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# MONOCHROMATIC KERNEL-PERFECTNESS OF SPECIAL CLASSES OF DIGRAPHS

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**Abstract**

In this paper, we introduce the concept of monochromatic kernel-perfect digraph, and we prove the following two results:

(1) If $D$ is a digraph without monochromatic directed cycles, then $D$ and each $\alpha_v, v \in V(D)$ are monochromatic kernel-perfect digraphs if and only if the composition over $D$ of $(\alpha_v)_{v \in V(D)}$ is a monochromatic kernel-perfect digraph.

(2) $D$ is a monochromatic kernel-perfect digraph if and only if for any $B \subseteq V(D)$, the duplication of $D$ over $B$, $D^B$, is a monochromatic kernel-perfect digraph.

**Keywords:** kernel, kernel by monochromatic paths, composition, duplication.

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## 1. Introduction

For general concepts we refer the reader to [1]. Let $D$ be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of $D$ respectively. Let $S_1, S_2 \subseteq V(D)$, an arc $(u_1, u_2)$ of $D$ will be called an $S_1S_2$-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph of $D$ induced by $S_1$. A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A *kernel* $N$ of $D$ is an independent set of vertices such that for each $z \in (V(D) - N)$ there exists a $zN$-arc in $D$. A digraph $D$ is called a *kernel-perfect digraph* when every induced subdigraph of $D$ has a kernel.

A digraph $D$ is said to be an $m$-coloured digraph, if its arcs are coloured with $m$ colours without loss of generality $\{1, 2, \ldots, m\}$. A directed path (or a directed cycle) is called *monochromatic* if all of its arcs are coloured alike.

A set $N \subseteq V(D)$ of vertices of $D$ is said to be a *kernel by monochromatic paths* of the $m$-coloured digraph $D$, if it satisfies the two following properties, (1) $N$ is independent by monochromatic paths; i.e., for any two different vertices $x, y \in N$, there is no monochromatic directed path between them, and (2) $N$ is *absorbent* by monochromatic paths; i.e., for each $u \in (V(D) - N)$ there exists a $uv$-monochromatic directed path, for some $v \in N$.

In this paper, we prove that if $D$ is a digraph without monochromatic directed cycles, then (i) $D$ has a kernel by monochromatic paths if and only if any composition over $D$ of a family of digraphs $(\alpha_v)_{v \in V(D)}$ each one of them having a kernel by monochromatic paths, has a kernel by monochromatic paths, and (ii) $D$ has a kernel by monochromatic paths if and only if for any $B \subseteq V(D)$ the duplication of $D$ over $B$, $D^B$, has a kernel by monochromatic paths.

As a consequence we obtain the two results mentioned in the abstract.

Clearly, $D$ has a kernel if and only if the $m$-coloured digraph $D$, in which every two different arcs have different colours, has a kernel by monochromatic paths. Sufficient conditions for the existence of a kernel in a digraph have been investigated by several authors, namely Von Neumann and Morgenstern [16], Richardson [13], Duchet and Meyniel [5] and Galeana-Sánchez and Neumann-Lara [6]. The concept of a kernel is very useful in applications, and clearly, the concept of a kernel by monochromatic paths generalizes that of kernel. Sufficient conditions for the existence of kernels by monochromatic paths in $m$-coloured digraphs have also been investigated by several authors; see for example [7, 9, 10, 14, 15, 18].

**Definition 1.1.** Let $D$ be an arc coloured digraph and $\alpha = (\alpha_v)_{v\in V(D)}$ a family of pairwise vertex disjoint arc coloured digraphs. We define the *composition of* $\alpha$ *over* $D$, denoted $\sigma(D,\alpha)$, by the following conditions:

(i) $V(\sigma(D,\alpha)) = \bigcup_{v\in V(D)} V(\alpha_v)$.

(ii) $A(\sigma(D,\alpha)) = \big(\bigcup_{v\in V(D)} A(\alpha_v)\big) \cup \{(x,y)$ coloured $i \,|\, x \in \alpha_u, y \in \alpha_v,$ $(u,v) \in F(D)$ coloured $i\}$.

