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SET-VALUED STRATONOVICH INTEGRAL

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Abstract

The purpose of the paper is to introduce *a set-valued Stratonovich integral* driven by a one-dimensional Brownian motion. We discuss the existence of this integral and investigate its properties.

Keywords: set-valued function, Hukuhara differential, selection of a set-valued map, semimartingale, Stratonovich integral.

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1. Introduction

In investigating some effects of real systems that are being researched in modern sciences, mathematical models described by stochastic differential equations often bear fruit. One of the methods of solving stochastic differential equations follows from E. Wong and M. Zakai. Using this method we approximate an Itô stochastic equation by ordinary differential equations. E. Wong and M. Zakai proved that solutions of such differential equations do not converge to a solution of an Itô stochastic equation, but to a Stratonovich one (see [14]). The most general definition of a single-valued Stratonovich integral and one-dimensional Stratonovich differential equation can be found in [13].

A lot of problems of controlled dynamic systems can be described using the set-valued analysis methods (see e.g., [2, 7]). Some of such problems depend on random parameters. This leads to a replacement of a stochastic Itô equation by a stochastic inclusion (see e.g., [1, 9, 10]). According to our research, the stochastic set-valued problems investigated till now describe only Itô type integrals (including martingale and semimartingale set-valued ones). The attempt to apply the Wong-Zakai method to stochastic control problems leads in a natural way to a set-valued Stratonovich integral, which has never been considered before. The aim of this paper is to introduce the notion of a set-valued Stratonovich integral driven by a one-dimensional Brownian motion and investigate its properties. These properties allow us to research the Stratonovich stochastic inclusion which will be discussed in the next paper. To define properly a set-valued Stratonovich integral we need some kind of differentiability of set-valued operators. In our consideration, the set-valued Stratonovich integral is connected with the Hukuhara derivative of set-valued functions.

2. Definitions and notation

Let E be a real Banach space and let $Cl(E)$ denote a space of all nonempty and closed subsets of E, while $Conv(E)$ denote a space of all nonempty compact and convex subsets of E.

Definition 2.1. Let $A, B \in Cl(E)$. *The Hausdorff metric* is a function

$$H(A,B) = max\{\bar{H}(A,B), \bar{H}(B,A)\},$$

where

$$\bar{H}(A,B) = \sup_{a\in A} dist(a,B) = \sup_{a\in A}\inf_{b\in B} \|a-b\|.$$

Definition 2.2. Let E be a real Banach space. Let $A, B \in Conv(E)$. The set $C := (A \div B) \in Conv(E)$ is *the Hukuhara difference* of A and B, if $A = B + C$, where " $+$ " is the algebraic sum of sets A, B.

Consider a set-valued mapping $F : R^1 \to Conv(R^1)$. We say that F admits *the Hukuhara differential* at $t_0 \in R^1$, if there exists a set denoted by $(DF)(t_0)$ such that the limits

$$\lim_{\triangle t \to 0^+} \frac{F(t_0+\triangle t) \div F(t_0)}{\triangle t}$$

and

$$\lim_{\triangle t \to 0^+} \frac{F(t_0) \div F(t_0-\triangle t)}{\triangle t}$$

exist and are equal to $(DF)(t_0)$. The limits are taken with respect to a Hausdorff metric. A set-valued function F is Hukuhara differentiable, it admits a Hukuhara differential in each $t \in R^1$.

3. Set-valued Stratonovich integral

Let $I = [0,T]$ and let $(\Omega, \mathbf{F}, (\mathbf{F}_s)_{s\in I}, P)$ be a complete filtered probability space satisfying the usual hypothesis, i.e., $(\mathbf{F}_s)_{s\in I}$ is an increasing and right continuous family of σ-subalgebras of $\mathbf{F}$ and $\mathbf{F}_0$ contains all P-null sets. Let $\mathcal{L}^2 = L^2(I\times\Omega, \mathcal{P}, \mathrm{dt}\otimes\mathrm{dP})$ be a Hilbert space. $\mathcal{P}$ is σ-algebra generated by a class of all subsets $R_+\times\Omega$ of the form $\{0\}\times A_0$ and $(s,t]\times A$, where $A_0 \in \mathbf{F}_0$ and $A \in \mathbf{F}_s$ for $s < t$ in R_+.

