

**SYSTEMS OF DIFFERENTIAL INCLUSIONS
IN THE ABSENCE OF MAXIMUM PRINCIPLES
AND GROWTH CONDITIONS**

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Abstract

This article investigates the existence of solutions to second-order boundary value problems (BVPs) for systems of ordinary differential inclusions. The boundary conditions may involve two or more points. Some new inequalities are presented that guarantee *a priori* bounds on solutions to the differential inclusion under consideration. These *a priori* bound results are then applied, in conjunction with appropriate topological methods, to prove some new existence theorems for solutions to systems of BVPs for differential inclusions. The new conditions allow the treatment of systems of BVPs in the absence of maximum principles and growth conditions. The results are also new for differential equations involving Carathéodory or even continuous right-hand sides.

Keywords: boundary value problem, systems of differential inclusions, existence of solutions, *a priori* bounds, two-point boundary conditions, three-point boundary conditions.

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1. INTRODUCTION

This paper considers the existence of solutions to the following second-order system of differential inclusions,

$$(1) \quad x'' \in F(t, x, x'), \quad \text{for a.e. } t \in [0, 1],$$

subject to either of the boundary conditions

$$(2) \quad x(0) + u_1x(c) = A_1, \quad x'(1) = A_2, \quad c \in (0, 1] \text{ is fixed;}$$

$$(3) \quad x'(0) = A_3, \quad x(1) + u_2x(d) = A_4, \quad d \in [0, 1) \text{ is fixed,}$$

where: $F : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ is a “multifunction” and $\mathcal{K}(\mathbb{R}^n)$ is the family of all nonempty convex and compact subsets of \mathbb{R}^n ($n > 1$); each A_i is a given constant in \mathbb{R}^n ; each $u_i \neq -1$ is a given constant in \mathbb{R} ; and “a.e.” stands for “almost every”.

The boundary conditions (2), (3) may involve two or three points, depending on where c and d lie in $[0,1]$ and, of course, whether any u_i is zero. Special cases of the boundary conditions include:

$$(4) \quad x(0) + u_1x(1) = A_1, \quad x'(1) = A_2,$$

$$(5) \quad x(0) = A_1, \quad x'(1) = A_2,$$

$$(6) \quad x'(0) = A_3, \quad x(1) + u_2x(0) = A_4,$$

$$(7) \quad x'(0) = A_3, \quad x(1) = A_4,$$

$$(8) \quad x(0) + u_1x(c) = A_1, \quad x'(1) = A_2, \quad c \in (0, 1) \text{ is fixed,}$$

$$(9) \quad x'(0) = A_3, \quad x(1) + u_2x(d) = A_4, \quad d \in (0, 1) \text{ is fixed.}$$

The study of differential inclusions has been motivated by their applications, for example, to the areas of control and in the treatment of differential equations with discontinuities in the right-hand side [3, 27, 30]. An integral aspect of the aforementioned applications are two-, three- and four-point boundary value problems involving differential inclusions, which have enjoyed much interest recently: [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 18, 19, 20, 21, 23, 22, 25, 26, 29, 31].

In the quest for existence of solutions to differential inclusions, the application of appropriate topologically inspired fixed-point methods generally relies on the obtention of *a priori* bounds on possible solutions to a family of

differential inclusions related to (1), and a corresponding family of boundary conditions [24, 28].

This paper formulates new, quite general and easily verifiable conditions, involving F , in the form of a single dynamic inequality such that the aforementioned *a priori* bounds on solutions are guaranteed. The new *a priori* bound results are then applied, in conjunction with appropriate topological methods, adapted from [16], to prove some novel results ensuring the existence of solutions. Several corollaries to the new results are presented that form new existence results in their own right for BVPs involving systems of differential equations.

For more on differential inclusions or BVPs, we refer the reader to [1, 3, 14, 15, 24, 30].

To present the new results, the necessary notation is defined as follows.