The composition of a family of graphs $\beta = (G_v)_{v\in V(G)}$ over a graph $G$ was studied in [3] and its definition was extended to digraphs in [17]. The existence of kernels in the composition $\sigma(D,\alpha)$ of a family of digraphs $\alpha = (\alpha_v)_{v\in V(D)}$ over a digraph $D$ was studied in [8], and the result was used to prove the existence of kernel-perfect digraphs with an arbitrarily large dichromatic number whose underlying graphs have no triangles.

In this paper, we study the existence of kernels by monchromatic paths in the composition $\sigma(D,\alpha)$ of a family of arc coloured digraphs $\alpha = (\alpha_v)_{v\in V(D)}$ over an arc coloured digraph $D$.

The duplication of a vertex of a graph was introduced in [4], and [11] gives the definition of the duplication of a subset of vertices of a graph as a generalization of the duplication of a vertex of a graph. This definition can be applied to arc coloured digraphs as follows:

**Definition 1.2.** Let $D$ be an arc coloured digraph, $B$ a proper subset of $V(D)$ and let $B'_D$ a digraph isomorphic to $D[B]$ with $V(B'_D)\cap V(D) = \emptyset$. A vertex belonging to $B'_D$ and corresponding to a vertex $x \in B$ will be denoted by $x'$. *The duplication of* $D$ *over* $B$ is the arc coloured digraph denoted $D^B$ and defined as follows:

$$V(D^B) = V(D) \cup V(B'_D)$$

and

$$A(D^B) = A(D) \cup A(B'_D) \cup A_0 \cup A$$

in which $A_0 = \{(x',y)$ coloured $i \,|\, x' \in V(B'_D), y \in V(D)$ and $(x,y) \in A(D)$ coloured $i\}$. $A_1 = \{(y,x')$ coloured $i \,|\, y \in V(D), x' \in V(B'_D)$ and $(y,x) \in A(D)$ coloured $i\}$.

We will denote $B' = V(B'_D)$. A vertex $x' \in B'$ (resp., a subset $S' \subseteq B'$) we will call the copy of the vertex $x \in B$ (resp., the copy of the subset $S \subseteq B$).

The vertex $x$ (resp., the subset $S$) will be named the original of the vertex $x'$ (resp., of the subset $S'$).

We will denote by Proy the function $\mathrm{Proy}\colon V(\sigma(D,\alpha)) \to V(D)$ such that $\mathrm{Proy}(x) = v$ if and only if $x \in V(\alpha_v)$.

The existence of kernels in the duplication of a digraph $D$ has been studied in [2]. In this paper, we study the existence of kernels by monochromatic paths in the duplication of an arc coloured digraph $D$ over a proper subset of vertices of $V(D)$.

The composition and the duplication are two operations in digraphs which have been considerated several times, see for example [3, 12, 17], and they constitute a powerful tool in the construction of many examples and counterexamples in digraphs.

Also we consider an extension of the concept of kernel perfectness of a digraph and obtain a large variety of monochromatic kernel-perfect digraphs.

## 2. Kernels by Monochromatic Paths in the Composition over $D$, and in the Duplication of $D$ over $B$

We start this section with a lemma which will be useful in the proof of Theorem 2.1. Its proof is easy and will be omitted.

**Lemma 2.1.** *Let $D$ be a digraph and $\alpha = (\alpha_v)_{v\in V(D)}$ a family of pairwise vertex disjoint digraphs. If $T = (x_0, x_1, \ldots, x_n)$ is a directed path in $\sigma(D,\alpha)$ such that $\{x_0, x_n\} \subseteq V(\alpha_v)$ for some $v \in V(D)$, then* $\mathrm{Proy}(T)$ *is a join of directed cycles of $D$ or a single vertex of $D$.*

**Theorem 2.1.** *Let $D$ be an arc coloured digraph which has no monochromatic directed cycle and $\alpha = (\alpha_v)_{v\in V(D)}$ a family of arc coloured pairwise vertex disjoint digraphs. A set $N^* \subseteq V(\sigma(D,\alpha))$ is a kernel by monochromatic paths of $\sigma(D,\alpha)$ if and only if there exists a kernel by monochromatic paths of $D$, say $N \subseteq V(D)$, such that $N^* = \bigcup_{v\in N} N_v$, in which $N_v$ is a kernel by monochromatic paths of $\alpha_v$.*

***Proof.*** Let $N \subseteq V(D)$ be a kernel by monochromatic paths of $D$ and $N_v$ a kernel by monochromatic paths of $\alpha_v$, $v \in N$. We will prove that $N^* = \bigcup_{v\in N} N_v$ is a kernel by monochromatic paths of $\sigma(D,\alpha)$.

