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CROSS ADDITIVITY - AN APPLICATION

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Abstract

We try to show that Discriminant Analysis can be considered as a branch of Statistical Decision Theory when viewed from a Bayesian approach. First we present the necessary measure theory results, next we briefly outline the foundations of Bayesian Inference before developing Discriminant Analysis as an application of Bayesian Estimation. Our approach renders Discriminant Analysis more flexible since it gives the possibility of classing an element as belonging to a group of populations. This possibility arises from the introduction of the concept of regions of controled posterior risk.

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We are going to apply the inference described in S.F. Saraiva (2006). We will consider a case study, see Fonseca *et al.* (2003), where the aim was to check the homogeneity of grapevine castes. In this case we had three random factors: location, origin and clone. The grapevines were grown in a rectangular lattice with rows following the fertility gradient.

Each line contained a single clone. Three groups of five lines were randomly chosen, thus having three sites and five repetitions for each factor combinations.

1. Preliminary results

Let there be L groups with $u_1, ..., u_L$ factors. The first factors in the different groups will have $a_l(1)$ levels.If $u_l > 1$, each level of the h-th factor nests $a_l(h+1)$ levels of the next factor, $l = 1, ..., L$. We also put $c_l(0) = 1, l = 1, ..., L$.

In group l there will be $c_l(h) = \prod_{k=1}^{h} a_l(k)$ level combinations for the h first factors each nesting

$$b_l(h) = \frac{c_l(u_l)}{c_l(h)}$$

level combinations of the remaining vectors, $h = 1, ..., u_l, l = 1, ..., L$.

Since factors that nest each other do not interact, effects and interactions in the model will correspond to sets of factors belonging to different groups. These sets of factors may be indicated by vectors $\mathbf{h}^L$ with components $h_l = 0, ..., u_l, l = 1, ..., L$.

If $h_l = 0$, no factor in the $l-$th group belongs to the set associated to $\mathbf{h}^L$, if $h_l \neq 0$, it will be index of the factor in the set taken from the $l-$th group.

Let Γ be the family of vectors $\mathbf{h}^L$. We take $\beta(\mathbf{0}) = \mu$, the general mean value. If $\mathbf{h}^L$ has only one non-null component, $\beta(\mathbf{h}^L)^{c(\mathbf{h}^L)}$ will correspond to effects of the factor indicated by that component; if $\mathbf{h}^L$ have more than one non null component $\beta(\mathbf{h}^L)^{c(\mathbf{h}^L)}$ will correspond to the interactions between the factors indicated by $\mathbf{h}^L$. If the factor or factors indicated by $\mathbf{h}^L$ have fixed effects, $\beta(\mathbf{h}^L)^{c(\mathbf{h}^L)}$ will belong to the fixed-effects part of the model, and we put $\mathbf{h}^L \in \Gamma_f$. Otherwise, $\beta(\mathbf{h}^L)^{c(\mathbf{h}^L)}$ will belong to the random effects part of the model and we put $\mathbf{h}^L \in \Gamma_r$.

We have cross additivity when we can admit, a priori, that are null the variance components for crossed interactions, this is, for interactions between one or more, fixed or random effects factors. Representing by Γ_c the set of vectors of Γ corresponding to crossed interactions we will have: $\sigma^2(\mathbf{h}^L) = 0, \quad \mathbf{h}^L \in \Gamma_c$. For $\mathbf{h}^L \in \Gamma$ we consider the sets

$$(1)\qquad \begin{cases} \varphi(\mathbf{h}^L) = \{l : h_l > 0\} \\ \\ \varphi_f(\mathbf{h}^L) = \{l : 0 < h_l \leq v_l\} \\ \\ \varphi_a(\mathbf{h}^L) = \{l : v_l < h_l\} \end{cases} .$$

We have $\mathbf{h}^L \in \Gamma_f[\Gamma_c; \Gamma_p = \Gamma_a \backslash \Gamma_c]$ when $\varphi_a(\mathbf{h}^L) = \emptyset[\varphi_a(\mathbf{h}^L) \neq \emptyset$ and $\varphi_f(\mathbf{h}^L) \neq \emptyset; \varphi_f(\mathbf{h}^L) = \emptyset]$.