Let $X = (X_s)_{s\in I}$ be a real valued stochastic process on the space $(\Omega, \mathbf{F}, (\mathbf{F}_s)_{s\in I}, P)$. The process X is adapted if X_s is $\mathbf{F}_s$-measurable for each $s \in I$. A stochastic process X is called a semimartingale, if it can be expressed as a sum: $X = N + A$, where N is a local martingale while A is a cádlág, adapted with paths of a finite variation process.

Definition 3.1. A set-valued function $F : R^1 \to Conv(R^1)$ is *bounded*, if there exists a constant $M \geq 0$ such that $sup_{a\in F(t)}\|a\| < M$ for every $t \in R$. F is integrably bounded, if there exists a function $m \in L^2(R^1)$ such that $H(F(t), \{0\}) \leq m(t)$ for each $t \in R^1$.

Let

$$\mathcal{S}_F(X) = \{f \in \mathcal{L}^2 : \quad f(s,\omega) \in F(X_s) \ \ dt\otimes dP - a.e. \ \}$$

mean a family of all jointly measurable, $(\mathbf{F}_s)_{s\in I}$-adapted and $\mathcal{L}^2$-integrable selectors of $F(X_s)$. Let $W = (W_s)_{s\in I}$ be a one-dimensional $(\mathbf{F}_s)_{s\in I}$-Brownian motion ($W_0 = 0$). The set-valued Itô type stochastic integral of $F(X_s)$ with respect to W is defined in the Aumann sense

$$\int F(X_s)dW_s = \left\{ \int f(s,\omega)dW_s : \quad f \in \mathcal{S}_F(X) \right\}$$

(see [8]).
Let

$$\mathcal{S}_{DF}(X) = \{f_1 \in \mathcal{L}^2 : \quad f_1(s,\omega) \in (DF)(X_s) \;\; dt \otimes dP - a.e. \;\; \}$$

mean a family of all jointly measurable, $(\mathbf{F}_s)_{s\in I}$-adapted and $\mathcal{L}^2$-integrable selections of $(DF)(X_s)$. Let $[X,W]$ denote the quadratic covariation process for X and W (see e.g., [12]). A set-valued stochastic integral of DF with respect to $[X,W]$ we define as

$$\int (DF)(X_s)d[X,W]_s = \left\{ \int f_1(s,\omega)d[X,W]_s : \right.$$

$$\left. f_1 \in \mathcal{S}_{DF}(X) \;\text{ and } X \text{ such that } \int f_1(s,\omega)d[X,W]_s \;\text{ is well defined} \right\}.$$

Definition 3.2. By a set-valued Stratonovich stochastic integral for set-valued function F with respect to the pair (X,W) we mean the set

$$\int F(X_s) \circ dW_s := \int F(X_s)dW_s + 1/2 \int (DF)(X_s)d[X,W]_s, \tag{1}$$

assuming both integrals from the right side exist.

Remark 3.3. A set-valued function $F : R^1 \to Conv(R^1)$ is (X,W)-integrable in the sense of Stratonovich if and only if sets from the right side of the equation (1) are nonempty.

Now we discuss conditions for the existence of the set-valued Stratonovich integral with respect to the pair (X,W).

Theorem 3.4. *Let X be a continuous semimartingale. Assume that a multifunction $F : R^1 \to Conv(R^1)$ is the Hukuhara differentiable and let the Hukuhara derivative DF be locally $L^2_{loc}(R^1)$-integrably bounded. Then the set-valued Stratonovich integral $\int F(X_s) \circ dW_s$ exists.*

Proof. We have to prove the existence of selectors $f \in F(X_s)$ and $f_1 \in (DF)(X_s)$, such that $\int_0^T f(s,\omega)dW_s$ and $\int_0^T f_1(s,\omega)d[X,W]_s$ exist.

Let us note that F, being Hukuhara differentiable, should be a continuous and convex valued set-valued function. Then F admits a continuous selector f by the Michael Selection Theorem (see e.g., [3]). Since $f(X_s)$ is a continuous process then the integral $\int_0^T f(X_s)dW_s$ exists and is finite. Moreover, $f(X_s) \in F(X_s)$ and therefore $f(X_s)$ is the needed selector of $F(X_s)$.

