

**ON GEOMETRY OF THE SET OF ADMISSIBLE
QUADRATIC ESTIMATORS OF QUADRATIC
FUNCTIONS OF NORMAL PARAMETERS**

KONRAD NEUMANN AND STEFAN ZONTEK

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra

prof. Z. Szafrana 4a, 65–516 Zielona Góra, Poland

e-mail: k.neumann@wmie.uz.zgora.pl

e-mail: s.zontek@wmie.uz.zgora.pl

Abstract

We consider the problem of admissible quadratic estimation of a linear function of μ^2 and σ^2 in n dimensional normal model $N(K\mu, \sigma^2 I_n)$ under quadratic risk function. After reducing this problem to admissible estimation of a linear function of two quadratic forms, the set of admissible estimators are characterized by giving formulae on the boundary of the set $\mathcal{D} \subset \mathcal{R}^2$ of components of the two quadratic forms constituting the set of admissible estimators. Different shapes and topological properties of the set \mathcal{D} are studied.

Keywords: linear estimator, quadratic estimator, Bayesian quadratic estimator, quadratic loss function, admissibility, quadratic subspace.

2000 Mathematics Subject Classification: 62F10, 62J10.

1. INTRODUCTION AND NOTATION

Let $Y \in \mathcal{R}^n$ be a normal random vector such that

$$E(Y) = K\mu \quad \text{and} \quad \text{cov}(Y) = \sigma^2 I_n,$$

where $\mu \in \mathcal{R}$ and $\sigma^2 > 0$ are unknown parameters, $K \in \mathcal{R}^n$ is a known vector and I_n is the $n \times n$ identity matrix. Such model can be considered for measurements of the intensity μ of a source of signals by a numbers of sensors. For a single observation we assume the following structure of measured intensity of the source by the sensor: $y = k\mu + e$, where k is known, while e denotes an unobservable random error (see Gnot *et al.*, 2001 and references therein).

In the paper we consider the problem of characterization of admissible quadratic estimators of the function

$$f' \theta = f_1 \mu^2 + f_2 \sigma^2,$$

where f_1, f_2 are given scalars, with the quadratic loss function.

The natural class of estimators of $f' \theta$ is the class of quadratic forms with respect to Y , that is $\{Y' A Y : A \in S_n\}$, where S_n denotes the space of $n \times n$ symmetric matrices.

It is known, that $K'Y$ and $Y' (I_n - \frac{1}{K'K} K K')$ Y are sufficient statistics for μ and σ^2 . So each quadratic estimator of $f' \theta$ is a linear combination of the following quadratic forms (see also Zmysłony, 1976)

$$Z_1 = \frac{1}{K'K} Y' B_1 Y$$

and

$$Z_2 = \frac{1}{n-1} Y' B_2 Y,$$

where $B_1 = \frac{1}{K'K} K K'$ and $B_2 = I_n - B_1$.

For K being the vector of ones (so called one way classification random model) Rukhin (1990) has shown that an estimator $g_1 Z_1 + g_2 Z_2$ is admissible for $f' \theta$ (see Remark in Section 2 of this paper) iff

$$(1) \quad \left(n f_2 - \frac{n(n+1)}{n-1} g_2 - g_1 \right) \left(5 g_1 - \frac{2n}{n-1} g_2 - f_1 \right) \\ \leq (g_1 - f_1) (n f_2 - n g_2 - 3 g_1)$$

and

$$(2) \quad \text{sgn}(g_1 - f_1) = \text{sgn}\left(nf_2 - \frac{n(n+1)}{n-1}g_2 - g_1\right).$$

For slightly more general case (arbitrary vector K) we will give formulae on the boundary of the set

$$\mathcal{D}_f = \{(g_1, g_2)' : g_1 Z_1 + g_2 Z_2 \text{ is admissible for } f'\theta\}.$$

The description of the set \mathcal{D}_f enables to prove whether or not a given estimator is admissible. We presented on figures different shapes of the set \mathcal{D}_f . Also there are given examples of a comparison of risk of an admissible estimator with the best unbiased estimator and the maximum likelihood estimator.

Congruent approach to characterization of admissible quadratic estimators in balanced two variance components model was initiated by LaMotte (1985). In more details it was analyzed by Neumann and Zontek (2004). Sets \mathcal{D}_f , considered in this paper, possess different properties then corresponding sets resulting from balanced two variance components model.