When cross additivity holds, to have $\sigma^2(\mathbf{h}^L) > 0$ it is necessary that $\mathbf{h}^L \in \Gamma_p$. We also put

$$(2)\qquad \begin{cases} \varphi_{f,1}(\mathbf{h}^L) = \{l : 0 < h_l \leq v_l = u_l\} \\ \\ \varphi_{f,2}(\mathbf{h}^L) = \{l : 0 < h_l \leq v_l < u_l\} \end{cases} ,$$

as well as $\Gamma_m = \Gamma_f \cup \Gamma_c$. Now $\mathbf{h}^L \in \Gamma_m$ if and only if $\emptyset \subset \varphi_f(\mathbf{h}^L)$. The non null vectors of Γ_m can be grouped in families $\Gamma_{m,1}$ e $\Gamma_{m,2}$ according to $\varphi_{f,2}(\mathbf{h}^L) \subset \varphi_f(\mathbf{h}^L)$ or to $\varphi_{f,2}(\mathbf{h}^L) = \varphi_f(\mathbf{h}^L)$. If $\mathbf{h}^L \in \Gamma_{m,1}[\Gamma_{m,2}]$ there does not exist [exists] $\mathbf{k}^L \in \Gamma_p$ such that $\mathbf{h}^L < \mathbf{k}^L$, therefore, when cross additivity holds,

$$(3)\qquad \gamma(\mathbf{h}^L) = \sigma^2, \quad \mathbf{h}^L \in \Gamma_{m,1}.$$

Moreover, we can group the vectors of Γ_p into classes $\Gamma_{p,1}$ e $\Gamma_{p,2}$, putting $\mathbf{h}^L \in \Gamma_{p,1}[\Gamma_{p,2}]$ if there exists [does not exist] l' such that $h_{l'} = v_l + 1 > 1$.

If $\mathbf{h}^L \in \Gamma_{m,1} \cup \Gamma_p$ we put $\alpha(\mathbf{h}^L) = \mathbf{h}^L$ and, if $\mathbf{h}^L \in \Gamma_{m,2}, \mathbf{h}^{0^L} = \alpha(\mathbf{h}^L)$ will have components

$$(4)\qquad \begin{cases} h_l^0 = h_l, \qquad l \notin \varphi_f(\mathbf{h}^L) \\ \\ h_l^0 = v_l + 1, l \in \varphi_f(\mathbf{h}^L) \end{cases} ,$$

we see that, if $\mathbf{h}^L \in \Gamma_{m,2}, \mathbf{h}^{0^L} = \alpha(\mathbf{h}^L) \in \Gamma_{p,1}$. Nextly we have

Proposition 1. *When cross additivity holds we have* $\gamma(\mathbf{h}^L) = \gamma(\mathbf{h}^{0^L})$ *with* $\mathbf{h}^{0^L} = \alpha(\mathbf{h}^L)$.

Proof. It suffices to consider the case in which $\mathbf{h}^L \in \Gamma_{m,2}$. Then, if $\mathbf{k}^L \in \Gamma_p$ and $\mathbf{h}^L < \mathbf{k}^L, \mathbf{h}^{0^L} = \alpha(\mathbf{h}^L) \leq \mathbf{k}^L$ so, the non null terms of $\gamma(\mathbf{h}^{0^L})$ will be the same ones of $\gamma(\mathbf{h}^L)$ and the thesis is established. ■

We define an equivalence relation α in Γ, writing $\mathbf{h}_1^L \alpha \mathbf{h}_2^L$ when $\alpha(\mathbf{h}_1^L) = \alpha(\mathbf{h}_2^L)$. With $[\mathbf{h}^L]_\alpha$ the α equivalence class that contains $\mathbf{h}^L$, let Γ/α be the family of these equivalence classes. To study Γ/α let us establish

Lemma 1. *If* $\mathbf{h}^L \in \Gamma_{m,1} \cup \Gamma_{p,2}$, $[\mathbf{h}^L]_\alpha$ *reduces to* $\mathbf{h}^L$.