(a) $N^*$ is absorbent by monochromatic paths.

Let $z \in (V(\sigma(D,\alpha)) - N^*)$. There exists $v_0 \in V(D)$ such that $z \in V(\alpha_{v_0})$. When $v_0 \in N$, we have $N_{v_0} \subseteq N^*$, in which $N_{v_0}$ is a kernel by monochromatic paths of $\alpha_{v_0}$; and there exists a $zN_{v_0}$-monochromatic directed path (as $z \in (V(\alpha_0) - N_{v_0})$).

When $v_0 \notin N$, we have $v_0 \in (V(D) - N)$ and thus, there exists a monochromatic directed path contained in $D$, say $T = (v_0, v_1, \ldots, v_{n-1}, u)$ with $u \in N$ (because $N$ is a kernel by monochromatic paths of $D$); since $z \in V(\alpha_0)$; taking $z_i \in V(\alpha_i)$ and $z_u \in N_u$, we have $T' = (z, z_1, z_2, \ldots, z_{n-1}, z_u)$; a $zz_u$-monochromatic directed path in $\alpha(D,\sigma)$ with $z_u \in N_u \subseteq N^*$.

(b) $N^*$ is independent by monochromatic paths.

We proceed by contradiction, suppose that there exist $x_0, x_n \in N^*$ and a $x_0x_n$-monochromatic directed path, say $T = (x_0, x_1, \ldots, x_n)$ contained in $\sigma(D,\alpha)$. We consider two possible cases:

*Case* (b.1). $\{x_0, x_n\} \subseteq V(\alpha_v)$, for some $v \in V(D)$.
When $T \subseteq \alpha_v$, we have that $T$ is an $x_0x_n$-monochromatic directed path contained in $\alpha_v$, with $\{x_0, x_n\} \subseteq N_v$, a contradiction.

When $T \nsubseteq \alpha_v$, we have from Lemma 2.1 that Proy$(T)$ is a join of monochromatic directed cycles contained in $D$, contradicting our hypothesis on $D$.

*Case* (b.2). $x_0 \in \alpha_v$ and $x_n \in \alpha_u$ with $u \neq v$.
In this case, it follows from the definition of $N^*$ that $x_0 \in N_v$ and $x_n \in N_u$ with $\{u, v\} \subseteq N$. Since $T$ is monochromatic, We have that Proy$(T)$ contains a $vu$-monochromatic path, which is contained in $D$, contradicting that $N$ is a kernel by monochromatic paths of $D$. We conclude that $N^*$ is a kernel by monochromatic paths.

Now let $N^*$ be a kernel by monochromatic paths of $\sigma(D,\alpha)$. We will prove that $N = \{v \in V(D) \,|\, N^* \cap \alpha_v \neq \emptyset\}$ is a kernel by monochromatic paths of $D$ and $N^* \cap V(\alpha_v) = N_v$ is a kernel by monochromatic paths of $\alpha_v$, for each $v \in N$.

$N$ is absorbent by monochromatic paths.

Let $v \in (V(D) - N)$ and $z_0 \in V(\alpha_v)$; since $v \notin N$ we have that $z_0 \notin N^*$; thus there exists a monochromatic directed path $T = (z_0, \ldots, z_n)$ with $z_n \in N^*$; now, $z_n \in V(\alpha_u)$ for some $u \in V(D)$; moreover, from the definition of $N$ we have $u \in N$ and then Proy$(T)$ contains a $vu$-monochromatic directed path with $u \in N$.

$N$ is independent by monochromatic paths.