2. CHARACTERIZATION OF THE ADMISSIBLE ESTIMATORS FOR $E(Z)$

First we characterize the class of admissible estimators of $E(Z)$, where $Z = (Z_1, Z_2)'$. It is easy to calculate that

$$E_\theta(Z) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu^2 \\ \sigma^2 \end{bmatrix}$$

and that

$$\text{cov}_\theta(Z) = 2 \begin{bmatrix} 2a\mu^2\sigma^2 + a^2\sigma^4 & 0 \\ 0 & b\sigma^4 \end{bmatrix},$$

where $a = \frac{1}{K'K}$ and $b = \frac{1}{n-1}$.

The total mean squared error of a linear estimator $L'Z$, where L is 2×2 matrix, of $E(Z)$ can be written as

$$TMSE_L(\mu^2, \sigma^2) = tr[L' cov(Z) L + (L - I_2)' E(Z) E(Z)' (L - I_2)],$$

where for a quadratic matrix A the symbol $tr(A)$ denotes the trace of A .

In order to construct an admissible estimator of $E(Z)$ we will use Bayesian approach. Let τ be a priori distribution on the parameter space such that $\Delta_\tau = \{\delta_{ij}\} = E_\tau[(\mu^2, \sigma^2)'(\mu^2, \sigma^2)]$ exists. Then the Bayesian risk of $L'Z$ has the form

$$r_\tau(L'Z) = tr[L' V_\tau L + (L - I_2)' \Phi_\tau (L - I_2)],$$

where

$$V_\tau = 2 \begin{bmatrix} a(2\delta_{12} + a\delta_{22}) & 0 \\ 0 & b\delta_{22} \end{bmatrix}$$

and

$$\Phi_\tau = \begin{bmatrix} \delta_{11} + a(2\delta_{12} + a\delta_{22}) & \delta_{12} + a\delta_{22} \\ \delta_{12} + a\delta_{22} & \delta_{22} \end{bmatrix}.$$

An estimator $L'Z$ of $E(Z)$ is said to be Bayesian estimator with respect to τ if $r_\tau(L'Z) \leq r_\tau(M'Z)$ for any 2×2 matrix M . It is known that an estimator $L'Z$ is Bayesian with respect to τ iff

$$(V_\tau + \Phi_\tau)L = \Phi_\tau$$

(see for example LaMotte, 1982). If $|V_\tau + \Phi_\tau| \neq 0$, then the above equation has a unique solution and

$$[(V_\tau + \Phi_\tau)^{-1} \Phi_\tau]' Z$$

is admissible for $E(Z)$. Note that the class of uniquely determined Bayesian estimators and their limits is equal to the class of all admissible estimators.

The Bayesian estimator with respect to τ is uniquely determined iff $\delta_{22} > 0$. Under this assumption, without loss of generality, we can assume that $\delta_{22} = 1$. Following Gnot and Kleffe (1983) we present Δ_τ in the following way

$$\Delta_{\tau} = \begin{bmatrix} u^2 + v^2 & u \\ u & 1 \end{bmatrix},$$

where $u, v \geq 0$. Then

$$V_{\tau} = 2 \begin{bmatrix} a(2u + a) & 0 \\ 0 & b \end{bmatrix}$$

and

$$\Phi_{\tau} = \begin{bmatrix} u^2 + a(2u + a) + v^2 & u + a \\ u + a & 1 \end{bmatrix}.$$

Simple algebra shows that for a fixed $u \geq 0$ and $v \geq 0$ the estimator $L'Z$ is Bayesian iff

$$(3) \quad L = \frac{1}{w} \begin{bmatrix} 2b(a+u)^2 + (2b+1)v^2 & 2b(a+u) \\ 2a(a+u)(a+2u) & 2a(a+2u) + v^2 \end{bmatrix},$$

where

$$w = 2bu^2 + 2a(3b+1)(a+2u) + (2b+1)v^2.$$

To the class of admissible estimators belong also limits of estimators given in (3). This implies (for arbitrary $u \geq 0$ and $v \rightarrow +\infty$) that the estimator of the following form is also admissible:

$$(4) \quad L = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2b+1} \end{bmatrix}.$$

All estimators determined by (3) and (4) constitute the class of all admissible estimators of $E(Z)$.

3. ADMISSIBLE ESTIMATION OF $f'\theta$

Admissible estimator for a function $f'\theta = f_1\mu^2 + f_2\sigma^2$ can be described in the following way. An estimator $g'Z = g_1Z_1 + g_2Z_2$ is admissible for $f'\theta$ iff

$$(5) \quad g \in \mathcal{D}_f = \{LH^{-1}f : L'Z \text{ is admissible for } E(Z)\},$$

where $H = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$. Note that $f'(H')^{-1}E(Z) = f'\theta$.