To deduce the nonemptiness of the second component of the integral let us note that the Hukuhara derivative $DF(X)$ is a measurable set-valued function. Then by the Kuratowki and Ryll-Nardzewski Selection Theorem DF admits a measurable selection $f_1 \in DF$ (see e.g., [7]). From the local $L^2_{loc}(R^1)$-integral boundedness of DF it follows that $f_1 \in L^2_{loc}(R^1)$. We show that the integral $\int_0^T f_1(X_s)d[X,W]_s$ exists.

By the Kunita-Watanabe inequality ([12]) we obtain

$$\left|\int_0^T f_1(X_s)d[X,W]_s\right| \leq \left(\int_0^T (f_1(X_s))^2 d[X,X]_s\right)^{\frac{1}{2}} \left(\int_0^T d[W,W]_s\right)^{\frac{1}{2}}$$

$$= \left(\int_0^T (f_1(X_s))^2 d[X,X]_s\right)^{\frac{1}{2}} (T)^{\frac{1}{2}}.$$

By the usual time-occupation formula and the continuity of the process X (see [12] Corollary 1 to Theorem IV.51) we have

$$\left|\int_0^T f_1(X_s)d[X,W]_s\right| \leq \left(\int_R (f_1(a))^2 L^a_T(X)da\right)^{\frac{1}{2}} (T)^{\frac{1}{2}},$$

where $L^a_T(X)$ is a local time of X. From the definition of the local time of a continuous semimartingale and its properties (see [12] Theorem IV.50 and Corollary 3 to Theorem IV.56) we deduce that for every $a > X^*_T = \sup_{s\leq T}|X_s|$, $L^a_T(X) = 0$, and $\sup_a \quad L^a_T(X) < \infty$ a.s.

Then

$$\left|\int_0^T f_1(X_s)d[X,W]_s\right| \le \left(\sup_a \ L_T^a(X)\ \ T\right)^{\frac{1}{2}} \left(\int_{-X_T^*}^{X_T^*} (f_1(a))^2 da\right)^{\frac{1}{2}} < \infty \ \ a.s.$$

This means that $\int_0^T F(X_s) \circ dW_s$ is a nonempty set.

4. Properties of set-valued Stratonovich integral

Let S^1 denote a Banach space of all $(\mathbf{F}_s)_{s\in I}$-adapted and cádlág processes $(X_s)_{s\in I}$, such that $\|X\|_{S^1} < \infty$, where $\|X\|_{S^1} = \|sup_{s\in I}|X_s|\|_{L^1}$, with $L^1 = L^1(\Omega, R^1)$.

Remark 4.1. Let a stochastic process X be any solution to a one-dimensional Stratonovich stochastic equation

$$X_t = x_0 + \int_0^t f(X_s) \circ dW_s.$$

Then X should be a continuous semimartingale of the following form

$$X_t = x_0 + \int_0^t a(s,\omega)dW_s + V_t$$

with a being an $(\mathbf{F}_s)_{s\in I}$-adapted and $\mathcal{L}^2$-integrable process and V being some adapted and continuous process having a path of finite variation (FV-process). For this reason it is quite natural to consider properties of the set-valued Stratonovich integral $\int F(X_s) \circ dW_s$ for the case of processes X of the above form.

Proposition 4.2. *Let $F : R^1 \to Conv(R^1)$ be the Hukuhara differentiable set-valued function, with Hukuhara derivative DF. Let a be an $(\mathbf{F}_s)_{s\in I}$-adapted and $\mathcal{L}^2$-integrable process and let V be an FV-process. Assume $X = \int a(s,\omega)dW_s + V_s$. If $F(X_s)$ and $DF(X_s)$ are $L^2(I\times\Omega)$-integrably bounded, then the set-valued Stratonovich integral $\int F(X_s) \circ dW_s$ is a bounded set in S^1.*

Proof.

$$
\begin{aligned}
&\left\|\int F(X_s)\circ dW_s\right\|_{S^1}\\
&= \left\|\int F(X_s)dW_s + 1/2\int (DF)(X_s)d[X,W]_s\right\|_{S^1}\\
&\leq \left\|\int F(X_s)dW_s\right\|_{S^1} + 1/2\left\|\int (DF)(X_s)a(s,\omega)d[W,W]_s\right\|_{S^1}\\
&+1/2\left\|\int (DF)(X_s)d[V,W]_s\right\|_{S^1}\\
&= \sup_{f\in\mathcal{S}_F(X)}\left\|\int f(s,\omega)dW_s\right\|_{S^1}\\
&+1/2\sup_{f_1\in\mathcal{S}_{DF}(X)}\left\|\int f_1(s,\omega)a(s,\omega)ds\right\|_{S^1}.
\end{aligned}
\tag{2}
$$

The third component vanished because V is an FV-process and W is a continuous semimartingale (see e.g., [12] Theorem II.26 and II.28).