We proceed by contradiction, suppose that there exist $v_0, v_n \in N$ and a $v_0v_n$-monochromatic directed path $T = (v_0, v_1, \ldots, v_n)$ contained in $D$. Since $v_0, v_n \in N$ there exist $z_0 \in V(\alpha_{v_0}) \cap N^*$ and $z_n \in V(\alpha_{v_n}) \cap N^*$; now taking any vertex $z_i \in V(\alpha_{v_i})$ for each $1 \leq i \leq n-1$; we have from the definition of $\sigma(D, \alpha)$ that $(z_0, z_1, \ldots, z_n)$ is a $z_0z_n$-monochromatic directed path with $\{z_0, z_n\} \subseteq N^*$, a contradiction.

Now; let $v \in V(D)$ be such that $N^* \cap V(\alpha_v) \neq \emptyset$. We will prove that $N_v = N^* \cap V(\alpha_v)$ is a kernel by monochromatic paths of $\alpha_v$.

$N_v$ is independent by monochromatic paths.

We proceed by contradiction. Suppose that there exist $u, x \in N_v$, $u \neq x$, and a monochromatic directed path $T$ between them, with $T \subseteq \alpha_v$; clearly, $T \subseteq \sigma(D, \alpha)$ and $\{u, x\} \subseteq N^*$, a contradiction (as $N^*$ is independent by monochromatic paths in $\sigma(D, \alpha)$).

$N$ is absorbent by monochromatic paths.

Let $u \in (V(\alpha_v) - N)$ clearly $u \in (V(\sigma(D, \alpha)) - N^*)$; thus there exists $z \in N^*$ and a $uz$-monochromatic directed path $T \subseteq \sigma(D, \alpha)$. Let $T = (u = u_0, u_1, \ldots, u_n = z)$, we will prove that $T \subseteq \alpha_v$. When $u_n = z \in V(\alpha_v)$; it follows from Lemma 2.1 that $\mathrm{Proy}(T)$ is a single vertex i.e., $T \subseteq \alpha_v$; otherwise $D$ contains a monochromatic directed cycle, contradicting our hypothesis on $D$. When $u_n \notin V(\alpha_v)$; we have $u_n \in \alpha_w$ for some $w \in V(D)$ and $w \in N^*$. Now take $x \in N_v^*$ (recall $N^* \cap \alpha_v \neq \emptyset$); it follows from the definition of $\alpha(D, \alpha)$ that $(x, u_1, u_2, \ldots, u_n = z)$ is a $xz$-monochromatic directed path in $\sigma(D, \alpha)$ with $x \neq z$, $x, z \in N^*$, a contradiction. ■

**Lemma 2.2.** *Let $D$ be an arc coloured digraph, $B \subset V(D)$; $D^B$ the duplication of $D$ over $B$; and $\psi\colon D[B] \to B'_D$ the isomorphism defined by the duplication (i.e., $\psi(x) = x'$ for any $x \in V(D[B])$); and denote by $\phi$ the function defined as follows: $\phi\colon D \to D^B - D[B]$*

$$\phi(x) = \begin{cases} x & \text{if } x \notin B, \\ \phi(x) = x' & \text{if } x \in B\,. \end{cases}$$

*Then $\phi$ is an isomorphism such that $(x, y)$ is coloured in $i$ if and only if $(\phi(x), \phi(y))$ is coloured in $i$; in particular $T \subseteq D$ is a monochromatic directed path if and only if $\phi(T) \subseteq D^B - D[B]$ is a monochromatic directed path.*

This Lemma is a direct consequence of the definition of $\phi$ and the definition of the duplication of $D$ over $B$.

**Theorem 2.2.** *Let $D$ be an arc coloured digraph which has no monochromatic directed cycles; $B \subset V(D)$ and $D^B$ the duplication of $D$ over $B$.*

*$D$ has a kernel by monochromatic paths if and only if $D^B$ has a kernel by monochromatic paths.*

***Proof.*** Let $D, B$ and $D^B$ be as in the hypothesis and suppose that $D$ has a kernel by monochromatic paths, say $N$.

We consider two possible cases:

*Case* 1. $N \cap B = \emptyset$.

In this case, we will prove that $N$ is a kernel by monochromatic paths of $D^B$.

$N$ is independent by monochromatic paths in $D^B$.

Let $x, y \in N$; $x \neq y$ and assume for a contradiction that there exists an $xy$-monochromatic directed path $T = (x = x_0, x_1, \ldots, x_n = y)$ contained in $D^B$.

