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MAGIC AND SUPERMAGIC DENSE BIPARTITE GRAPHS

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Abstract

A graph is called magic (supermagic) if it admits a labelling of the edges by pairwise different (and consecutive) positive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. In the paper we prove that any balanced bipartite graph with minimum degree greater than $|V(G)|/4 \geq 2$ is magic. A similar result is presented for supermagic regular bipartite graphs.

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1. Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every } v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index-mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G).$$

A magic labelling f of G is called a *supermagic labelling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) whenever there exists a supermagic (magic) labelling of G.

The concept of magic graphs was introduced by Sedláček [11]. The regular magic graphs are characterized in [2]. Two different characterizations of all magic graphs are given in [7] and [8]. Supermagic graphs were introduced by Stewart [12]. It is easy to see that the classical concept of a magic square of n^2 boxes corresponds to the fact that the complete bipartite graph $K_{n,n}$ is supermagic for every positive integer $n \neq 2$ (see also [12]). Stewart [13] characterized supermagic complete graphs. In [5] supermagic regular complete multipartite graphs and supermagic cubes are characterized. In [6] there are given characterizations of magic line graphs of general graphs and supermagic line graphs of regular bipartite graphs. In [10] supermagic labellings of the Möbius ladders are constructed. Some constructions of supermagic labellings of various classes of regular graphs are described in [4] and [5]. More comprehensive information on magic and supermagic graphs can be found in [3].

A graph G is called *bipartite* if its vertex set can be partitioned into disjoint parts $V_1(G)$, $V_2(G)$ such that every edge of G joins vertices of different parts. If $|V_1(G)| = |V_2(G)|$ the graph G is called *balanced*. In the paper we deal with magic and supermagic bipartite graphs.

2. Magic Graphs

In this section we show that dense balanced bipartite graphs are magic.

A *cross-bridge* of a balanced bipartite graph G is a pair of its edges e_1, e_2 such that $G-\{e_1, e_2\}$ has components H_1, H_2 which are balanced bipartite graphs and e_i, for $i \in \{1,2\}$, joins a vertex of $V_i(H_1)$ with a vertex of $V_{3-i}(H_2)$. As usual, for $S \subset V(G)$, $N(S)$ denotes the set of vertices adjacent to a vertex in S. In the sequel we will use the following assertion proved in [7].

Proposition 1 ([7]). *A connected bipartite graph G is magic if and only if*

(i) *it is balanced,*

(ii) $|N(S)| > |S|$ *for all* $S \subset V_1(G)$, $\emptyset \neq S \neq V_1(G)$, *and*

(iii) *it contains no cross-bridge.*

For dense bipartite graphs it holds

Theorem 1. *Let G be a balanced bipartite graph with minimum degree $\delta(G)$. If $\delta(G) > |V(G)|/4 \geq 2$, then G is a magic graph.*

Proof. Put $n := |V(G)|/2$. As $\delta(G) > n/2 \geq 2$, the graph G is Hamiltonian (see [9]) and so connected. Moreover, its edge-connectivity is greater than 2. Therefore, G contains no cross-bridge.

Now, suppose that S is a subset of $V_1(G)$ and consider the following cases.

I. $0 < |S| \leq n/2$. If v is an arbitrary vertex of S then

$$|N(S)| \geq |N(\{v\})| = \deg(v) \geq \delta(G) > \frac{n}{2} \geq |S|.$$

II. $n/2 < |S| < n$. If u is an arbitrary vertex of $V_2(G)$ then

$$|N(\{u\})| + |S| \geq \delta(G) + |S| > n = |V_1(G)|.$$

It implies $N(\{u\}) \cap S \neq \emptyset$, i.e., $u \in N(S)$. Thus $N(S) = V_2(G)$. In this case we have

$$|N(S)| = n > |S|.$$

According to Proposition 1, G is a magic graph. ■

It is easy to see that the bound $\delta(G) > |V(G)|/4$ in the previous theorem can be replaced by $\deg(u) + \deg(v) > |V(G)|/2$ for all non-adjacent vertices $u \in V_1(G)$ and $v \in V_2(G)$.