Proof. If $\mathbf{h}^L \in \Gamma_{m,1} \cup \Gamma_{p,2}$, $\alpha(\mathbf{h}^L) = \mathbf{h}^L$. If there is $\mathbf{k}^L \neq \mathbf{h}^L$ such that $\alpha(\mathbf{k}^L) = \alpha(\mathbf{h}^L) = \mathbf{h}^L$, we will have $\mathbf{k}^L \in \Gamma_{m,2} \cup \Gamma_{p,1}$ so that $\alpha(\mathbf{k}^L) \in \Gamma_{p,1}$ which is contradictory. ■
We now establish

Lemma 2. *If* $\mathbf{h}^L \in \Gamma_{p,1}$, *there exists* $\mathbf{k}^L \in \Gamma_{m,2}$ *such that* $\alpha(\mathbf{k}^L) = \alpha(\mathbf{h}^L) = \mathbf{h}^L$, *then we have* $\mathbf{k}^L \in [\mathbf{h}^L]_\alpha$.

Proof. If $\mathbf{h}^L \in \Gamma_{p,1}$, there exists l' such that $h_{l'} = v_l + 1 > 1$. Now we may take $\mathbf{k}^L$ with components $k_l = h_l, l \neq l'$ and $k_{l'} = v_{l'}$. We see that $\mathbf{k}^L \in \Gamma_{m,2}$ and that $\alpha(\mathbf{k}^L) = \mathbf{h}^L$, so the thesis is established. ■
We still have

Lemma 3. $\alpha(\mathbf{h}^L) \in [\mathbf{h}^L]_\alpha$ *holds.*

Proof. It suffices to consider the case in which $\mathbf{h}^L \neq \mathbf{h}^{0^L} = \alpha(\mathbf{h}^L)$. Then we have $\mathbf{h}^L \in \Gamma_{m,2}$ and $\mathbf{h}^{0^L} \in \Gamma_{p,1}$, so $\alpha(\mathbf{h}^{0^L}) = \alpha(\mathbf{h}^L) = \mathbf{h}^{0^L}$ and consequently $\mathbf{h}^{0^L} \alpha \mathbf{h}^L$. ■
We may now establish

Proposition 2. *If* $\mathbf{h}^L \in \Gamma_{m,1} \cup \Gamma_{p,2}, [\mathbf{h}^L]_\alpha$ *reduces to* $\alpha(\mathbf{h}^L) = \mathbf{h}^L$*; if* $\mathbf{h}^L \in \Gamma_{m,2} \cup \Gamma_{p,1}$, $[\mathbf{h}^L]_\alpha$ *intersects* $\Gamma_{m,2}$ *and contains a only one vector* $\alpha(\mathbf{h}^L) \in \Gamma_{p,1}$.

Proof. On account of the previous Lemmas, 1, 2 and 3, it is enough to show that, if $\mathbf{h}^L \in \Gamma_{m,2} \cup \Gamma_{p,1}$, $\alpha(\mathbf{h}^L)$ is the only vector in $[\mathbf{h}^L]_\alpha \cap \Gamma_{p,1}$. This unicity follows from, if $\mathbf{k}^L \in [\mathbf{h}^L]_\alpha \cap \Gamma_{p,1}$, due to Lemma 3, $\alpha(\mathbf{k}^L) \in [\mathbf{h}^L]_\alpha$, so that $\alpha(\mathbf{h}^L) = \alpha(\mathbf{k}^L) = \mathbf{k}^L$ and the thesis is established. ■

Corollary 1. *With* $\alpha(\Gamma) = \{\alpha(\mathbf{h}^L), \mathbf{h}^L \in \Gamma\} = \Gamma_{m,1} \cup \Gamma_p$, *which* $[\mathbf{h}^L]_\alpha$ *contains one and only one vector of* $\alpha(\Gamma)$*.*

Corollary 2. $\mathbf{k}^L \in [\mathbf{h}^L]_\alpha$ *implicate* $\gamma(\mathbf{k}^L) = \gamma(\mathbf{h}^L)$.