In order to describe the boundary of \mathcal{D}_f we introduce a new parameter l (instead of v) defined in the proof of the following lemma.

Lemma 1. *An estimator $L'Z$ is admissible for $E(Z)$ iff*

$$(6) \quad L = \begin{bmatrix} 1 - \frac{a(2b+1)(a+2u)l}{a(3b+1)(a+2u)+bu^2} & \frac{b(a+u)l}{a(3b+1)(a+2u)+bu^2} \\ \frac{a(a+u)(a+2u)l}{a(3b+1)(a+2u)+bu^2} & \frac{1}{1+2b} \left(1 - \frac{b(a+u)^2l}{a(3b+1)(a+2u)+bu^2} \right) \end{bmatrix},$$

where $u \geq 0$, $0 \leq l \leq 1$.

Proof. Define l as

$$(7) \quad l = \frac{2[bu^2 + a(3b+1)(a+2u)]}{2[bu^2 + a(3b+1)(a+2u)] + (2b+1)v^2}.$$

Note that $l \in (0, 1]$ for any fixed $u \geq 0$, and that $l \rightarrow 0$ when $v \rightarrow +\infty$. Simple calculations show that substituting l in formula (6) we get (3). For $l = 0$ we get matrix of the form (4). This ends the proof. \blacksquare

Introducing the new parametrization we obtain simpler and compact formula on matrices constituting the class of admissible estimators of $E(Z)$.

Note that entries of (6) linearly depend on the new parameter l and that l belongs to the closed interval. This remark play important role in the proof of the main result of the paper.

According to (5) we get the following corollary.

Corollary 1. *An estimator $g'Z = g_1Z_1 + g_2Z_2$ is admissible for $f'\theta = f_1\mu^2 + f_2\sigma^2$ if and only if*

$$(8) \quad g_1 = \frac{a^2(3b+1)(1-l) + a[2(3b+1) - (5b+2)l]u + bu^2}{a(3b+1)(a+2u) + bu^2} f_1 + \frac{b(a+u)l}{a(3b+1)(a+2) + bu^2} f_2,$$

$$(9) \quad g_2 = \frac{a^2(3b+1)(l-1) + a[(8b+3)l - 2(3b+1)]u + [(5b+2)l - b]u^2}{(2b+1)[a(3b+1)(a+2u) + bu^2]} a f_1 + \frac{b(1-l)u^2 + a[b(3-l) + 1](a+2u)}{(2b+1)[a(3b+1)(a+2u) + bu^2]} f_2,$$

where $u \geq 0$, $0 \leq l \leq 1$.

Since g_1 and g_2 given by (8) and (9), respectively, are linear combinations of elements of (6), then they also linearly depend on l .

Theorem 1. *An estimator $g_1Z_1 + g_2Z_2$ is admissible for $f'\theta = f_1\mu^2 + f_2\sigma^2$ if and only if $(g_1, g_2)' \in \mathcal{D}_f$, where \mathcal{D}_f is the area with the boundary $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, where*

$$\mathcal{A} = \left\{ \left[\begin{array}{c} \frac{b(a+u)(uf_1+f_2)}{a(3b+1)(a+2u)+bu^2} \\ \frac{a(a+2u)(uf_1+f_2)}{a(3b+1)(a+2u)+bu^2} \end{array} \right] : u \geq 0 \right\},$$

\mathcal{B} is the interval connecting point $B = (f_1, \frac{2af_1}{b})'$ with point $A_\infty = (f_1, \frac{2af_1}{b})'$ and \mathcal{C} is the interval connecting point B with point $A_0 = (\frac{bf_2}{a(3b+1)}, \frac{f_2}{3b+1})'$; such that $\mathcal{A} \subset \mathcal{D}_f$, $\mathcal{C} \subset \mathcal{D}_f$ and $\mathcal{D}_f \cap \mathcal{B} = \{B\}$.