We obtain

$$
\begin{aligned}
&\left\|\int F(X_s)\circ dW_s\right\|_{S^1} \leq \sup_{f\in S_F(X)}\left\|\sup_{0\leq t\leq T}\left|\int_0^t f(s,\omega)dW_s\right|\ \right\|_{L^1}\\
&+1/2\sup_{f_1\,\in\, S_{DF}(X)}\left\|\sup_{0\leq t\leq T}\left|\int_0^t f_1(s,\omega)a(s,\omega)ds\right|\ \right\|_{L^1}.
\end{aligned}
\tag{3}
$$

Using Doob and Hölder inequalities together with an Itô isometry we get

$$\begin{aligned}
&\Big\| \int F(X_s) \circ dW_s \Big\|_{S^1} \\
&\le \sup_{f \in S_F(X)} \|f\|_{L^2(I\times\Omega)} + 1/2 \sup_{f_1 \in S_{DF}(X)} \|f_1\|_{L^2(I\times\Omega)} \cdot \|a\|_{L^2(I\times\Omega)} \qquad (4)\\
&= \|F(X_s)\| + 1/2 \|DF(X_s)\| \cdot \|a\|
\end{aligned}$$

and therefore, $\int F(X_s) \circ dW_s$ is a bounded set in S^1.

Remark 4.3. Let us notice that if the assumptions of Proposition 4.2 are satisfied, then the signed set-valued integral $\int_0^t F(X_s) \circ dW_s$ is a conditionally weakly compact set in $L^2(\Omega)$.

Lemma 4.4. *Let Y denote the space of all Hukuhara differentiable set-valued functions acting from R^1 into $Conv(R^n)$. Then the operator $D_H(\cdot)$ is linear on the space Y.*

Proof. The proof follows from the definition of the Hukuhara difference and properties of the Hausdorff metric. Indeed, let Z_1 and Z_2 denote the following sets:

$$Z_1 = h^{-1}([F(t_0 + h) + G(t_0 + h)] \div [F(t_0) + G(t_0)]),$$

$$Z_2 = h^{-1}([F(t_0) + G(t_0)] \div [F(t_0 - h) + G(t_0 - h)]).$$

To prove the additivity of the Hukuhara derivative we need to show that the Hausdorff distance of sets Z_1 and Z_2 from the set $((DF)(t_0) + (DG)(t_0))$ tends to zero with $h \to 0^+$. To this end, we use the equality

$$\begin{aligned}
&[F(t_0 + h) + G(t_0 + h)] \div [F(t_0) + G(t_0)] \\
&= [F(t_0 + h) \div F(t_0)] + [G(t_0 + h) \div G(t_0)], \qquad (5)
\end{aligned}$$

which is easily deduced from the definition of the Hukuhara difference.

Using the above equality together with the property of a Hausdorff metric $H(A+B, C+D) \leq H(A,C) + H(B,D)$ (see e.g., Proposition 1.3.3 [11]) we get

$$
\begin{aligned}
& H(\ Z_1,\ (DF)(t_0) + (DG)(t_0)\ ) \\
& \leq H(\ h^{-1}[F(t_0+h) \div F(t_0)],\ (DF)(t_0)\ ) \qquad (6) \\
& + H(\ h^{-1}[G(t_0+h) \div G(t_0)],\ (DG)(t_0)\ ),
\end{aligned}
$$

which tends to zero with $h \to 0^+$.

Similarly, we prove that

$$\lim_{h \to 0^+} H(\ Z_2,\ (DF)(t_0) + (DG)(t_0)\ ) = 0.$$

The homogeneity of the Hukuhara derivative follows in a similar manner, because of positive homogeneity of the Hausdorff metric H.

Other properties and applications of the Hukuhara differentiable set-valued functions can be found in [4, 6, 11].