For $\mathbf{h}^L \in \Gamma_r$, we take $(\mathbf{h}^L)_r = [\mathbf{h}^L]_\alpha \cap \Gamma_r$, so we have $(\mathbf{h}^L)_r = [\mathbf{h}^L]_\alpha = \{\mathbf{h}^L\}$, when $\mathbf{h}^L \in \Gamma_{p,2}$.

We will now consider $S(\mathbf{k}^L) = \|\widetilde{\eta}(\mathbf{k}^L)^{g(\mathbf{k}^L)}\|^2$, that will be, see Mexia (1995, p. 52), the product by $\gamma(\mathbf{k}^L)$ of a chi-square with $g(\mathbf{k}^L)$ degrees of freedom and non-centrality parameter

$$\delta(\mathbf{k}^L) = \frac{\|\eta(\mathbf{k}^L)^{g(\mathbf{k}^L)}\|^2}{\gamma(\mathbf{k}^L)}, \mathbf{k}^L \in \Gamma_f, \tag{5}$$

so we have $S(\mathbf{k}^L) \sim \gamma(\mathbf{k}^L)\chi^2_{g(\mathbf{k}^L),\delta(\mathbf{k}^L)}, \mathbf{k}^L \in \Gamma_f$.

Likewise, $S \sim \sigma^2 \chi^2_g$ and we have the UMVUE

$$\tilde{\sigma}^2 = \frac{S}{g} \tag{6}$$

for σ^2.

Let us put

$$\begin{cases} S_r(\mathbf{h}^L) = \sum_{\mathbf{k}^L \in (\mathbf{h}^L)_r} S(\mathbf{k}^L) \\ \\ g_r(\mathbf{h}^L) = \sum_{\mathbf{k}^L \in (\mathbf{h}^L)_r} g(\mathbf{k}^L) \end{cases}, \mathbf{h}^L \in \Gamma_r. \tag{7}$$

Due to the reprodutibility of the chi-squares, $S_r(\mathbf{h}^L) \sim \gamma(\mathbf{h}^L)\chi^2_{g_r(\mathbf{h}^L)}$, $\mathbf{h}^L \in \Gamma_p$, so we have the UMVUE

$$\widetilde{\gamma}_r(\mathbf{h}^L) = \frac{S_r(\mathbf{h}^L)}{g_r(\mathbf{h}^L)}, \mathbf{h}^L \in \Gamma_p, \tag{8}$$

since $\widetilde{\eta}(\mathbf{h}^L)^{g(\mathbf{h}^L)}, \mathbf{h}^L \in \Gamma_f$ and the $S(\mathbf{k}^L), \mathbf{k}^L \in \Gamma_r$ continue to constitute, along the general mean, one minimal complete sufficient statistic.

Let us point out that, if $\mathbf{h}^L \in \Gamma_{m,1} \cap \Gamma_r$, there does not exist $\mathbf{h}^L \in \Gamma_p$ such that $\mathbf{h}^L < \mathbf{k}^L$ so we have $\gamma(\mathbf{h}^L) = \sigma^2$ when cross additivity holds.

Putting

$$\begin{cases} S_r = S + \displaystyle\sum_{\mathbf{h}^L \in \Gamma_{m,1} \cap \Gamma_r} S(\mathbf{h}^L) \\ \\ g_r = g + \displaystyle\sum_{\mathbf{h}^L \in \Gamma_{m,1} \cap \Gamma_r} g(\mathbf{h}^L) \end{cases}, \tag{9}$$

we will have, as before, the UMVUE

$$\sigma_r^2 = \frac{S_r}{g_r}. \tag{10}$$

In S.F. Saraiva (2006), it is established

$$\sigma^2(\mathbf{u}^L) = \frac{1}{r}\left(\gamma(\mathbf{u}^L) - \sigma^2\right), \tag{11}$$

when $\mathbf{u}^L \notin \Gamma_p$, so we get the UMVUE

$$\widetilde{\sigma}_r^2(\mathbf{u}^L) = \frac{1}{r}(\widetilde{\gamma}_r(\mathbf{u}^L) - \sigma^2). \tag{12}$$