Proof. First note that for any fixed $u \geq 0$ components of g given by (8) and (9) linearly depend on l and that l belongs to the closed interval $[0, 1]$. So the set \mathcal{D}_f consists of intervals connecting a point A from line \mathcal{A} obtained for $l = 1$ that is

$$A = \left(\frac{b(a+u)(uf_1+f_2)}{a(3b+1)(a+2u)+bu^2}, \frac{a(a+2u)(uf_1+f_2)}{a(3b+1)(a+2u)+bu^2} \right)', \quad u \geq 0$$

with the point B obtained for $l = 0$. Now it is enough to notice that the interval \mathcal{C} we obtain for $u = 0$ and the interval \mathcal{B} is the set of limits of the points $(g_1, g_2)'$ when $u \rightarrow +\infty$. This ends the proof. ■

Notice that for estimation of $f'\theta$ the set \mathcal{D}_f is bounded, and sometimes closed.

Remark 1. The description of the set of admissible estimators of a linear combination of μ^2 and σ^2 given in Theorem 1 is different from the one given by Rukhin (1990, formulae (1) and (2) in this paper). For $k = \mathbf{1}_n$ the line \mathcal{A} is a part of the curve obtained by putting equality in inequality (1). Intervals \mathcal{B} and \mathcal{C} are contained in straight lines $\text{sgn}(g_1 - f_1) = 0$ and $\text{sgn}\left(nf_2 - \frac{n(n+1)}{n-1}g_2 - g_1\right) = 0$, respectively (compare condition (2)). However, condition (2) excludes the admissible estimators corresponding to interval \mathcal{C} except the point B . So the condition (2) is too restrictive.

For the fixed model (i.e., a and b are fixed) the shape of the set \mathcal{D}_f depends on estimated function $f_1\mu^2 + f_2\sigma^2$. Different shapes of the set \mathcal{D}_f correspond to three areas given by the following lines:

$$(a) : f_2 = 0,$$

$$(b) : f_2 = \frac{a(3b+1)}{b} f_1,$$

$$(c) : f_2 = \frac{a(5b+2)}{b} f_1.$$

We also consider three cases corresponding to boundary lines (a), (b) and (c).

Different forms of the set \mathcal{D}_f we present on figures. Because the shape of area \mathcal{D}_f does not depend on the parameters a and b , at calculations for figures we accepted $a = 0.25$ and $b = 0.1$ (for all figures).

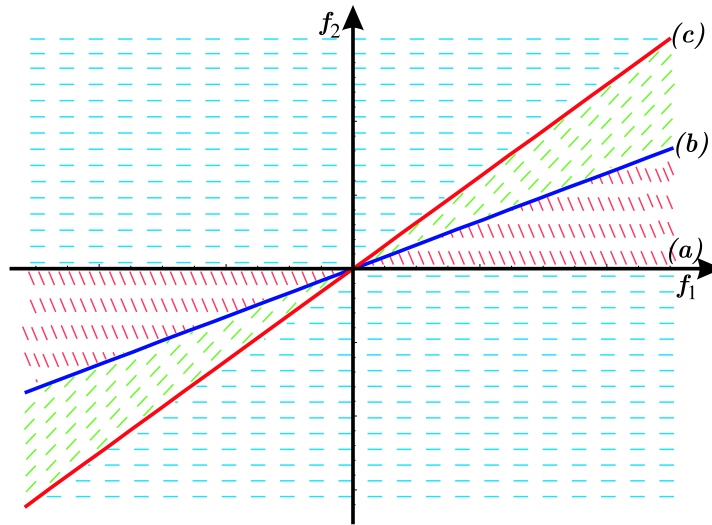


Figure 1. Areas of vectors f under considerations.

On Figures 2–5 presenting set \mathcal{D}_f , the position of points corresponding to the best unbiased estimator (point \mathcal{P}) and the maximum likelihood estimator (point \mathcal{S} , derived under normality) are marked. These estimators are given by $g_1 = f_1$, $g_2 = f_2 - a f_1$ and $g_1 = f_1$, $g_2 = \frac{1}{b+1} f_2$, respectively.

On Figure 2 there is presented the shape of \mathcal{D}_f for $f = (f_1, 0)'$, $f_1 > 0$. The set \mathcal{D}_f is the sum of intervals connecting points from \mathcal{A} with the point B . So the set \mathcal{D}_f is bounded, but not closed and not convex.

