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NON-CENTRAL GENERALIZED *F* DISTRIBUTIONS

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Abstract

The quotient of two linear combinations of independent chi-squares will have a generalized F distribution. Exact expressions for these distributions when the chi-square are central and those in the numerator or in the denominator have even degrees of freedom were given in Fonseca *et al.* (2002). These expressions are now extended for non-central chi-squares. The case of random non-centrality parameters is also considered.

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1. Introduction

Quotients of linear combinations of chi-squares have relevant applications. For instance, the statistics of the generalized F tests are such quotients.

These tests were introduced by Michalski & Zmyślony (1996) and (1999), first for variance components and later for linear combinations of parameters in mixed linear models.

When the chi-squares in the numerator or in the denominator are even and all coefficients are non-negative an exact expression was obtained, see Fonseca *et al.* (2002), for the distribution of these quotients when the chi-squares are central. We now intend to extend that result to the non-central case. In carrying out this extension we will first use the Robbins (1948) and Robbins & Pitman (1949) method mixtures for fixed non-centrality parameters. Nextly we will consider randomization of the non-centrality parameters and obtain the corresponding extension.

Thus, our aim is mainly theoretical. We must point out that if practical applications are the main goal, an excellent alternative for our treatment is given by Imhof (1961). The algoritm presented by Davies (1980) could also be used. In this way the shortcomings of previous approaches such as the one given by Satterthwaite (1946) and Gaylor & Hopper (1969) may be overcame.

2. Generalized F and related distributions

In this section for vectors m^{r+s} we consider upper sub-vectors m_1^r and lower sub-vectors m_2^s, with $m_1, ..., m_{r+s}$ the components of m^{r+s} and $m_{1,1}, ..., m_{1,r}$ $[m_{2,1}, ..., m_{2,s}]$ the components of m_1^r $[m_2^s]$. Moreover, $U \sim \sigma^2\chi^2_{g,\delta}$ indicates the product by σ^2 of a chi-square with g degrees of freedom and non-centrality parameter δ. When $\delta = 0$ we write simply $U \sim \sigma^2\chi^2_g$.

Given the vectors a_1^r and a_2^s, with non-negative components and at least one non null, and the independent random variables $U_i \sim \chi^2_{g_{1,i}}$, $i = 1, ..., r$, and $V_j \sim \chi^2_{g_{2,j}}$, $j = 1, ..., s$, the distribution of

$$\frac{\sum_{i=1}^{r} a_{1,i}U_i}{\sum_{j=1}^{s} a_{2,j}V_j}$$

will be $F^+(z|a_1^r, a_2^s, g_1^r, g_2^s)$. Since

$$\frac{\sum_{i=1}^{r} a_{1,i}U_i}{\sum_{j=1}^{s} a_{2,j}V_j} = \frac{\sum_{i=1}^{r} a_{1,i}}{\sum_{j=1}^{s} a_{2,j}} \frac{\sum_{i=1}^{r} c_{1,i}U_i}{\sum_{j=1}^{s} c_{2,j}V_j}, \tag{2.1}$$

where

$$c_{1,i} = \frac{a_{1,i}}{\sum_{i'=1}^{r} a_{1,i'}}, \quad i = 1, ..., r$$

and

$$c_{2,j} = \frac{a_{2,j}}{\sum_{j'=1}^{s} a_{2,j'}}, \quad j = 1, ..., s,$$

with

$$k = \frac{\sum_{i=1}^{r} a_{1,i}}{\sum_{j=1}^{s} a_{2,j}}$$

we will have

$$F^+(z|a_1^r, a_2^s, g_1^r, g_2^s) = F^+\left(\frac{z}{k}|c_1^r, c_2^s, g_1^r, g_2^s\right), \tag{2.2}$$

which show that we can consider only, if it is convenient, quotients of convex combinations of central chi-squares.

Let $(v^m)^{-1}$ be the vector whose components are the inverses of the components of v^m. The generalized central F distribution will be

$$F(z|g_1^r, g_2^s) = F^+(z|(g_1^r)^{-1}, (g_2^s)^{-1}, g_1^r, g_2^s).$$

Another interesting case of $F^+(z|a_1^r, a_2^s, g_1^r, g_2^s)$ will be

$$\overline{F}(z|g_1^r, g_2^s) = F^+(z|(1^r, 1^s, g_1^r, g_2^s).$$

If $r = s = 1$, in the first case we will have the usual central F distribution while for the second case we will have the $\overline{F}$ distribution, defined for the quotient of independent chi-squares with g_1 and g_2 degrees of freedom.