Theorem 4.5. *Let $F, G : R^1 \to Conv(R^1)$ be Hukuhara differentiable set-valued functions, with Hukuhara derivatives DF and DG. Let a be an $(\mathbf{F}_s)_{s \in I}$-adapted and $\mathcal{L}^2$-integrable process and let V be an FV-process. Assume $X = \int a(s,\omega) dW_s + V_s$. If F, G, DF and DG are $L^2_{loc}(R^1)$-integrably bounded, then the set-valued Stratonovich integral is linear with respect to the S^1 closure. It means that*

$$
\begin{aligned}
& cl_{S^1}\left(\int (\alpha F(X_s) + \beta G(X_s)) \circ dW_s \right) \\
& = cl_{S^1}\left(\alpha \int F(X_s) \circ dW_s + \beta \int G(X_s) \circ dW_s \right)
\end{aligned}
$$

for every real constant α and β.

Proof. First we prove that

$$\begin{aligned}&\int (F(X_s)+G(X_s))\circ dW_s \\ &\subseteq cl_{S^1}\left(\int F(X_s)\circ dW_s+\int G(X_s)\circ dW_s\right).\end{aligned} \tag{7}$$

Let us denote $H(X_s)=F(X_s)+G(X_s)$. Since $F(X_s)$ and $G(X_s)$ are compact sets, the algebraic sum of compact sets is a compact set, then, $H\in Conv(R)$, that is $clH(X_s)=H(X_s)$. Let z be an arbitrary element of the set $\int H(X_s)\circ dW_s$. Then

$$z=\int h(s,\omega)dW_s+\frac{1}{2}\int h_1(s,\omega)a(s,w)ds, \tag{8}$$

where $h\in\mathcal{S}_H(X)$ and $h_1\in\mathcal{S}_{DH}(X)$.

Since $h(s,\omega)\in H(X_s)$, $h_1(s,\omega)\in(DH)(X_s)$, $H(X_s)=F(X_s)+G(X_s)$ and $(DH)(X_s)=(DF)(X_s)+(DG)(X_s)$, then by Theorem 1.4 of [5] we obtain:

$$\mathcal{S}_H(X)=cl_{L^2(I\times\Omega)}(\mathcal{S}_F(X)+\mathcal{S}_G(X)) \tag{9}$$

and

$$\mathcal{S}_{DH}(X)=cl_{L^2(I\times\Omega)}(\mathcal{S}_{DF}(X)+\mathcal{S}_{DG}(X)). \tag{10}$$

This means that there exist sequences $f_n\in\mathcal{S}_F(X)$, $g_n\in\mathcal{S}_G(X)$, $f_n^1\in\mathcal{S}_{DF}(X)$, $g_n^1\in\mathcal{S}_{DG}(X)$ such that

$$\begin{cases} h=\lim_{n\to\infty}(f_n+g_n) \\ h_1=\lim_{n\to\infty}(f_n^1+g_n^1)\end{cases}, \tag{11}$$

where the limit is taken with respect to $L^2(I\times\Omega)$ norm.

Now we will prove that $z \in cl_{S^1}(\int F(X_s) \circ dW_s + \int G(X_s) \circ dW_s)$ or equivalently that

$$\lim_{n\to\infty} \Big\| \int h(s,\omega) dW_s + 1/2 \int h_1(s,\omega) a(s,w) ds$$

$$- \Big(\int f_n(s,\omega) dW_s + 1/2 \int f_n^1(s,\omega) a(s,w) ds \tag{12}$$

$$+ \int g_n(s,\omega) dW_s + \frac{1}{2} \int g_n^1(s,\omega) a(s,w) ds \Big) \ \Big\|_{S^1} = 0.$$

We obtain

$$\Big\| \int h(s,\omega) - (f_n(s,\omega) + g_n(s,\omega)) dW_s$$

$$+1/2 \int (h_1(s,\omega) - (f_n^1(s,\omega) + g_n^1(s,\omega))) a(s,w) ds \ \Big\|_{S^1}$$

$$\leq \Big\| \sup_{0\leq t\leq T} \Big| \int_0^t h(s,\omega) - (f_n(s,\omega) + g_n(s,\omega)) dW_s \Big| \ \Big\|_{L^1(\Omega)}$$

$$+1/2 \Big\| \sup_{0\leq t\leq T} \Big| \int_0^t (h_1(s,\omega) - (f_n^1(s,\omega) + g_n^1(s,\omega))) a(s,w) ds \Big| \Big\|_{L^1(\Omega)} \tag{13}$$