Likewise, using the following Proposition, when $\mathbf{h}^L \in \Gamma \backslash \mathbf{u}^L$, $m(\mathbf{h}^L) > 0$ and

Proposition 3.

$$\sigma^2(\mathbf{h}^L) = \frac{1}{b(\mathbf{h}^L)} \sum_{\mathbf{k}^L \in \odot(\mathbf{h}^L)} (-1)^{m(\mathbf{h}^L, \mathbf{k}^L)} \gamma(\mathbf{k}^L) = \sigma^2(\mathbf{h}^L)^+ - \sigma^2(\mathbf{h}^L)^-,$$

with

$$(13) \qquad \begin{cases} \sigma^2(\mathbf{h}^L)^+ = \dfrac{1}{b(\mathbf{h}^L)} \displaystyle\sum_{\mathbf{k}^L \in \odot(\mathbf{h}^L)^+} \gamma(\mathbf{k}^L) \\ \\ \sigma^2(\mathbf{h}^L)^- = \dfrac{1}{b(\mathbf{h}^L)} \displaystyle\sum_{\mathbf{k}^L \in \odot(\mathbf{h}^L)^-} \gamma(\mathbf{k}^L) \end{cases},$$

we get the UMVUE

$$(14) \qquad \widetilde{\sigma}_r^2(\mathbf{h}^L) = \frac{1}{b(\mathbf{h}^L)} \sum_{\mathbf{k}^L \in \odot(\mathbf{h}^L)} (-1)^{m(\mathbf{h}^L, \mathbf{k}^L)} \widetilde{\gamma}_r(\mathbf{k}^L), \quad \mathbf{h}^L \in \Gamma_p \backslash \{\mathbf{u}^L\}.$$

Going over to the fixed effects part of the model, the inference is extremely simplified. Namely, we will have

$$(15) \qquad \begin{cases} \gamma(\mathbf{h}^L) = \sigma^2, \quad \mathbf{h}^L \in \Gamma_f \cap \Gamma_{m,1} \\ \\ \gamma(\mathbf{h}^L) = \gamma(\mathbf{h}^{0^L}), \quad \mathbf{h}^L \in \Gamma_f \cap \Gamma_{m,2} \end{cases},$$

with $\mathbf{h}^{0^L} = \alpha(\mathbf{h}^L) \in \Gamma_{p,1}$. Thus, we will have $\widetilde{\eta}(\mathbf{h}^L)^{g(\mathbf{h}^L)}$ independent of $\widetilde{\gamma}_r(\mathbf{h}^L)$, with $\widetilde{\eta}(\mathbf{h}^L)^{g(\mathbf{h}^L)} \sim \mathcal{N}(\eta(\mathbf{h}^L)^{g(\mathbf{h}^L)}, \gamma_r(\mathbf{h}^L)\mathbf{I}_{g(\mathbf{h}^L)})$. If $\mathbf{h}^L \in \Gamma_f \cap \Gamma_{m,1}$ it is easily seen that $\widetilde{\gamma}_r(\mathbf{h}^L) = \widetilde{\sigma}_r^2$ and that $g_r(\mathbf{h}^L) = g$. Thus it will be straightforward to build confidence ellipsoids and carry out selective or not selective $\mathcal{F}$ tests.

2. An application

We now assume that location and origin have fixed effects and that cross additivity is verified. Considering $u_1 = 1, u_2 = 2$ e $v_1 = 1, v_2 = 1$, we have

$$\Gamma = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \begin{bmatrix} 0 \\ 2 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

with

$$\Gamma_f = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\},$$

$$\Gamma_a = \left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix}; \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}, \quad \Gamma_c = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}, \quad \Gamma_p = \left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

and

$$\left\{ \Gamma_{p,1} = \left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} \quad \Gamma_{p,2} = \emptyset \right. .$$

