_{i=m+1}^{w+1} b_{i,j}\sigma_i^2 = \sigma^2 + \sum_{i=1}^{w} b_{i,j}\sigma_i^2, j = 1, ..., w+1. \tag{5}$$

Moreover, if we can group the matrices from the two basis in pairs $(\boldsymbol{M}_i, \boldsymbol{Q}_i), i = 1, ..., w+1$, so that:

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_{1,1} & \mathbf{0} \\ \boldsymbol{B}_{2,1} & \boldsymbol{B}_{2,2} \end{bmatrix} \tag{6}$$

with $\boldsymbol{B}_{1,1}$ and $\boldsymbol{B}_{2,2}$ square matrices, these matrices will be regular and the model, see Ferreira (2006), will be segregated with $\varphi = \{1, ..., m\}$ the set of index for the fixed effects. Then the pairs of matrices $(\boldsymbol{M}_i, \boldsymbol{Q}_i), i = 1, ..., m$ will be associated to the fixed effects part of the model which will behave as a sub-model segregated from the global model. So, considering $\boldsymbol{\sigma}^2_{(1)} = \mathbf{0}^m$, and putting

$$\begin{cases} \boldsymbol{\gamma}_{(1)} = (\gamma_1, ..., \gamma_m) \\ \boldsymbol{\gamma}_{(2)} = (\gamma_{m+1}, ..., \gamma_{w+1}) \\ \boldsymbol{\sigma}^2_{(2)} = (\sigma^2_{m+1}, ..., \sigma^2_{w+1}) \end{cases} \tag{7}$$

we have

$$\boldsymbol{\gamma}_{(j)} = \boldsymbol{B}^\top_{2,j}\boldsymbol{\sigma}^2_{(2)}, j = 1, 2, \tag{8}$$

so that, since $\boldsymbol{B}^\top_{2,2}$ is regular,

$$(9) \qquad \begin{cases} \boldsymbol{\sigma}^2_{(2)} = \left(\boldsymbol{B}^\top_{2,2}\right)^{-1} \boldsymbol{\gamma}_{(2)} \\ \\ \boldsymbol{\gamma}_{(1)} = \boldsymbol{B}^\top_{2,1} \left(\boldsymbol{B}^\top_{2,2}\right)^{-1} \boldsymbol{\gamma}_{(2)} \end{cases} .$$

With $\boldsymbol{A}_j$ the matrix whose g_j row vectors are the eigenvectors of $\boldsymbol{Q}_j$ corresponding to eigenvalues equal to 1, $j = 1, ..., w+1$, we show, see Fonseca *et al.* (2006), that the statistics

$$(10) \qquad \begin{cases} S_j = \|\widetilde{\boldsymbol{\eta}}^{g_j}_j\|^2, j = 1, ..., w+1 \\ \\ \widetilde{\boldsymbol{\eta}}^{g_j}_j = \boldsymbol{A}_j \boldsymbol{Y}^n, j = 1, ..., m \end{cases} .$$

are sufficient, complete and independent.

With $\boldsymbol{\eta}_j = \boldsymbol{A}_j \boldsymbol{\mu}^n$, $j = 1, ..., w$, and $g_j = rank(\boldsymbol{Q}_j)$, $j = 1, ..., m$, we have

$$(11) \qquad \widetilde{\boldsymbol{\eta}}_j \sim \mathcal{N}(\boldsymbol{\eta}_j, \gamma_j \boldsymbol{I}_{g_j}), j = 1, ..., m,$$

as well as

$$(12) \qquad S_j \sim \gamma_j \chi^2_{g_j}, j = m+1, ..., w.$$

Using now the Blackwell-Lehmann-Scheffé theorem, the estimators $\widetilde{\boldsymbol{\eta}}_j$, $j = 1, ..., m$, and $\widetilde{\gamma}_j = \dfrac{s_j}{g_j}$, $j = m+1, ..., w+1$ will be UMVUE of $\boldsymbol{\eta}_j$, $j = 1, ..., m$, and of γ_j, $j = m+1, ..., w+1$. So we have the UMVUE

$$(13) \qquad \widetilde{\boldsymbol{\gamma}}_{(2)} = (\widetilde{\gamma}_{m+1}, ..., \widetilde{\gamma}_{w+1})$$

from which we get the

$$\begin{cases} \widetilde{\boldsymbol{\sigma}}^2_{(2)} = (\boldsymbol{B}_{2,2}^\top)^{-1}\widetilde{\boldsymbol{\gamma}}_{(2)} \\ \\ \widetilde{\boldsymbol{\gamma}}_{(1)} = \boldsymbol{B}_{2,1}^\top(\boldsymbol{B}_{2,2}^\top)^{-1}\widetilde{\boldsymbol{\gamma}}_{(2)} \end{cases} . \tag{14}$$

3. Joint estimation

Let us assume that there are L experiments involving the same factors with fixed and with random effects. The factor levels will be different from experiment to experiment since they were designed separately, according to orthogonal segregated mixed models.