When $V(T) \cap B = \emptyset$, we have $T \subseteq D^B - D[B]$, and from Lemma 2.1 $\phi^{-1}(T)$ is an $xy$-monochromatic directed path contained in $D$, contradicting that $N$ is independent by monochromatic paths. When $V(T) \cap B \neq \emptyset$, we denote $I = V(T) \cap B$; say $I = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, $i_1 < i_2 < \cdots < i_k$, we also denote by $T(I) = (x_0, \ldots, x_{i_1 - 1}, x'_{i_1}, x_{i_1 + 1}, \ldots, x_{i_2 - 1}, x'_{i_2}, \ldots, x_n = y)$ (the succesion obtained from $T$ by substituting $x_{i_j}$, for $x'_{i_j}$ in $T$, for each $j \in \{1, \ldots, k\}$). It follows from the definition of $D^B$ that $T(I)$ is a monochromatic directed path contained in $D^B - D[B]$; and from Lemma 2.2 $\phi^{-1}(T(I))$ is an $xy$-monochromatic directed path contained in $D$, a contradiction.

$N$ is absorbent by monochromatic paths in $D^B$.

Let $z \in (V(D^B) - N)$. If $z \notin B'_D$, then $z \in (V(D) - N)$ and there exists a $zN$-monochromatic directed path, say $T$, with $T \subseteq D \subseteq D^B$.

If $z \in V(B'_D)$, then there exists $y \in B$ such that $z = y' \in V(B'_D)$; we have $y \notin N$ because $N \cap B = \emptyset$; thus there exists a $yN$-monochromatic directed path, say $T = (y, x_1, \ldots, x_n)$ and then from definition of $D^B$ we have that $T' = (y', x_1, \ldots, x_n)$ is a $zN$-monochromatic directed path in $D^B$.

*Case* 2. $N \cap B \neq \emptyset$.

Let $Z = N \cap B$, and denote by $Z' = \{z' \in B'_D \mid z \in Z\}$.

We will prove that $N^* = N \cup Z'$ is a kernel by monochromatic paths of $D^B$.

$N^*$ is independent by monochromatic paths.

Let $x, y \in N^*$, $x \neq y$, and assume for a contradiction that there exists an $xy$-monochromatic directed path $T = (x = x_0, x_1, \ldots, x_n = y)$ contained in $D^B$. Here we consider several possible cases:

*Case* 2.a. $x, y \in N$.
Let $I' = V(T) \cap V(B'_D) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ and denote by $y_{i_j}$ the original of $x_{i_j}$ (i.e., $y_{i_j} = \psi^{-1}(x_{i_j})$. Now let $T'$ be the succession obtained from $T$ by substituting each $x_{i_j}$ for $y_{i_j}$. It follows from the definition of $\psi$ and from the definition of $D^B$ that $T'$ contains an $xy$-monochromatic directed path contained in $D$, with $x, y \in N$, a contradiction.

*Case* 2.b. $x \in N$, $y \in Z'$ and $x \notin B$. In this case, we proceed as in Case 2.a to get a contradiction.

*Case* 2.c. $x \in N \cap B$, $y \in Z'$.
When $x$ is the original vertex of $y$, taking the succession $T'$ defined in Case 2.a we have that $T'$ contains a monochromatic directed cycle, contradicting our hypothesis on $D$; as $T' \subseteq D$.

When $x$ is not the original vertex of $y$; taking again the succession $T'$ defined in Case 2.a, we have that $T'$ contains an $xz$-monochromatic directed path, in which $z$ is the original vertex of $y$ and $x, z \in N$ with $x \neq z$, contradicting that $N$ is independent by monochromatic paths.

*Case* 2.d. $x, y \in Z'$.
Let $\overline{x}$ (resp., $\overline{y}$) be the original vertex of $x$ (resp., $y$); clearly, in this case $T'$ (defined in Case 2.a) contains an $\overline{xy}$-monochromatic directed path which is contained in $D$; with $\overline{x} \neq \overline{y}$, $\overline{x}, \overline{y} \in N$, a contradiction. So, we conclude that $N^*$ is independent by monochromatic paths.

Now we prove that $N^*$ is absorbent by monochromatic paths.