Now we have

Table 1. Indexes set

	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$\varphi_f(\mathbf{h}^L)$	$\emptyset$	$\{1\}$	$\{2\}$	$\emptyset$	$\{1,2\}$	$\{1\}$
$\varphi_{f,1}(\mathbf{h}^L)$	$\emptyset$	$\{1\}$	$\emptyset$	$\emptyset$	$\{1\}$	$\{1\}$
$\varphi_{f,2}(\mathbf{h}^L)$	$\emptyset$	$\emptyset$	$\{2\}$	$\emptyset$	$\{2\}$	$\emptyset$

as well

$$\Gamma_m = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

and

$$\begin{cases} \Gamma_{m,1} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \\ \Gamma_{m,2} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \end{cases}.$$

Considering (4) we obtain

Table 2. α Equivalence classes

	$\Gamma_{m,1}$	$\Gamma_{m,2}$	Γ_p	$\Gamma_{m,1}$	$\Gamma_{m,1}$
$\mathbf{h}^L$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$\mathbf{h}^{0^L}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$[\mathbf{h}^L]_\alpha$	$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$	$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$	$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$	$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$	$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Now, according to Proposition 1 we have

$$\begin{cases} \gamma\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \gamma\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \\ \gamma\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \gamma\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ \gamma\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \gamma\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \\ \gamma\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \gamma\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\ \gamma\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \gamma\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \end{cases}.$$

We recall that, see Lemma 1, the equivalence classes to $\mathbf{h}^L \in \Gamma_{m,1}$ will be, in agreement with the previous table,

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)_\alpha = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \ \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)_\alpha = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\},$$

$$\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)_\alpha = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

and according to Lemma 2, we have

$$\alpha\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \alpha\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We see that

$$\Gamma_{m,1} \cap \Gamma_c = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\},$$

as well as

$$S_a = S + S\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 26,433 + 3,4158 = 29,591$$

and

$$g_a = g + g\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 72 + 8 = 80,$$

so we obtain $\widetilde{\sigma}_a^2 = \dfrac{S_a}{g_a} = \dfrac{29,591}{80} = 0,3731$. We point out that, for the only vector of Γ_p we have

$$\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)_a = \left\{\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right\},$$

coming

$$\widetilde{\gamma}_a\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \widetilde{\gamma}\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \frac{6,2590}{4} = 1,5648,$$

so we have

$$\widetilde{\sigma}_a^2\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \frac{\widetilde{\gamma}_a\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) - \widetilde{\sigma}_a^2}{b\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)} = \frac{1}{15}(1,5648 - 0,3731) = 0,0794.$$

Passing to the part of the fixed effects, we have to consider the two first factors and their interaction

$$\begin{cases} \Gamma_f \cap \Gamma_{m,1} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \\ \Gamma_f \cap \Gamma_{m,2} = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \end{cases},$$

so we have, according to expression (15),

$$\begin{cases} \gamma\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \gamma\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \sigma^2 \\ \gamma\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \gamma\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \end{cases}.$$

Thus, the non selectives $\mathcal{F}$ tests for the nullity of effects and interactions will have the statistics

$$\mathcal{F}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\frac{g_a}{g\left(\begin{bmatrix}1\\0\end{bmatrix}\right)}\frac{S\left(\begin{bmatrix}1\\0\end{bmatrix}\right)}{S_a}=0,90043$$

$$\mathcal{F}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\frac{g_a}{g\left(\begin{bmatrix}1\\1\end{bmatrix}\right)}\frac{S\left(\begin{bmatrix}1\\1\end{bmatrix}\right)}{S_a}=4,6184$$

$$\mathcal{F}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\frac{g\left(\begin{bmatrix}0\\2\end{bmatrix}\right)}{g\left(\begin{bmatrix}0\\1\end{bmatrix}\right)}\frac{S\left(\begin{bmatrix}0\\1\end{bmatrix}\right)}{S\left(\begin{bmatrix}0\\2\end{bmatrix}\right)}=6,1419.$$

Then we conclude that, at the level of 5%, the test is not significant nor for the first factor nor for the second factor. For the interaction at the level of 5%, the test is significant.

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