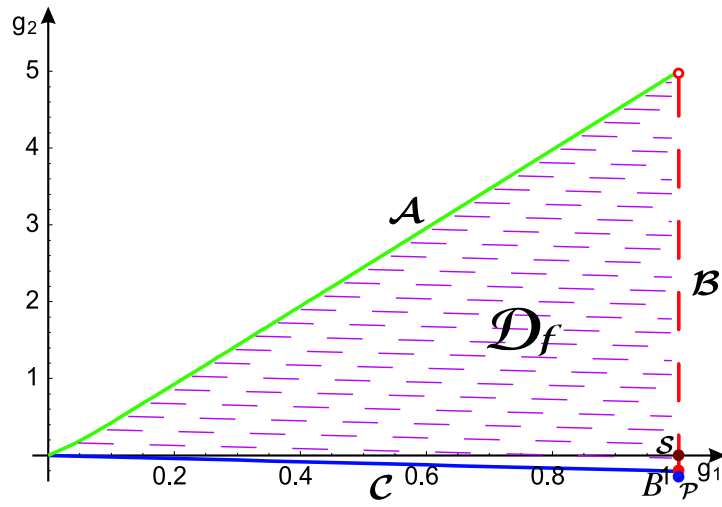


Figure 2. Admissible estimators for $f'\theta$, where $f = (1, 0)'$.

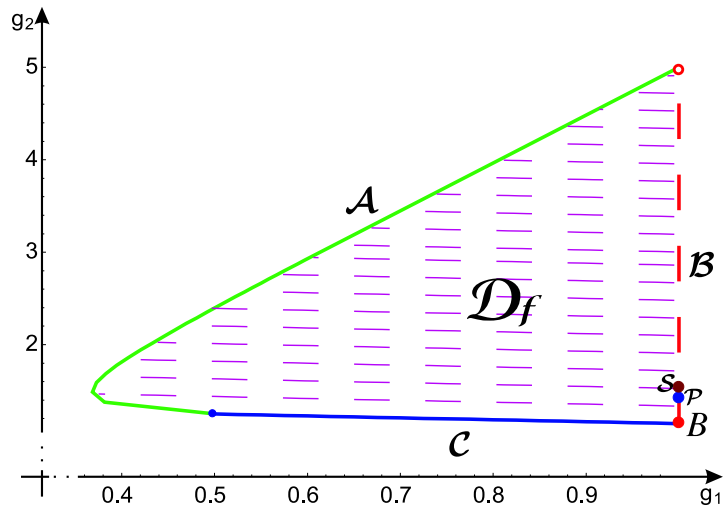


Figure 3. Admissible estimators for $f'\theta$, where $f = (1, 1.625)'$.

The shape of \mathcal{D}_f on Figure 3 we obtain for $f = (f_1, f_2)'$, when $0 < f_2 < \frac{a(3b+1)f_1}{b}$, $f_1 > 0$. This set is not closed but bounded and convex.

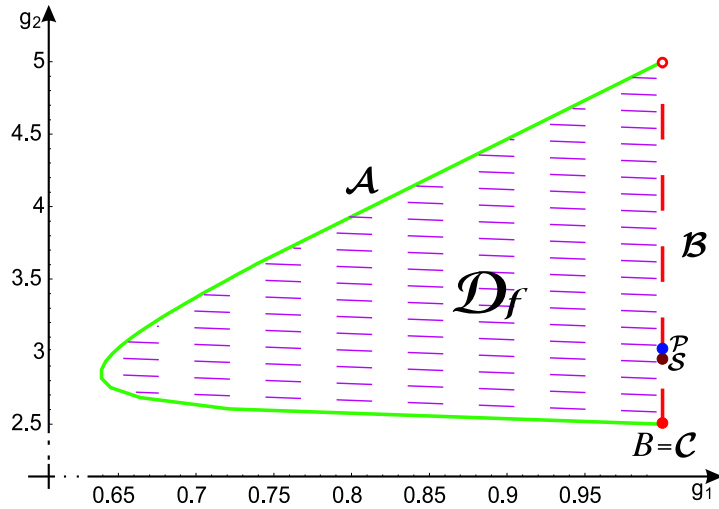


Figure 4. Admissible estimators for $f'\theta$, where $f = (1, 3.25)'$.