In Fonseca *et al.* (2002) the exact expressions of $F^+(z|a_1^r, a_2^s, g_1^r, g_2^s)$ is given when the degrees of freedom in the numerator or in the denominator are even. Moreover, the second case reduces to the first one since

$$F^+\Big(z|a_1^r, a_2^s, g_1^r, 2m^s\Big) = 1 - F^+\Big(z^{-1}|a_2^s, a_1^r, 2m^s, g_1^r\Big). \tag{2.3}$$

To simplify the notation we represent by $a_1, ..., a_r$ $[g_1, ..., g_r]$ the components of $a_1^r[g_1^r]$ and by $a_{r+1}, ..., a_{r+s}$ $[g_{r+1}, ..., g_{r+s}]$ the components of $a_2^s[g_2^s]$. Let us remember that the $X_i \sim a_i\chi^2_{g_i}$, $i = 1, ..., r+s$, if $g_1^r = 2m^r$, have densities

$$\begin{cases} f_i(x) = \dfrac{x^{m_i-1}}{(m_i-1)!(2a_i)^{m_i}} e^{-\frac{x}{2a_i}};\ x > 0; i = 1, ..., r \\ \\ f_i(x) = \dfrac{x^{\frac{g_i}{2}-1}}{\Gamma(\frac{g_i}{2})(2a_i)^{\frac{g_i}{2}}} e^{-\frac{x}{2a_i}};\ x > 0; i = r+1, ..., r+s \end{cases}, \tag{2.4}$$

so that we have

$$F^+\Big(z|a_1^r, a_2^s, 2m^r, g_2^s\Big) = pr\left(\sum_{i=1}^{r} X_i \le z\sum_{i=r+1}^{r+s} X_i\right)$$

$$= \int_0^{+\infty} ... \int_0^{+\infty} \int_0^{z\sum_{i=r+1}^{r+s} x_i} ... \left(\int_0^{z\sum_{i=r+1}^{r+s} x_i - \sum_{i=2}^{r} x_i} f_1(x_1)dx_1\right) \tag{2.5}$$

$$...f_r(x_r)dx_r...f_{r+s}(x_{r+s})dx_{r+s}.$$

Putting $y = z \sum_{i=r+1}^{r+s} x_i$ as well as $w_i = m_i - 1$, $i = 1, ..., r$, we get

$$\int_0^{y-\sum_{i=2}^{r} x_i} f_1(x_1)dx_1 = \int_0^{y-\sum_{i=2}^{r} x_i} \frac{x_1^{w_1}}{w_1!(2a_1)^{w_1+1}} e^{-\frac{x_1}{2a_1}} dx_1$$

$$(2.6)\quad \begin{aligned} &= \sum_{k_1=0}^{1} (-1)^{k_1} e^{-\frac{k_1}{2a_1}\left(y-\sum_{i=2}^{r} x_i\right)} \sum_{j_1=0}^{k_1 w_1} \frac{\left(y-\sum_{i=2}^{r} x_i\right)^{j_1}}{(2a_1)^{j_1} j_1!} \\ &= \sum_{k_1=0}^{1} (-1)^{k_1} e^{-\frac{k_1}{2a_1}\left(y-\sum_{i=2}^{r} x_i\right)} \sum_{j_1=0}^{k_1 w_1} \frac{1}{(2a_1)^{j_1}} \\ &\times \sum_{\left(\sum_{i=1}^{r} t_{1,i} = j_1\right)} (-1)^{j_1 - t_{1,1}} \frac{y^{t_{1,1}}}{t_{1,1}!} \prod_{i=2}^{r} \frac{x_i^{t_{1,i}}}{t_{1,i}!}. \end{aligned}$$