$$\leq \Big(4E \Big| \int_0^T (h(s,\omega) - f_n(s,\omega) - g_n(s,\omega)) dW_s \Big|^2 \Big)^{1/2}$$

$$+1/2 \int_{I\times\Omega} \Big| (h_1(s,\omega) - (f_n^1(s,\omega) + g_n^1(s,\omega))) a(s,w) \Big| ds \otimes dP$$

$$\leq 2 \Big\| h - f_n - g_n \Big\|_{L^2(I\times\Omega)} + \frac{1}{2} \Big\| h_1 - f_n^1 - g_n^1 \Big\|_{L^2(I\times\Omega)} \cdot \Big\| a \Big\|_{L^2(I\times\Omega)},$$

which tends to zero together with $n \to \infty$ because of (11).

Therefore

$$z \in cl_{S^1}\left(\int F(X_s)\circ dW_s + \int G(X_s)\circ dW_s\right). \tag{14}$$

To complete the proof, it remains to observe that

$$\int F(X_s)\circ dW_s + \int G(X_s)\circ dW_s \subseteq \int (F(X_s)+G(X_s))\circ dW_s. \tag{15}$$

Let us take an arbitrary element $z \in \int F(X_s)\circ dW_s + \int G(X_s)\circ dW_s$. Then there exist selections $f \in \mathcal{S}_F(X),\ g \in \mathcal{S}_G(X),\ f_1 \in \mathcal{S}_{DF}(X),\ g_1 \in \mathcal{S}_{DG}(X)$ such that

$$z = \int f(s,\omega)dW_s + 1/2\int f_1(s,\omega)a(s,\omega)ds$$

$$+\int g(s,\omega)dW_s + 1/2\int g_1(s,\omega)a(s,\omega)ds.$$

Let $h = f+g$ and $h_1 = f_1+g_1$. Using again Theorem 1.4 from [5] together with Lemma 4.4 we obtain

$$\begin{aligned} h = f+g \in \mathcal{S}_F(X)+\mathcal{S}_G(X) &\subseteq cl_{L^2(I\times\Omega)}(\mathcal{S}_F(X)+\mathcal{S}_G(X)) \\ &= \mathcal{S}_{cl(F+G)}(X) = \mathcal{S}_{F+G}(X). \end{aligned} \tag{16}$$

Similarly, we deduce $h_1 = f_1 + g_1 \in \mathcal{S}_{D(F+G)}(X)$ and therefore, $z \in \int (F+G)(X_s)\circ dW_s$.

Now we prove the homogeneity of the set-valued Stratonovich integral.

Let $z \in \lambda \int F(X_s)\circ dW_s$. Then

$$z = \lambda\left(\int f(s,\omega)dW_s + \frac{1}{2}\int f_1(s,\omega)a(s,\omega)ds\right), \tag{17}$$

where $f \in \mathcal{S}_F(X)$ and $f_1 \in \mathcal{S}_{DF}(X)$. If we introduce $g := \lambda f$ and $g_1 := \lambda f_1$, then $g \in \mathcal{S}_{\lambda F}(X)$ and $g_1 \in \mathcal{S}_{D(\lambda F)}(X) = \mathcal{S}_{\lambda DF}(X)$ and therefore, $z \in \int \lambda F(X_s) \circ dW_s$. The opposite inclusion we deduce in a similar way.

Corollary 4.6. *Under the assumption of Theorem* 4.5 *the set-valued integral* $\int F(X_s) \circ dW_s$ *is a convex set.*

Proof. Let us take two arbitrary elements a and b from the set $\int F(X_s) \circ dW_s$. Then, there exist functions $f, g \in \mathcal{S}_F(X)$ and $f_1, g_1 \in \mathcal{S}_{DF}(X)$ such that

$$\begin{aligned} a &= \int f(s,\omega) dW_s + 1/2 \int f_1(s,\omega) a(s,\omega) ds, \\ b &= \int g(s,\omega) dW_s + 1/2 \int g_1(s,\omega) a(s,\omega) ds. \end{aligned} \tag{18}$$

Let us take an arbitrary $\lambda \in [0,1]$. Since set-valued functions F and DF have compact and convex values, then from Theorem 1.5 of [5] we deduce

$$h = \lambda f + (1-\lambda) g \in \overline{co}\mathcal{S}_F(X) = \mathcal{S}_{\overline{co}F}(X) = \mathcal{S}_F(X). \tag{19}$$

In a similar way

$$h_1 = \lambda f_1 + (1-\lambda) g_1 \in \mathcal{S}_{DF}(X). \tag{20}$$

Therefore $\lambda a + (1-\lambda) b$ belongs to the set $\int F(X_s) \circ dW_s$.