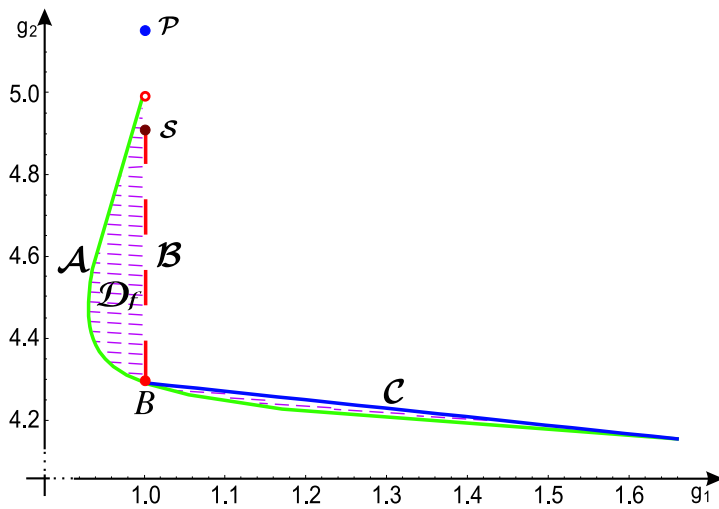


Figure 5. Admissible estimators for $f'\theta$, where $f = (1, 5.4)'$.

Figure 4 presents the shape of \mathcal{D}_f for $f = (f_1, f_2)'$, where $f_2 = \frac{a(3b+1)f_1}{b}$, $f_1 > 0$. Then interval \mathcal{C} is degenerated to one point and the convex set \mathcal{D}_f is between line \mathcal{A} and interval \mathcal{B} . It is bounded and convex but not closed.

The shape of \mathcal{D}_f on Figure 5 we obtain for $f = (f_1, f_2)'$, where $\frac{a(3b+1)f_1}{b} \leq f_2 < \frac{a(5b+2)f_1}{b}$, $f_1 > 0$. This set consists of two parts and is bounded by line \mathcal{A} , interval \mathcal{B} and interval \mathcal{C} . The set \mathcal{D}_f is bounded but not closed and not convex.

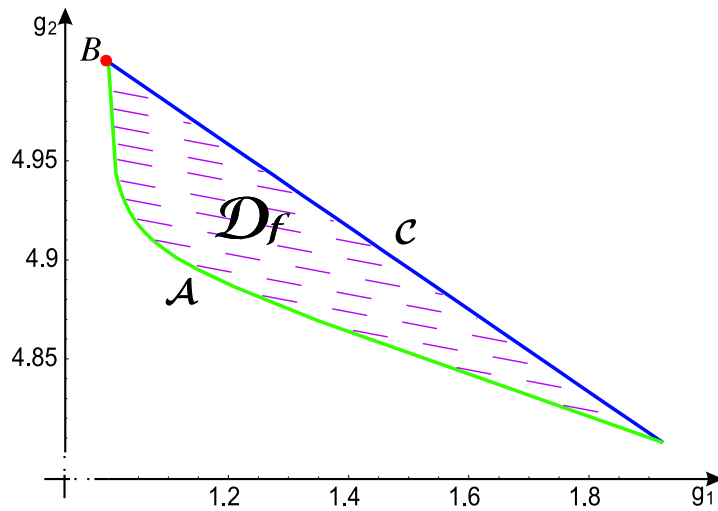


Figure 6. Admissible estimators for $f'\theta$, where $f = (1, 6.25)'$.

For $f = (f_1, f_2)'$, where $f_2 = \frac{a(5b+2)f_1}{b}$, $f_1 > 0$, we obtain the shape of \mathcal{D}_f in the form presented on Figure 6. Then interval \mathcal{B} is degenerated to one point and the convex set \mathcal{D}_f is between line \mathcal{A} and interval \mathcal{C} . It is bounded, convex and closed. On Figure 6 and 7 we do not mark points \mathcal{S} and \mathcal{P} because the position of these points go beyond the scale of picture.

In the case when $f_2 > \frac{a(5b+2)f_1}{b}$, $f_1 > 0$ the set \mathcal{D}_f is not convex.

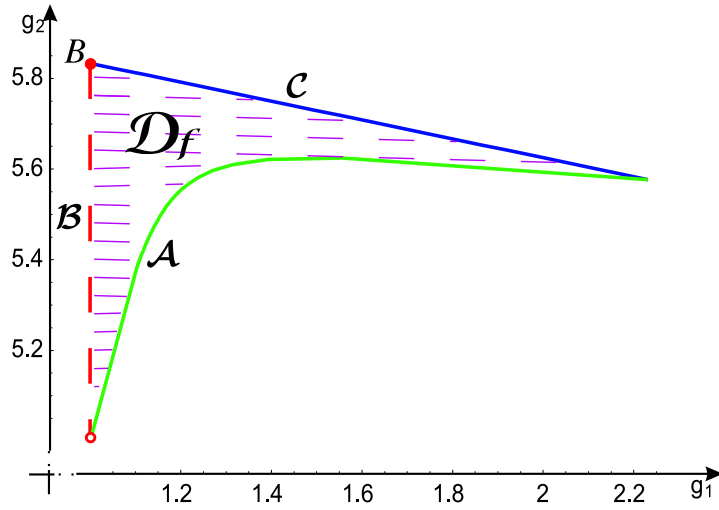


Figure 7. Admissible estimators for $f'\theta$, where $f = (1, 7.25)'$.