Let $z \in (V(D^B) - N^*)$. When $z \in B'$, we have $z = y'$ in which $y \in B$ is the original vertex of $z$. Since $N$ is a kernel by monochromatic paths of $D$; there exists a $yN$-monochromatic directed path in $D$, say, $T = (y = x_0, x_1, \ldots, x_n)$; thus $T' = (y' = z, x_1, \ldots, x_n)$ is a $zN^*$-monochromatic directed path contained in $D^B$. When $z \notin B'$, we have $z \in (V(D) - N)$ and there exists a $zN$-monochromatic directed path contained in $D$; say, $T$. Clearly, $T$ is a $zN^*$-monochromatic directed path contained in $D^B$.

We conclude that $N^*$ is a kernel by monochromatic paths of $D^B$. Now suppose that $D^B$ has a kernel by monochromatic paths and let $N^*$ be a kernel by monochromatic paths of $D^B$. We will prove that $D$ has a kernel by monochromatic paths.

Let $Z$ be such that $Z' = N^* \cap V(B'_D)$ in which $Z'$ is defined by the process introduced in the construction of $B'_D$, when $Z' = \emptyset$ we define $Z = \emptyset$. Denote by $N = (N^* - Z') \cup Z$. We will show that $N$ is a kernel by monochromatic paths of $D$.

$N$ is independent by monochromatic paths in $D$.
Assume by contradiction that there exist $x, y \in N$; $x \neq y$; and an $xy$-monochromatic directed path $T = (x = x_0, x_1, \ldots, x_n = y)$ contained in $D$.

Let $\overline{x}$ and $\overline{y}$ be defined as follows: $\overline{x} = x$ if $x \in (N^* - Z')$ and $\overline{x}$ is the copy of $x$ if $x \in Z$, $\overline{y} = y$ if $y \in (N^* - Z^*)$ and $\overline{y}$ is the copy of $y$ if $y \in Z$. Clearly, $T' = (\overline{x}, x_1, \ldots, x_{n-1}, \overline{y})$ is a monochromatic directed path in $D^B$ with $\overline{x} \neq \overline{y}$ and $\overline{x}, \overline{y} \in N^*$, a contradiction.

$N$ is absorbent by monochromatic paths in $D$.

Let $z \in (V(D) - N)$, then from the definition of $N$, we have $z \in (V(D^B) - N)$, thus there exists a $zN^*$-monochromatic directed path, say $T = (z = x_0, x_1, \ldots, x_n)$ contained in $D^B$. Let $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} = V(T) \cap V(B'_D)$; $y_{i_j}$ the original vertex of $x_{i_j}$ and $T'$ the succession obtained from $T$ by substituting $x_{i_j}$ for $y_{i_j}$ for each $1 \leq j \leq k$ in $T$. Clearly, $T'$ contains a $zN$-monochromatic directed path, and $T' \subseteq D$.

We conclude that $N$ is absorbent by monochromatic paths. ■

## 3. Monochromatic Kernel Perfectness of Composition and Duplication

The following definition is a generalization of the concept of kernel perfectness of a digraph.

**Definition 3.1.** Let $D$ be an arc coloured digraph, $D$ is said to be a monochromatic kernel perfect digraph whenever for every nonempty subset $B$ of vertices of $D$, the digraph $D[B]$ has a kernel by monochromatic paths.

**Theorem 3.1.** *Let $D$ be an arc coloured digraph which has no monochromatic directed cycle and $\alpha = (\alpha_v)_{v \in V(D)}$ a family in which the $\alpha_v$ are mutually disjoint arc coloured digraphs.*

*$D$ and each $\alpha_v$, $v \in V(D)$ are monochromatic kernel perfect digraphs if and only if $\sigma(D, \alpha)$ is a monochromatic kernel perfect digraph.*

***Proof.*** Theorem 3.1 follows directly from Theorem 2.1 and the two following assertions: (1) The disjoint union of monochromatic kernel perfect digraphs is also a monochromatic kernel perfect digraph. (2) Every connected induced subdigraph of $\sigma(D,\alpha)$ has the form $\sigma(D',\alpha')$ for a suitable $D'$ and $\alpha' = (\alpha'_v)_{v\in V(D)}$ (actually $D'$ is an induced subdigraph of $D$ and $\alpha'_v$ is an induced subdigraph of $\alpha_v$ for each $v \in V(D')$). ■