In the last step we used

$$(2.7)\quad \left(y - \sum_{i=2}^{r} x_i\right)^{j_1} = \sum_{\left(\sum_{i=1}^{r} t_{1,i} = j_1\right)} \frac{j_1!}{\prod_{i=1}^{r} t_{1,i}!} y^{t_{1,1}} \prod_{i=2}^{r} (-x_i)^{t_{1,i}}$$

as well as $\sum_{i=2}^{r} t_{1,i} = j_1 - t_{1,1}$, to obtain the power of coefficient -1. Moreover, with $k^n = (k_1, ..., k_n)$ and ℓ_n the largest index for non-null components of k^n, we take $d(k^n) = (2a_{\ell_n})^{-1}$, $n = 2, ...$ Besides this,

$$(2.8)\quad d(k^n) + k_{n+1}\left(\frac{1}{2a_{n+1}} - d(k^n)\right) = d(k^{n+1}).$$

So, we have

$$\int_0^y \dots \int_0^{y-\sum_{i=2}^{r} x_i} \prod_{v=1}^{r} f_v(x_v) \prod_{v=1}^{r} dx_v$$

$$= \frac{1}{\prod_{v=2}^{r}(2a_v)^{w_v+1}} \sum_{k_1=0}^{1} \sum_{j_1=0}^{k_1 w_1^+} \sum_{\left(\sum_{i=1}^{r+1-1} t_{1,i} = j_1\right)} \dots \sum_{k_r=0}^{1} \sum_{j_r=0}^{k_r w_r^+} \sum_{\left(\sum_{i=1}^{r+1-r} t_{r,i} = j_r\right)}$$

$$\times \frac{(-1)^{\sum_{i=1}^{r}(k_i+j_i-t_{i,1})}}{(2a_1)^{j_1}} \prod_{v=2}^{r} \left[\frac{w_v^+!}{w_v! \prod_{i=1}^{v-1} t_{i,v+1-i}!} \left(\frac{1}{2a_v} - d(k^{v-1}) \right)^{j_v - w_v^+ - 1} \right]$$

$$\times \frac{e^{-d(k^r)y} \; y^{\sum_{i=1}^{r} t_{i,1}}}{\prod_{i=1}^{r} t_{i,1}!},$$

(2.9)

with $w_j^+ = w_j + \sum_{u=1}^{j-1} t_{u,j+1-u}$, $j = 1, \dots$ The exact expression of $F^+(z|a_1^r, a_2^s, 2m^r, g_2^s)$, see Fonseca *et al.* (2002), will be

$$F^+(z|a_1^r, a_2^s, 2m^r, g_2^s)$$

$$= \frac{1}{\prod_{v=2}^{r}(2a_v)^{w_v+1}} \sum_{k_1=0}^{1} \sum_{j_1=0}^{k_1 w_1^+} \sum_{\left(\sum_{i=1}^{r} t_{1,i} = j_1\right)} \dots \sum_{k_r=0}^{1} \sum_{j_r=0}^{k_r w_r^+} \sum_{\left(\sum_{i=1}^{1} t_{r,i} = j_r\right)} \times$$

$$\times \frac{(-1)^{\sum_{v=1}^{r}(k_v+j_v-t_{v,1})}}{(2a_1)^{j_1}} \prod_{i=2}^{r} \left[\frac{w_i^{+}!}{w_i! \prod_{v=1}^{i-1} t_{v,i+1-v}!} \left(\frac{1}{2a_i} - d(k^{i-1}) \right)^{j_i - w_i^{+} - 1} \right]$$

$$\times \frac{\left(\sum_{i=1}^{r} t_{i,1}\right)!}{\prod_{i=1}^{r} t_{i,1}!} \sum_{\left(\sum_{i=r+1}^{r+s} h_i = \sum_{v=1}^{r} t_{v,1}\right)} \prod_{i=r+1}^{r+s} \frac{\Gamma(h_i + \frac{g_i}{2})}{h_i! \Gamma(\frac{g_i}{2})}$$

$$\times \frac{z^{\sum_{i=1}^{r} t_{i,1}}}{(2a_i)^{\frac{g_i}{2}}} \left(\frac{1}{2a_i} + d(k^r) z \right)^{-(h_i + \frac{g_i}{2})} . \tag{2.10}$$

3. NON-CENTRAL GENERALIZED F DISTRIBUTIONS

Distributions $\chi^2_{g,\delta}$ is a mixture of the χ^2_{g+2j}, $j = 0, ...$ The coefficients in this mixture are the probabilities for non-negative integers of the Poisson distribution with parameter $\frac{\delta}{2}$, $P_{\delta/2}$. Thus, if $U \sim \chi^2_{g,\delta}$, we may assume there is an indicator variable $J \sim P_{\delta/2}$ such that, when $J = \ell$, $U \sim \chi^2_{g+2\ell}$, $\ell = 0, ...$