Figures 2–7 present different shapes of \mathcal{D}_f for $f_1 \geq 0$. For completeness notice that the set \mathcal{D}_{-f} is symmetric to the set \mathcal{D}_f with respect to point $(0, 0)'$. Also notice that for $f = (0, 0)'$ the set \mathcal{D}_f reduced to $\{(0, 0)'\}$.

4. CONCLUSIONS

In the paper we give explicitly described the boundary of the set of admissible quadratic estimators for a linear combination of parameters in n dimensional normal model $N(K\mu, \sigma^2 I_n)$. The shape of appointed set of admissible quadratic estimators strongly depends on estimated function $f'\theta = f_1 \mu^2 + f_2 \sigma^2$. Some examples we present on a series of figures. We obtain bounded set \mathcal{D}_f (convex or not), which usually is not closed. Only in one case, when we estimate $f'\theta = f_1 \mu^2 + f_2 \sigma^2$ with $f_2 = \frac{a(5b+2)f_1}{b}$, we obtain closed set \mathcal{D}_f of admissible quadratic estimators.

It is easy to prove that the best unbiased estimator (UE) is admissible only in the case when we estimate $f'\theta$ with $f_2 = a f_1$, similarly the maximum likelihood estimator (MLE) is admissible only in the case when we

estimate $f' \theta$ with $f_2 = \frac{-a(b+1)}{b} f_1$. In other cases we can find an admissible estimator, which is better than UE and MLE. Of course we can suspect that an admissible estimator can be much better than UE and MLE only for small values n .

For the estimation of linear combination μ^2 and σ^2 we should find estimators which have small risk. The minimal risk is attained by estimators corresponding to $l_{\max} = 1$. For selected $u = 0$ the smallest risk we have for $\mu^2 \ll \sigma^2$. When we choose somewhat larger u the suitable estimator have minimal risk for $\mu^2 \approx \sigma^2$. In the case when $\mu^2 \gg \sigma^2$ profits are relatively small when $u \rightarrow +\infty$.

In details we will consider two main cases: estimation of μ^2 and estimation of σ^2 . For both cases we choose an estimator corresponding to point B on line \mathcal{B} obtained for $l = 0$. This point is given by $B = \left(f_1, \frac{-a f_1 + f_2}{2b+1} \right)'$. Simple calculations show that the estimator corresponding to this point in both cases is better than UE and MLE.

In calculations we consider n dimensional model with $K = \mathbf{1}_n$. For such model $a = \frac{1}{n}$ and $b = \frac{1}{n-1}$.

4.1. Estimation of μ^2

Since in this case ($f_2 = 0$)

$$MSE_{\mathcal{P}}(\mu^2, \sigma^2) - MSE_{\mathcal{S}}(\mu^2, \sigma^2) = a^2 (2b - 1) \sigma^4 f_1^2,$$

the maximum likelihood estimator is worse than the unbiased one for $n > 3$. So we compare the risk of the unbiased estimator with the risk of the admissible quadratic estimator corresponding to the point B . The ratio of risks has following form

$$\frac{MSE_{\mathcal{P}}(\mu^2, \sigma^2)}{MSE_B(\mu^2, \sigma^2)} = \frac{(2b + 1) (2\mu^2 + a(b + 1)\sigma^2)}{(4b + 2)\mu^2 + a(3b + 1)\sigma^2}.$$

Table 1. Ratio of risks of estimators of μ^2

$\sigma^2 \setminus n$	2	4	6	8	10
0.0	1.000	1.000	1.000	1.000	1.000
0.1	1.018	1.002	1.001	1.000	1.000
0.2	1.038	1.004	1.001	1.000	1.000
0.3	1.062	1.007	1.002	1.001	1.000
0.4	1.091	1.010	1.003	1.001	1.001
0.5	1.125	1.014	1.004	1.002	1.001
0.6	1.167	1.020	1.006	1.003	1.001
0.7	1.219	1.029	1.009	1.004	1.002
0.8	1.286	1.042	1.014	1.006	1.003
0.9	1.375	1.064	1.023	1.011	1.006
1.0	1.500	1.111	1.050	1.029	1.019

MSE of *UE* relative to *MSE* corresponding to *B* for *n* dimensional normal model, calculated for $(1 - \sigma^2, \sigma^2)'$ with $\sigma^2 = 0(0.1)1$, respectively.