Definition 1.1. Suppose that E and G are Banach spaces and $X \subset E$ and $Y \subset G$ are subsets. Denote the family of all nonempty convex and compact subsets of Y by $\mathcal{K}(Y)$. A multivalued map $\Gamma : X \rightarrow \mathcal{K}(Y)$ is called upper semi-continuous (u.s.c.) if $\{x \in X : \Gamma(x) \subset U\}$ is an open subset of X for any open U in Y .

Definition 1.2. In what follows, consider the following Banach function spaces:

$$C([0, 1]; \mathbb{R}^n) = \{u : [0, 1] \rightarrow \mathbb{R}^n : u \text{ is continuous on } [0, 1]\}$$

with the norm $\|u\|_\infty = \sup_{t \in [0, 1]} \|u(t)\|$ where $\|\cdot\|$ denotes the usual Euclidean norm in \mathbb{R}^n and $\langle \cdot, \cdot \rangle$ will denote the usual inner product on \mathbb{R}^n ;

$$L^2([0, 1]; \mathbb{R}^n) = \{u : [0, 1] \rightarrow \mathbb{R}^n : \|u(t)\| \text{ is } L^2\text{-integrable}\}$$

with the norm

$$\|u\|_2 = \left(\int_0^1 \|u(t)\|^2 dt \right)^{1/2};$$

$$H^k([0, 1]; \mathbb{R}^n)$$

$$= \{u : [0, 1] \rightarrow \mathbb{R}^n : u \text{ has weak derivatives } u^{(i)} \in L^2([0, 1]; \mathbb{R}^n) \text{ for } 0 \leq i \leq k\}$$

with the norm

$$\|u\|_{2;k} = \max\{\|u^{(i)}\|_2 : 0 \leq i \leq k\}.$$

The spaces $H^k([0, 1]; \mathbb{R}^n)$ are the usual Sobolev spaces of vector functions, denoted also by $W^{k,2}([0, 1]; \mathbb{R}^n)$ (for more details see [14]).

The following notion of a Carathéodory map or multifunction will be central in the results to follow.

Definition 1.3. A multifunction $F : [0, 1] \times \mathbb{R}^m \rightarrow \mathcal{K}(\mathbb{R}^n)$ is said to be a *Carathéodory multifunction* in case it satisfies the following conditions:

- (i) the map $t \rightarrow F(t, u)$ is Lebesgue measurable for each $u \in \mathbb{R}^m$;
- (ii) the map $u \rightarrow F(t, u)$ is u.s.c. for each $t \in [0, 1]$;
- (iii) for any $r \geq 0$ there is a function $\psi_r \in L^2[0, 1]$ such that for all $t \in [0, 1]$, $u \in \mathbb{R}^m$ with $\|u\| \leq r$ and $y \in F(t, u)$ we have $\|y\| \leq \psi_r(t)$.

The following general existence theorem will be very useful for minimizing the length of existence proofs in the remainder of the paper. The proof of the result is closely linked with that of [16, Theorem 3.1] through an application of topological transversality [24] and thus is omitted.

Theorem 1.4. *Suppose that $F : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ is a Carathéodory multifunction and that $a_1, \dots, a_4, b_1, \dots, b_4$ are given constants in \mathbb{R} with $\sum_{i=1}^4 a_i^2 > 0$ and $\sum_{i=1}^4 b_i^2 > 0$ and that A, B are given constants in \mathbb{R}^n . If there exists a positive constant R (independent of λ) such that:*

$$(10) \quad \max\{\|x\|_\infty, \|x'\|_\infty\} < R,$$

for all solutions x to

$$(11) \quad x'' \in \lambda F(t, x, x'), \quad \text{for a.e. } t \in [0, 1],$$

$$(12) \quad a_1 x(0) + a_2 x(c) + a_3 x'(0) + a_4 x'(1) = \lambda A, \quad c \text{ fixed in } (0, 1],$$

$$(13) \quad b_1 x(1) + b_2 x(d) + b_3 x'(0) + b_4 x'(1) = \lambda B, \quad d \text{ fixed in } [0, 1),$$

for $\lambda \in [0, 1]$; and if the only solution to (11)–(13) for $\lambda = 0$ is the zero solution; then, for $\lambda = 1$, (11)–(13) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.