Let G be a graph which we obtain from two disjoint copies of complete bipartite graphs $K_{k,k}$ by adding two new edges joining vertices of distinct parts of the copies. Evidently, $|V(G)| = 4k$, $\delta(G) = k$ and there are non-adjacent vertices $u \in V_1(G)$, $v \in V_2(G)$ such that $\deg(u) + \deg(v) = 2k$. As the pair of new edges is a cross-bridge of G, the graph G is not magic. Therefore, the considered bounds are the best possible.

3. Supermagic Graphs

The similar problem for supermagic graphs seems to be very difficult. Next, it is solved for regular bipartite graphs. Clearly, they are balanced. However, the result is not straightforward even in this restricted case. In [5] there is proved that $|V(G)| \equiv 2 \pmod 4$ for every supermagic regular graph G of odd degree. So, no regular graph of odd degree and order $4k$ is supermagic. We believe that the following conjecture is true.

Conjecture 1. Let G be a d-regular bipartite graph of order $2n$. If $d > n/2$, then G is supermagic except for $n \equiv 0 \pmod 2$ and $d \equiv 1 \pmod 2$.

However, now we are able to prove only the next result.

Theorem 2. *Let G be a d-regular bipartite graph of order $2n$ such that one of the following conditions is satisfied:*

(i) $d \equiv 0 \pmod 4$ *and* $d - 2 > n/2$,

(ii) $d \equiv 1 \pmod 4$, $n \equiv 1 \pmod 2$, $d - 11 > n/2$ *and* $d \geq (3n+2)/4$,

(iii) $d \equiv 2 \pmod 4$, $n \equiv 1 \pmod 2$ *and* $d - 8 > n/2$,

(iv) $d \equiv 2 \pmod 4$, $n \equiv 0 \pmod 2$, $d - 8 > n/2$ *and* $d \geq (3n+2)/4$,

(v) $d \equiv 3 \pmod 4$, $n \equiv 1 \pmod 2$, $d - 5 > n/2$ *and* $d \geq (3n+2)/4$.

Then G is a supermagic graph.

In the proof of this theorem we will use the following assertions.

Proposition 2 ([4, 5])**.** *Let $F_1, F_2, \ldots, F_k$ be mutually edge-disjoint supermagic factors of a graph G which form its decomposition. Then G is supermagic.*

Let $p \geq 3$ be an integer. By M_p we denote the graph with the vertex set $\cup_{i=1}^{p}\{u_i, v_i\}$ and the edge set $\cup_{i=1}^{p}\{u_iv_i, u_iv_{i+1}, u_{i+1}v_i\}$, the indices are being taken modulo p. Note, that M_p is a bipartite 3-regular graph isomorphic to either the Möbius ladder, for p odd, or the graph of p-side prism, for p even.

Proposition 3 ([10]). *Let $p \geq 3$ be an odd integer. Then the Möbius ladder M_p is a supermagic graph.*

Lemma 1. *Let G be a balanced bipartite graph of order $2n$ with minimum degree $\delta(G)$. If $\delta(G) \geq (3n+2)/4$ and $n \geq 3$, then G contains a factor isomorphic to M_n.*