**Theorem 3.2.** *Let $D$ be an arc coloured digraph which has no monochromatic directed cycle, $B \subset V(D)$ and $D^B$ the duplication of $D$ over $B$. Then $D$ is a monochromatic kernel perfect digraph if and only if $D^B$ is a monochromatic kernel perfect digraph.*

***Proof.*** Clearly, an arc coloured digraph $D$ is a monochromatic kernel perfect digraph if and only if each induced subdigraph of $D$ is a monochromatic kernel perfect digraph. Thus if $D^B$ is a monochromatic kernel perfect digraph, then $D$ is a monochromatic kernel perfect digraph.

Now suppose that $D$ is a monochromatic kernel perfect digraph and let $A \subseteq V(D^B)$. We will prove that $D^B[A]$ has a kernel by monochromatic paths. Here we consider two possible cases:

*Case* 1. $A \cap V(B'_D) = \emptyset$.
In this case, $A \subseteq V(D^B - V(B'_D))$ and $D^B[A] \cong D[A]$ and since $D[A]$ has a kernel by monochromatic paths; it follows that $D^B[A]$ has a kernel by monochromatic paths.

*Case* 2. $A \cap V(B'_D) \neq \emptyset$.
Let $C' = \{x' \in V(D^B) \,|\, x' \in A \cap V(B'_D)\}$ and $E = A - C'$ be, thus $A = C' \cup E$.

*Case* 2.1. $E \cap C = \emptyset$. (In Which $C = \psi^{-1}(C')$).
In this case, we have $D^B[E \cup C'] \cong D^B[E \cup C] \cong D[E \cup C]$ and then $D^B[A] \cong D[E \cup C]$ has a kernel by monochromatic paths.

*Case* 2.2. $E \cap C \neq \emptyset$.
It follows from the hypothesis that $D[E \cup C]$ has a kernel by monochromatic paths, say N.

When $N \cap B = \emptyset$ it follows as in Case 1 of the proof of Theorem 2.2 that $N$ is a kernel by monochromatic paths of $D^C[E \cup C]$ (the duplication of $D[E \cup C]$ over $C$); therefore $N$ is independent by monochromatic paths in $D^B[E \cup C']$. Since $N$ is absorbent by monochromatic paths in $D[E \cup C]$ it follows that $N$ is absorbent by monochromatic paths in $D^B[E \cup C']$.

(Clearly, to each monochromatic directed path in $D[E \cup C]$, say $T$ there corresponds an unique monochromatic directed path in $D^B[E \cup C']$, $T'$ obtained from $T$ by substituting each vertex $x$ in $V(T) \cap (C - E)$ for its copy $x'$ in $C'$).

When $N \cap B \neq \emptyset$, we denote by $Z = N \cap B$; we have proved in Case 2 of the proof of Theorem 2.2 that $N \cup Z'$ is a kernel by monochromatic paths of $D^C[E \cup C]$ (the duplication of $D[E \cup C]$ over $C$). So $N \cup Z'$ is independent by monochromatic paths in $D^B[E \cup C']$. Now, let $z \in (V(D^B[E \cup C']) - (N \cup Z'))$; clearly, $z \in (V(D^C[E \cup C] - (N \cup Z')))$ and then there exists a $z \in (N \cup Z')$-monochromatic directed path, say $T = (z = x_0, x_1, \ldots, x_n)$; if $T \cap (C - E) = \{x_{i_1}, \ldots, x_{i_k}\}$ then let $T'$ be the succession obtained from $T$ by substituting each $x_{i_j}$, $1 \leq j \leq k$ for its copy $x'_{i_j}$ in $C'$. Since $D$ has no monochromatic directed cycles we have $x'_{i_j} \notin V(T)$ for each $1 \leq j \leq k$. Therefore from the definition of $D^B$ we have that $T'$ is a monochromatic directed path contained in $D^B[E \cup C']$ from $z$ to $(N \cup Z')$. We conclude that $N \cup Z'$ is a kernel by monochromatic paths of $D^B[E \cup C']$. ■

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