2. A PRIORI BOUNDS

In order to apply Theorem 1.4, some new *a priori* bound results for solutions to differential inclusions are now presented. The inequalities used do not rely on maximum principles or on growth conditions.

Lemma 2.1. *Let N be a positive constant. If*

$$(14) \quad \inf\{\langle x', w \rangle : w \in F(t, x, x'), \|x'\| = N\} > 0,$$

$$(15) \quad \|A_2\| < N,$$

then all solutions to (1), (2) satisfy

$$\|x(t)\| < \frac{\|A_1\| + N(1 + |u_1|)}{|1 + u_1|}, \quad \|x'(t)\| < N, \text{ for a.e. } t \in [0, 1].$$

Proof. Let x be a solution to (1), (2). We use “proof by contradiction” and assume that there exists a $t_0 \in [0, 1]$ such that $\|x'(t_0)\| \geq N$. Obviously, from (15) we see that $t_0 \in [0, 1)$. Next, define the function

$$r(t) = \|x'(t)\|^2 - N^2,$$

and assume that r attains its non-negative maximum value on $[0, 1]$ at $t_0 \in [0, 1)$. By (15) and the continuity of r , there must exist a $t_1 \in [t_0, 1)$ such that $r(t_1) = 0$ (so $\|x'(t_1)\| = N$) and

$$\begin{aligned} 0 &\geq r'(t_1) = 2\langle x'(t_1), x''(t_1) \rangle \\ &\geq 2 \inf\{\langle x'(t_1), w(t_1) \rangle : w(t_1) \in F(t_1, x(t_1), x'(t_1)), \|x'(t_1)\| = N\} \\ &> 0, \quad \text{by (14),} \end{aligned}$$

and a contradiction is reached. Therefore $\|x'(t)\| < N$, for a.e. $t \in [0, 1]$.

For a.e. $t \in [0, 1]$,

$$\begin{aligned} |1 + u_1| \|x(t)\| - \|x(0) + u_1 x(c)\| &\leq \|x(t) - x(0) + u_1(x(t) - x(c))\| \\ &\leq \left\| \int_0^t x'(s) ds + u_1 \int_c^t x'(s) ds \right\| \\ &< (1 + |u_1|)N, \end{aligned}$$

and rearrange to obtain

$$\|x(t)\| < \frac{\|A_1\| + N(1 + |u_1|)}{|1 + u_1|}, \quad \text{for a.e. } t \in [0, 1]. \quad \blacksquare$$

Lemma 2.2. *Let N be a positive constant. If*

$$(16) \quad \inf\{\langle x', w \rangle : w \in F(t, x, x'), \|x'\| = N\} < 0,$$

$$(17) \quad \|A_3\| < N,$$

then all solutions to (1), (3) satisfy

$$\|x(t)\| < \frac{\|A_4\| + N(1 + |u_2|)}{|1 + u_2|}, \quad \|x'(t)\| < N, \quad \text{for a.e. } t \in [0, 1].$$

Proof. The proof is virtually identical to that of Lemma 2.1 and so is only briefly discussed. Define r as in the proof of Lemma 2.1. Arguing by contradiction, there exists a $t_0 \in (0, 1]$ such that $r(t_0) \geq 0$ and there exists a $t_2 \in (0, t_0]$ such that $r(t_2) = 0$ with $0 \leq r'(t_2)$. A contradiction arises from (16). The *a priori* bound on x follows a similar line as in the proof of Lemma 2.1. \blacksquare

Remark 2.3. If F is a singleton set, consisting of one Carathéodory vector function, that is, when $F(t, x, x') = \{f(t, x, x')\}$ and $f : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ ($n > 1$), then conditions (14) and (16) respectively reduce to

$$(18) \quad \langle x', f(t, x, x') \rangle > 0, \quad \text{for a.e. } t \in [0, 1], \quad \forall x \in \mathbb{R}^n, \quad \|x'\| = N,$$

$$(19) \quad \langle x', f(t, x, x') \rangle < 0, \quad \text{for a.e. } t \in [0, 1], \quad \forall x \in \mathbb{R}^n, \quad \|x'\| = N.$$

3. EXISTENCE OF SOLUTIONS

In this section, some new existence theorems are presented for solutions to (1) subject to either (2) or (3). The *a priori* bound lemmas from Section 2 are utilised, in conjunction with Theorem 1.4, to produce the new results. The conditions of the new existence theorems do not involve any maximum principles or growth restrictions on F and therefore are applicable to a wider class of certain problems than those dealt with in [16].