For the next properties we recall the notion of the semimartingale space $\mathcal{H}^2$. We assume that a semimartingale X has a decomposition $X = N + A$, where N is a local martingale, and A is a predictable, cádlág, adapted process with paths of finite variation. The space $\mathcal{H}^2$ consists of all semimartingales with a finite $\mathcal{H}^2$ norm:

$$\left\| X \right\|_{\mathcal{H}^2} = \left\| [N,N]_T^{1/2} \right\|_{L^2(\Omega)} + \left\| \int_0^T \left| dA_t \right| \right\|_{L^2(\Omega)}.$$

Theorem 4.7. *Assume that a set-valued function $F : R^1 \to Conv(R^1)$ is Hukuhara differentiable, and its Hukuhara derivative DF is $L^2_{loc}(R^1)$-integrably bounded. Let $g : R^1 \to R^1$ be an absolutly continuous function with $L^2_{loc}(R^1)$-integrably bounded derivative g'. Let $X = \int a(s,\omega)dW_s + V_s$, where a is an $(\mathbf{F}_s)_{s\in I}$-adapted and $\mathcal{L}^2$-integrable process and V is an FV-process. Then*

$$dist^2_{\mathcal{H}^2}\left(\int g(X_s)\circ dW_s, \int F(X_s)\circ dW_s\right)$$

$$\leq 2\sqrt{T}\left\|\int dist^2_R(g(X_s),F(X_s))dW_s\right\|_{\mathcal{H}^2}$$

$$+1/2\cdot T\left\|\int dist^2_R(g'(X_s),(DF)(X_s))\Big|a(s,w)\Big|^2 ds\right\|_{\mathcal{H}^2}.$$

Proof. Let f and f_1 denote arbitrary elements from the sets $\mathcal{S}_F(X)$ and $\mathcal{S}_{DF}(X)$ respectively.

$$dist^2_{\mathcal{H}^2}\left(\int g(X_s)\circ dW_s, \int F(X_s)\circ dW_s\right)$$

$$= \inf_{f,f_1}\Bigg\|\int g(X_s)dW_s + 1/2\int g'(X_s)a(s,w)ds$$

$$-\int f(s,w)dW_s - 1/2\int f_1(s,w)a(s,w)ds\Bigg\|^2_{\mathcal{H}^2}$$

$$\leq 2\left(\inf_f\left\|\int g(X_s)-f(s,w)dW_s\right\|^2_{\mathcal{H}^2}\right. \tag{21}$$

$$\left.+\inf_{f_1}\left\|1/2\int g'(X_s)a(s,w)-f_1(s,w)a(s,w)ds\right\|^2_{\mathcal{H}^2}\right)$$

$$=2\left(\inf_f E\left(\int_0^T\Big|g(X_s)-f(s,w)\Big|^2 ds\right)\right.$$

$$\left.+1/4\inf_{f_1} E\left(\int_0^T\Big|(g'(X_s)-f_1(s,w))a(s,w)\Big|ds\right)^2\right).$$

Applying Hölder's inequality to the second term we obtain

$$
\begin{aligned}
& dist^2_{\mathcal{H}^2}\left(\int g(X_s)\circ dW_s, \int F(X_s)\circ dW_s\right) \\
(22) \qquad & \leq 2E\int_0^T dist^2_R(g(X_s),F(X_s))ds \\
& +1/2\cdot TE\int_0^T dist^2_R(g'(X_s)a(s,w),(DF)(X_s)a(s,w))ds.
\end{aligned}
$$

Let us note that the first term of the inequality (22) can be estimated as follows

$$
\begin{aligned}
& \left\|\int_0^T dist^2_R(g(X_s),F(X_s))ds\right\|_{L^1(\Omega)} \\
(23) \qquad & \leq \left\|\int_0^T dist^2_R(g(X_s),F(X_s))ds\right\|_{L^2(\Omega)} \\
& \leq \sqrt{T}\left\|\int dist^2_R(g(X_s),F(X_s))dW_s\right\|_{\mathcal{H}^2}.
\end{aligned}
$$