Proof. As $\delta(G) \geq (3n+2)/4 > n/2$, G is Hamiltonian (see [9]) and so it contains a 1-factor F. Let $e_1 = x_1y_1, \ldots, e_n = x_ny_n$ be edges of F. Define a graph H with the vertex set $\{e_1, \ldots, e_n\}$, where vertices e_i and e_j of H are adjacent if and only if $\{x_iy_j, x_jy_i\} \subset E(G)$. Note that every edge of H corresponds to a pair of edges in G. Evidently,

$$\deg_H(e_k) = |\{i : x_i \in N(\{y_k\}), y_i \in N(\{x_k\})\}| \geq \deg(y_k) + \deg(x_k) - n - 1$$

and so $\deg_H(e_k) \geq n/2$ for every $k \in \{1, \ldots, n\}$. By the well known Dirac's result, H is a Hamiltonian graph, i.e., it contains a Hamilton cycle C. Clearly, the edges of F together with n pairs of edges corresponding to edges of C induce a factor of G isomorphic to M_n. ■

If $n \geq 4$ is an even integer, then the graph $2C_n$ consisting of two disjoint cycles of order n is a factor of M_n. Thus, by Lemma 1 we have immediately

Lemma 2. *Let G be a balanced bipartite graph of order $2n$ with minimum degree $\delta(G)$. If $\delta(G) \geq (3n+2)/4$ and $4 \leq n \equiv 0 \pmod 2$, then G contains a factor isomorphic to $2C_n$.*

For sufficiently large n the bound $\delta(G) \geq (3n+2)/4$ in the previous lemma can be replaced by $\delta(G) \geq 1 + n/2$ (see [1]).

In [4] there is proved that every 4-regular bipartite graph which can be decomposed into two edge-disjoint Hamilton cycles is supermagic. The following lemmas extend this result.

Lemma 3. *Let G be a $4k$-regular bipartite graph which can be decomposed into two edge-disjoint connected $2k$-factors. Then G is a supermagic graph.*

Proof. Put $n := |V(G)|/2$ and choose a vertex $w \in V_1(G)$. Then $|E(G)| = 4kn$. Let F_1 and F_2 be two edge-disjoint connected $2k$-factors of G. Clearly, they are Eulerian. Therefore, there is an ordering $e^i_1, e^i_2, \dots, e^i_{2kn}$ of $E(F_i)$ which forms an Eulerian trail of F_i, $i \in \{1, 2\}$, starting (and finishing) at w. Consider a bijection $f : E(G) \longrightarrow \{1, 2, \dots, 4kn\}$ defined by

$$f(e^i_j) = \begin{cases} \frac{j+1}{2} & \text{if } i = 1 \text{ and } j \equiv 1 \pmod 2, \\ 1 + 4kn - \frac{j}{2} & \text{if } i = 1 \text{ and } j \equiv 0 \pmod 2, \\ 1 + 2kn - \frac{j+1}{2} & \text{if } i = 2 \text{ and } j \equiv 1 \pmod 2, \\ 2kn + \frac{j}{2} & \text{if } i = 2 \text{ and } j \equiv 0 \pmod 2. \end{cases}$$

Since each of the Eulerian trails passes through every vertex of G exactly k times and $f(e^i_{2r-1}) + f(e^i_{2r}) = 1 + 4kn$ for $r \in \{1, \dots, kn\}$, $i \in \{1, 2\}$, $f^*(u) = 2k(1+4kn)$ for every $u \in V_2(G)$. Similarly, $f(e^1_{2kn}) + f(e^1_1) = 2+3kn$, $f(e^2_{2kn}) + f(e^2_1) = 5kn$, $f(e^1_{2r}) + f(e^1_{2r+1}) = 2 + 4kn$ and $f(e^2_{2r}) + f(e^2_{2r+1}) = 4kn$ for $r \in \{1, \dots, kn-1\}$. Thus, $f^*(u) = 2k(1+4kn)$ for every $u \in V_1(G)$. Hence, f is a supermagic labelling. ■

Lemma 4. *Let G be a 6-regular bipartite graph of order $2n$ which can be decomposed into three edge-disjoint Hamilton cycles. If n is odd, then G is supermagic.*