The CJA associated to those models will have principal basis $\{\boldsymbol{Q}_1(l), ..., \boldsymbol{Q}_w(l)\}$ and transition matrices

$$\boldsymbol{B}(l) = \begin{bmatrix} \boldsymbol{B}_{1,1}(l) & \boldsymbol{B}_{1,2}(l) \\ \\ \boldsymbol{B}_{2,1}(l) & \boldsymbol{B}_{2,2}(l) \end{bmatrix}, \quad l = 1, ..., L, \tag{15}$$

so that we will have

$$\begin{cases} \boldsymbol{\sigma}^2_{(2)}(l) = \left(\boldsymbol{B}_{2,2}(l)^\top\right)^{-1}\boldsymbol{\gamma}_{(2)}(l), \quad l = 1, ..., L \\ \\ \boldsymbol{\gamma}_{(1)}(l) = \boldsymbol{B}_{2,1}(l)^\top\left(\boldsymbol{B}_{2,2}(l)^\top\right)^{-1}\boldsymbol{\gamma}_{(2)}(l), \quad l = 1, ..., L \end{cases} , \tag{16}$$

as well, with $\boldsymbol{Q}_j(l) = \boldsymbol{A}_j(l)^\top\boldsymbol{A}_j(l), l = 1, ..., L, j = 1, ..., w$, we have

$$\boldsymbol{\eta}_j(l) = \boldsymbol{A}_j(l)\boldsymbol{\mu}_l, l = 1, ..., L, j = 1, ..., w, \tag{17}$$

where $\boldsymbol{\mu}_l, l = 1, ..., L$, is the mean vector of the observations vector for the l-th experiment, $l = 1, ..., L$. For the different models we have the complete sufficient statistics $[\widetilde{\boldsymbol{\eta}}_1(l), ..., \widetilde{\boldsymbol{\eta}}_m(l), S_{m+1}(l), ..., S_w(l)]$, $l = 1, ..., L$, which pooled together give sufficient complete statistics for the joint model.

Since the factor levels will not be the same the joint analysis will have to be based on the random effects part of the models mainly on the variance components. We are this led to take

$$\boldsymbol{\gamma}_2(l) = \boldsymbol{\gamma}_2, l = 1, ..., L. \tag{18}$$

Actually we may validate thus assumption testing the equality of the corresponding components of the $\boldsymbol{\gamma}_2(l), l = 1, ..., L$.

Once this assumption is validated we may combine the estimators $\widetilde{\boldsymbol{\gamma}}_2(l), l = 1, ..., L$. Now $S_{m+j}(l) \sim \gamma_{m+j}\ \chi^2_{g_{m+j}(l)}, l = 1, ..., L, j = 1, ..., w-m$, so

$$S_{m+j} = \sum_{l=1}^{L} S_{m+j}(l) \sim \gamma_{m+j}\ \chi^2_{g_{m+j}}, \quad j = 1, ..., w-m, \tag{19}$$

where

$$g_{m+j} = \sum_{l=1}^{L} g_{m+j}(l), \quad j = 1, ..., w-m.$$

Using again the Blackwell-Lehmann-Scheffé theorem, we get UMVUE, first the

$$\widetilde{\gamma}_{m+j} = \frac{S_{m+j}}{g_{m+j}}, \quad j = 1, ..., w-m, \tag{20}$$

and then the

$$(21)\qquad \begin{cases} \widetilde{\boldsymbol{\sigma}}^2_{(2)}(l) = \left(\boldsymbol{B}_{2,2}(l)^\top\right)^{-1}\widetilde{\boldsymbol{\gamma}}_{(2)}, & l = 1, ..., L \\ \\ \widetilde{\boldsymbol{\gamma}}_{(1)}(l) = \boldsymbol{B}_{2,1}(l)^\top\left(\boldsymbol{B}_{2,2}(l)^\top\right)^{-1}\widetilde{\boldsymbol{\gamma}}_{(2)}, & l = 1, ..., L \end{cases}.$$

As for the fixed effects factors and their interactions for the joint model we may consider the

$$(22)\qquad \boldsymbol{\eta}_j = \left[\boldsymbol{\eta}_j(1)^\top, ..., \boldsymbol{\eta}_j(L)^\top\right]^\top, \quad j = 1, ..., m,$$

for which we have the UMVUE

$$(23)\qquad \widetilde{\boldsymbol{\eta}}_j = \left[\widetilde{\boldsymbol{\eta}}_j(1)^\top, ..., \widetilde{\boldsymbol{\eta}}_j(L)^\top\right]^\top, \quad j = 1, ..., m.$$

With $\boldsymbol{D}(\boldsymbol{U}_1, ..., \boldsymbol{U}_L)$ the blockwise diagonal matrix with principal blocks $\boldsymbol{U}_1, ..., \boldsymbol{U}_L$, we will have

$$(24)\qquad \boldsymbol{\Sigma}(\widetilde{\boldsymbol{\eta}}_j) = \boldsymbol{D}(\gamma_j(1)\boldsymbol{I}_{g_j(1)}, ..., \gamma_j(L)\boldsymbol{I}_{g_j(L)}), \quad j = 1, ..., m,$$

and we can also use the estimators $\widetilde{\gamma}_j(l), l = 1, ..., L, j = 1, ..., m$, given by the second of expressions (21).

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