Since the ratio $MSE_{\mathcal{P}}(t\mu^2, t\sigma^2) / MSE_B(t\mu^2, t\sigma^2)$, $t > 0$, does not depend on t , we can compare risks on the interval $\{(\mu^2, \sigma^2)' : \mu^2 = 1 - \sigma^2, 0 \leq \sigma^2 \leq 1\}$. In Table 1 there are given ratios of risks calculated for $(1 - \sigma^2, \sigma^2)'$ with σ^2 from the set $\{0, 0.1, \dots, 0.9, 1\}$. Results of calculations confirm that even for not large values of n the estimator corresponding to *B* is only a little better than UE. For example when $n = 10$ the admissible estimator is only $0 \div 2\%$ better than UE.

4.2. Estimation of σ^2

Since in this case ($f_1 = 0$)

$$MSE_{\mathcal{P}}(\mu^2, \sigma^2) - MSE_S(\mu^2, \sigma^2) = \frac{b^2 (3 + 2b) \sigma^4 f_2^2}{(1 + b)^2} \geq 0,$$

then the maximum likelihood estimator is as good as the unbiased one. So we compare the risk of the unbiased estimator with the risk of the admissible quadratic estimator corresponding to the point *B*. The ratio of risks has the following form

$$\frac{MSE_S(\mu^2, \sigma^2)}{MSE_B(\mu^2, \sigma^2)} = \frac{(2 + b) (1 + 2b)}{2(1 + b)^2}.$$

Note that this ratio does not depend on the model parameters.

Calculated values of ratio $MSE_S(\mu^2, \sigma^2)$ and $MSE_B(\mu^2, \sigma^2)$ for different n are given in Table 2.

Table 2. Ratio of risks of estimators of σ^2

n	2	5	10	15	20	25	30	35
$\frac{MSE_S(\mu^2, \sigma^2)}{MSE_B(\mu^2, \sigma^2)}$	1.125	1.080	1.045	1.031	1.024	1.019	1.016	1.014

MSE of UE relative to MSE corresponding to B for n dimensional normal model

Only for small numbers n the estimator corresponding to point B is much better than MLE. For sufficiently large n , the maximum likelihood estimator is "almost admissible". For example the admissible estimator corresponding to B is only 1.4% better than MLE, when $n = 35$.

REFERENCES

- [1] S. Gnot, E. Rafałłowicz and A. Urbańska-Motyka, *Statistical inference in a linear model for spatially located sensors and random input*, Ann. Inst. Statist. Math. **53**, 2 (2001), 370–379.
- [2] S. Gnot and J. Kleffe, *Quadratic estimation in mixed linear models with two variance components*, J. Statist. Plann. Inference **8** (1983), 267–279.
- [3] D.A. Harville, *Quadratic unbiased estimation of two variance components for the one-way classification*, Biometrika **56** (1969), 313–326.
- [4] L.R. LaMotte, *Admissibility in linear model*, Ann. Statist. **19** (1982), 245–256.
- [5] L.R. LaMotte, *Admissibility, unbiasedness, and nonnegativity in the balanced, random, one-way anova model*, Linear statistical inference (Poznań, 1984), Lecture Notes in Statist. **35** (1985), 184–199.
- [6] K. Neumann and S. Zontek, *On geometry of the set of admissible invariant quadratic estimators in balanced two variance components model*, Statistical Papers **45** (2004), 67–80.
- [7] A.L. Rukhin, *Quadratic estimators of quadratic functions of normal parameters*, J. Statist. Plann. Inference **15** (1987), 301–310.

- [8] A.L. Rukhin, *Admissible polynomial estimates for quadratic polynomials of normal parameters* (in russian), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. **184**, Issued. Mat. Statist. **9** (1990), 234–247.
- [9] R. Zmyślony, *Quadratic admissible estimators*, (in polish) Roczniki Polskiego Towarzystwa Matematycznego, Seria III: Matematyka Stosowana VII, (1976), 117–122.
- [10] S. Zontek, *Admissibility of limits of the unique locally best linear estimators with application to variance components models*, Probab. Math. Statist. **9, 2** (1988), 29–44.

Received 10 November 2005

Revised 20 October 2006