Theorem 3.1. *Let $F : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction and let N be a positive constant. If (14) and (15) hold, then (1), (2) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Proof. Consider the following family of BVPs associated with (1), (2)

$$(20) \quad x'' \in \lambda F(t, x, x'), \quad \text{for a.e. } t \in [0, 1],$$

$$(21) \quad x(0) + u_1 x'(c) = \lambda A_1, \quad x'(1) = \lambda D, \quad c \in (0, 1] \text{ is fixed,}$$

where $\lambda \in [0, 1]$ and see that this is in the form (11), (12). Also see that, for $\lambda = 1$, (20), (21) is equivalent to the BVP (1), (2).

All that is required is to show that the conditions of Theorem 1.4 hold.

Since $u_1 \neq -1$ it is easy to see by direct computation that the only solution to (20), (21) is the zero solution.

We now show that (20), (21) satisfy the conditions of Lemma 2.1. Consider $w \in F(t, x, x')$ and let $w_1 = \lambda w$ where $w_1 \in \lambda F(t, x, x')$. See that, for $\lambda = 0$, the only solution to (20), (21) is the zero solution (and thus an *a priori* bound holds), so assume from now on that $\lambda \in (0, 1]$.

Since (14) holds we then have

$$\begin{aligned} 0 &< \lambda \inf\{\langle x', w \rangle : w \in F(t, x, x'), \|x'\| = N\} \\ &= \inf\{\langle x', \lambda w \rangle : w \in F(t, x, x'), \|x'\| = N\} \\ &= \inf\{\langle x', w_1 \rangle : w_1 \in \lambda F(t, x, x'), \|x'\| = N\}. \end{aligned}$$

Also, from (15),

$$N > \|A_2\| \geq \|\lambda A_2\|.$$

So we have an *a priori* bound on x and x' by Lemma 2.1.

Hence, if we define $R > 0$ by

$$R := \max \left\{ \frac{\|A_1\| + N(1 + |u_1|)}{|1 + u_1|}, N \right\} + 1$$

then all of the conditions of Theorem 1.4 are satisfied and the family (20), (21) has a solution for $\lambda = 1$. Then so must the BVP (1), (2). ■

Inequality (14) shows that there is a delicate dependency on F in x' for Theorem 3.1 to hold. However, the main focus of this paper is on those F that do feature x' and do not exhibit standard growth conditions.

Theorem 3.2. *Let $F : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction and let N be a positive constant. If (16) and (17) hold, then (1), (3) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Proof. In view of Lemma 2.2, the proof is virtually identical to that of Theorem 3.1. ■

Corollary 3.3. *Let $F : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction and let N be a positive constant. If (14) and (15) hold, then (1), (4) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Proof. Immediate from Theorem 3.1. ■

Corollary 3.4. *Let $F : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction and let N be a positive constant. If (14) and (15) hold, then (1), (5) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Proof. Immediate from Theorem 3.1. ■

Corollary 3.5. *Let $F : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction and let N be a positive constant. If (16) and (17) hold, then (1), (6) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Proof. Immediate from Theorem 3.2. ■

Corollary 3.6. *Let $F : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction and let N be a positive constant. If (16) and (17) hold, then (1), (7) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Proof. Immediate from Theorem 3.2. ■

Corollary 3.7. *Let $F : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction and let N be a positive constant. If (14) and (15) hold, then (1), (8) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Proof. Immediate from Theorem 3.1. ■

Corollary 3.8. *Let $F : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction and let N be a positive constant. If (16) and (17) hold, then (1), (9) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Proof. Immediate from Theorem 3.2. ■

4. ON DIFFERENTIAL EQUATIONS

If F is a singleton set consisting of one Carathéodory vector function, that is, when $F(t, x, x') = \{f(t, x, x')\}$ and $f : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ ($n > 1$), then (1) reduces to

$$(22) \quad x'' = f(t, x, x'), \quad \text{for a.e. } t \in [0, 1],$$

and the following new existence results follow as corollaries to the results of Section 3. The proofs involve applying the respective theorems to (1) for the special case $F(t, x, x') = \{f(t, x, x')\}$.