Proof. Let C^1, C^2 and C^3 be edge-disjoint Hamilton cycles of G. Choose a vertex $x \in V_1(G)$. By y we denote the vertex of G such that the distance between x and y in C^1 is $n-1$. Clearly, $y \in V_1(G)$. We denote the edges of the cycle C^1 successively $e^1_1, e^1_2, \dots, e^1_{2n}$ such that e^1_1, e^1_{2n} are incident with x and e^1_{n-1}, e^1_n are incident with y. Similarly, we denote the edges of the cycle C^i starting at x, for $i = 2$, (y, for $i = 3$) by $e^i_1, e^i_2, \dots, e^i_{2n}$. Now consider a bijection $f : E(G) \longrightarrow \{1, 2, \dots, 6n\}$ given by

$$f(e^i_j) = \begin{cases} 1 + j & \text{if } i = 1,\ 1 \le j < n \text{ and } j \equiv 1 \pmod 2, \\ 1 + 6n - j & \text{if } i = 1,\ 1 \le j < n \text{ and } j \equiv 0 \pmod 2, \\ 1 + j - n & \text{if } i = 1,\ n \le j \le 2n \text{ and } j \equiv 1 \pmod 2, \\ 1 + 7n - j & \text{if } i = 1,\ n \le j \le 2n \text{ and } j \equiv 0 \pmod 2, \\ 1 + in - \frac{j+1}{2} & \text{if } i \in \{2, 3\} \text{ and } j \equiv 1 \pmod 2, \\ (6 - i)n + \frac{j}{2} & \text{if } i \in \{2, 3\} \text{ and } j \equiv 0 \pmod 2. \end{cases}$$

As $f(e^i_{2r-1}) + f(e^i_{2r}) = 1 + 6n$ for $r \in \{1, \dots, n\}$, $i \in \{1, 2, 3\}$, $f^*(u) = 3(1 + 6n)$ for every $u \in V_2(G)$. Similarly, $f(e^1_{2r}) + f(e^1_{2r+1}) = 3 + 6n$, if $(n - 1)/2 \neq r \neq n$, and $f(e^i_{2r}) + f(e^i_{2r+1}) = 6n$ for $i \in \{2, 3\}$, $r \in \{1, \dots, n - 1\}$. Thus, $f^*(u) = 3(1 + 6n)$ for every $u \in V_1(G) - \{x, y\}$.

Moreover, $f(e^1_{2n}) + f(e^1_1) = f(e^1_{n-1}) + f(e^1_n) = 3 + 5n$ and $f(e^2_{2n}) + f(e^2_1) = f(e^3_{2n}) + f(e^3_1) = 7n$. Thus, $f^*(x) = f^*(y) = 3(1 + 6n)$. Hence, f is a supermagic labelling. ■

Lemma 5. *Let G be a 6-regular bipartite graph of order $2n$ which can be decomposed into three edge-disjoint 2-factors where the first is isomorphic to $2C_n$ and the others are Hamilton cycles. Then G is a supermagic graph.*

Proof. Since G is bipartite and the cycle of order n is its subgraph, n is even.

Let $x, y \in V_1(G)$ be vertices belonging to distinct cycles of the first factor. We denote the edges of the first cycle in the first factor successively $e^1_1, e^1_2, \dots, e^1_n$ such that the vertex x is incident with e^1_1 and e^1_n. Similarly, we denote the edges of the second cycle successively $e^1_{n+1}, e^1_{n+2}, \dots, e^1_{2n}$ such that the vertex y is incident with e^1_{n+1} and e^1_{2n}. By $e^i_1, e^i_2, \dots, e^i_{2n}$ we denote the edges of the cycle of the i-th factor starting at x, for $i = 2$, (y, for $i = 3$). Now consider a bijection $f : E(G) \longrightarrow \{1, 2, \dots, 6n\}$ given by