Then we obtain:

$$
\begin{aligned}
& dist^2_{\mathcal{H}^2}\left(\int g(X_s)\circ dW_s, \int F(X_s)\circ dW_s\right) \\
(24) \qquad & \leq 2\sqrt{T}\left\|\int dist^2_R(g(X_s),F(X_s))dW_s\right\|_{\mathcal{H}^2} \\
& +1/2\cdot T\left\|\int dist^2_R(g'(X_s),(DF)(X_s))\big|a(s,w)\big|^2 ds\right\|_{\mathcal{H}^2}.
\end{aligned}
$$

Theorem 4.8. *Let $F, G : R^1 \to Conv(R^1)$ be Hukuhara differentiable set-valued functions, with Hukuhara derivatives DF and DG. Let a be an $(\mathbf{F}_s)_{s\in I}$-adapted and $\mathcal{L}^2$-integrable process and let V be an FV-process.*

Assume $X = \int a(s,\omega)dW_s + V_s$. *If* F, G, DF *and* DG *are* $L^2_{loc}(R^1)$-*integrably bounded, then the Hausdorff distance of set-valued integrals satisfies the inequality*

$$H^2_{\mathcal{H}^2}\left(\int F(X_s)\circ dW_s, \int G(X_s)\circ dW_s\right)$$

$$\leq 2\sqrt{T}\left\|\int H^2_R(F(X_s),G(X_s))dW_s\right\|_{\mathcal{H}^2}$$

$$+1/2\cdot T\left\|\int \left|a(s,w)\right|^2 H^2_R((DF)(X_s),(DG)(X_s))ds\right\|_{\mathcal{H}^2}.$$

Proof. Let f and f_1 denote arbitrary elements of the sets $\mathcal{S}_F(X)$ and $\mathcal{S}_{DF}(X)$ respectively. Let $\overline{H}$ denote Hausdorff submetric in $\mathcal{H}^2$. By Theorem 4.7 we obtain

$$\overline{H}^2\left(\int F(X_s)\circ dW_s, \int G(X_s)\circ dW_s\right)$$

$$= \sup_{z\in\int F(X_s)\circ dW_s} dist^2_{\mathcal{H}^2}\left(z, \int G(X_s)\circ dW_s\right)$$

$$= \sup_{f,f_1} dist^2_{\mathcal{H}^2}\left(\int f(s,w)dW_s + 1/2\int f_1(s,w)a(s,w)ds,\right.$$

(25)

$$\left.\int G(X_s)dW_s + 1/2\int (DG)(X_s)a(s,w)ds\right)$$

$$\leq 2\sup_f E\int_0^T dist^2_R(f(s,w),G(X_s))ds$$

$$+1/2T\sup_{f_1} E\int_0^T \left|a(s,w)\right|^2 dist^2_R(f_1(s,w),(DG)(X_s))ds.$$

Using Theorem 2.2 from [5] we obtain

$$
\begin{aligned}
&\overline{H}^2\left(\int F(X_s)\circ dW_s, \int G(X_s)\circ dW_s\right)\\
&\le 2E\int_0^T \sup_{x\in F(X_s)} dist_R^2(x, G(X_s))ds\\
&+1/2TE\int_0^T \Big|a(s,w)\Big|^2 \sup_{y\in(DF)(X_s)} dist_R^2(y,(DG)(X_s))ds \qquad (26)\\
&\le 2E\int_0^T H_R^2(F(X_s), G(X_s))ds\\
&+1/2TE\int_0^T \Big|a(s,w)\Big|^2 H_R^2((DF)(X_s),(DG)(X_s))ds.
\end{aligned}
$$

Rearranging the first term of the inequality (26) and applying Hölder's inequality we get

$$
\begin{aligned}
&\overline{H}^2\left(\int F(X_s)\circ dW_s, \int G(X_s)\circ dW_s\right)\\
&\le 2\sqrt{T}\left\|\int H_R^2(F(X_s), G(X_s))dW_s\right\|_{\mathcal{H}^2} \qquad (27)\\
&+1/2\cdot T\left\|\int \Big|a(s,w)\Big|^2 H_R^2((DF)(X_s),(DG)(X_s))ds\right\|_{\mathcal{H}^2}.
\end{aligned}
$$

A similar inequality we obtain for submetric $\overline{H}^2(\int G(X_s)\circ dW_s, \int F(X_s)\circ dW_s)$ which completes the proof.

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