Corollary 4.1. *Let $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Carathéodory function and let N be a positive constant. If (18) and (15) hold, then (22), (2) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Corollary 4.2. *Let $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Carathéodory function and let N be a positive constant. If (19) and (17) hold, then (22), (3) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Corollary 4.3. *Let $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Carathéodory function and let N be a positive constant. If (18) and (15) hold, then (22), (4) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Corollary 4.4. *Let $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Carathéodory function and let N be a positive constant. If (18) and (15) hold, then (22), (5) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Corollary 4.5. *Let $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Carathéodory function and let N be a positive constant. If (19) and (17) hold, then (22), (6) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Corollary 4.6. *Let $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Carathéodory function and let N be a positive constant. If (19) and (17) hold, then (22), (7) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Corollary 4.7. *Let $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Carathéodory function and let N be a positive constant. If (18) and (15) hold, then (22), (8) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Corollary 4.8. *Let $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Carathéodory function and let N be a positive constant. If (19) and (17) hold, then (22), (9) has at least one solution in $H^2([0, 1]; \mathbb{R}^n)$.*

Remark 4.9. In fact, Corollaries 4.7 and 4.8 are new, even in the case when $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous, scalar-valued function.

5. EXAMPLES

In this final section, the applicability of the new results are highlighted through two examples.

Example 5.1. Let $x = (x_1, x_2)$ and $p = (p_1, p_2)$. Consider (1), (5) for $n = 2$, where

$$\begin{aligned} F(t, x, p) &= F(t, x_1, x_2, p_1, p_2) \\ &= \cup_{\kappa \in [1, 2]} \begin{pmatrix} p_2 e^{x_2 p_2} + \kappa(x_1^2 + 1)p_1^3 + 1 \\ -p_1 e^{x_2 p_2} + \kappa(x_2^2 + 1)p_2^3 + 1 \end{pmatrix}, \quad t \in [0, 1], \\ \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} x_1'(1) \\ x_2'(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

There is no growth condition applicable to F and thus the theorems of [16] do not apply. We will apply Theorem 3.1.

For N to be chosen below, consider, for all $f(t, x, p) \in F(t, x, p)$

$$\begin{aligned} \langle p, f(t, x, p) \rangle &= \kappa[(x_1^2 + 1)p_1^4 + (x_2^2 + 1)p_2^4] + p_1 + p_2 \\ &\geq p_1^4 + p_1 + p_2^4 + p_2 \\ &> 0 \quad \text{for } \|p\| = 4, \end{aligned}$$

thus (14) holds for $N = 4$.

It is easy to see that (15) holds. Thus Theorem 3.1 is applicable and the BVP has a solution. ■

As previously pointed out in Remark 4.9 the results contained in this paper are new, for three-point BVPs with continuous f .

Example 5.2. Consider the following three-point scalar BVP

$$\begin{aligned}x'' &= f(t, x, p) = (t^2 + 1 + x^2)pe^p, \quad t \in [0, 1], \\x(0) + x(1/2) &= 1, \quad x'(1) = 1.\end{aligned}$$

We will apply Corollary 4.1.

For N to be chosen below, consider,

$$\begin{aligned}\langle p, f(t, x, p) \rangle &= (t^2 + 1 + x^2)p^2e^p \\&> 0 \quad \text{for } \|p\| = N = 2,\end{aligned}$$

thus (18) holds for $N = 2$. It is easy to see that (15) holds. Corollary 4.1 is applicable and the BVP has a solution. ■

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