$$f(e^i_j) = \begin{cases} 1 + j & \text{if } i = 1,\ 1 \le j \le n \text{ and } j \equiv 1 \pmod 2, \\ 1 + 6n - j & \text{if } i = 1,\ 1 \le j \le n \text{ and } j \equiv 0 \pmod 2, \\ j - n & \text{if } i = 1,\ n < j \le 2n \text{ and } j \equiv 1 \pmod 2, \\ 7n - j & \text{if } i = 1,\ n < j \le 2n \text{ and } j \equiv 0 \pmod 2, \\ 1 + in - \frac{j+1}{2} & \text{if } i \in \{2, 3\} \text{ and } j \equiv 1 \pmod 2, \\ (6 - i)n + \frac{j}{2} & \text{if } i \in \{2, 3\} \text{ and } j \equiv 0 \pmod 2. \end{cases}$$

As in the previous proof, it can be seen that f is a supermagic labelling for index $3(1 + 6n)$. ■

4. PROOF OF THEOREM 2

Suppose that G is a d-regular bipartite graph of order $2n$. Consider the following cases.

A. $d \equiv 0 \pmod 4$ and $d - 2 > n/2$. Then there is a Hamilton cycle C^1 in G. The graph G_1 which we obtain from G by deleting the edges of C^1 is regular of degree $d-2$. Therefore, there is a Hamilton cycle C^2 in G_1. Let G_2 be a graph which we obtain from G_1 by deleting the edges of C^2. The graph G_2 is regular bipartite of degree $d-4$ and so there exist mutually edge-disjoint 1-factors $F_1, F_2, \dots, F_{d-4}$ which form its decomposition. Let H_i, $i \in \{1,2\}$, be a factor of G which contains the edges of C^i and edges of F_k for all $k \equiv i \pmod 2$. Clearly, H_1 and H_2 are edge-disjoint $(d/2)$-regular connected factors of G. According to Lemma 3, G is a supermagic graph.

B. $d \equiv 2 \pmod 4$, $n \equiv 1 \pmod 2$ and $d - 8 > n/2$. As in the previous case, it is easy to see that there are three edge-disjoint Hamilton cycles C^1, C^2 and C^3 in G. Let H_1 be a factor of G which contains the edges of these cycles. Let H_2 be a factor of G containing the edges of G which are not in H_1. By Lemma 4 the graph H_1 is supermagic. As H_2 is a $(d-6)$-regular bipartite graph, by the case A, it is supermagic. According to Proposition 2, G is supermagic.

C. $d \equiv 2 \pmod 4$, $n \equiv 0 \pmod 2$, $d - 8 > n/2$ and $d \geq (3n+2)/4$. By Lemma 2 the graph G contains a factor C^1 isomorphic to $2C_n$. Let G_1 be a $(d-2)$-factor of G containing no edge of C^1. Then G_1 includes two edge-disjoint Hamilton cycles C^2 and C^3. Let H_1 be a 6-factor of G which contains the edges of C^1, C^2 and C^3. Let H_2 be a $(d-6)$-factor of G containing the edges of G which are not in H_1. By Lemma 5 and the case A, H_1 and H_2 are supermagic graphs. So, Proposition 2 implies that G is a supermagic graph.

D. $d \equiv 3 \pmod 4$, $n \equiv 1 \pmod 2$, $d - 5 > n/2$ and $d \geq (3n+2)/4$. According to Lemma 1, the graph G includes a 3-factor F isomorphic to M_n. By H we denote a $(d-3)$-factor of G containing these edges of G which are not in F. Therefore, the factors F and H form a decomposition of G. Combining Proposition 3, the case A and Proposition 2 we get that G is supermagic.

E. $d \equiv 1 \pmod 4$, $n \equiv 1 \pmod 2$, $d - 11 > n/2$ and $d \geq (3n+2)/4$. As in the previous case, it is easy to see that G can be decomposed into factors F and H where F is isomorphic to the Möbius ladder M_n. Combining Proposition 3, the case B and Proposition 2 we obtain that G is a supermagic graph.

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