Likewise if the $U_i \sim \chi^2_{g_{1,i},\delta_{1,i}}$, $i = 1, ..., r$, and $V_j \sim \chi^2_{g_{2,j},\delta_{2,j}}$, $j = 1, ..., s$, are independent, their joint distribution $\chi^2_{g_1^r, g_2^s, \delta_1^r, \delta_2^s} = \prod_{i=1}^{r} \chi^2_{g_{1,i},\delta_{1,i}} \prod_{j=1}^{s} \chi^2_{g_{2,j},\delta_{2,j}}$ will be a mixture with coefficients

$$c(\ell_1^r, \ell_2^s, \delta_1^r, \delta_2^s) = \prod_{i=1}^{r} e^{-\frac{\delta_{1,i}}{2}} \frac{\left(\frac{\delta_{1,i}}{2}\right)^{\ell_{1,i}}}{\ell_{1,i}!} \prod_{j=1}^{s} e^{-\frac{\delta_{2,j}}{2}} \frac{\left(\frac{\delta_{2,j}}{2}\right)^{\ell_{2,j}}}{\ell_{2,j}!} \tag{3.1}$$

of the $\chi^2_{g_1^r + 2\ell_1^r, g_2^s + 2\ell_2^s} = \prod_{i=1}^{r} \chi^2_{g_{1,i}+2\ell_{1,i}} \prod_{j=1}^{s} \chi^2_{g_{2,j}+2\ell_{2,j}}$.

Moreover, see Robbins (1948) and Robbins & Pitman (1949), using the mixtures method, the distribution of

$$Z = \frac{\sum_{i=1}^{r} a_{1,i} U_i}{\sum_{j=1}^{s} a_{2,j} V_j}$$

will be

$$\begin{aligned} & F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \delta_1^r, \delta_2^s) \\ \text{(3.2)} \quad & = \sum_{\ell_{1,1}=0}^{+\infty} \cdots \sum_{\ell_{1,r}=0}^{+\infty} \\ & \times \sum_{\ell_{2,1}=0}^{+\infty} \cdots \sum_{\ell_{2,s}=0}^{+\infty} c(\ell_1^r, \ell_2^s, \delta_1^r, \delta_2^s) F^+(z|a_1^r, a_2^s, g_1^r + 2\ell_1^r, g_2^s + 2\ell_2^s). \end{aligned}$$

If, as above, we consider indicator variables, the conditional distribution of Z, when $J_{1,i} = \ell_{1,i}$, $i = 1, ..., r$ and $J_{2,j} = \ell_{2,j}$, $j = 1, ..., s$, will be $F^+(z|a_1^r, a_2^s, g_1^r + 2\ell_1^r, g_2^s + 2\ell_2^s)$. Thus, desconditioning in order to the indicator variables, we would obtain the expression of $F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \delta_1^r, \delta_2^s)$.

We now consider monotonicity properties for these distributions. We will have, with $\delta_{1,\ell}$ the ℓ-th component of δ_1^r,

$$\begin{aligned} & \frac{\partial F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \delta_1^r, \delta_2^s)}{\partial \delta_{1,\ell}} \\ \text{(3.3)} \quad & = \frac{F^+(z|a_1^r, a_2^s, g_1^r + 2q_\ell^r, g_2^s, \delta_1^r, \delta_2^s) - F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \delta_1^r, \delta_2^s)}{2} < 0, \end{aligned}$$

where q_ℓ^r has all the components null, but the ℓ-th that is 1, as well as

$$\begin{aligned} & \frac{\partial F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \delta_1^r, \delta_2^s)}{\partial \delta_{2,h}} \\ \text{(3.4)} \quad & = \frac{F^+(z|a_1^r, a_2^s, g_1^r, g_2^s + 2q_h^s, \delta_1^r, \delta_2^s) - F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \delta_1^r, \delta_2^s)}{2} > 0. \end{aligned}$$

Nextly, with $r = 1$, $a_1 = 1$ and $a_2, ..., a_{s+1}$ $[g_2, ..., g_{s+1}]$ the components of $a_2^s[g_2^s]$, we will obtain the exact expression of $F^+(z|1, a_2^s, g_1, g_2^s, \delta)$, the distribution of

$$\frac{\chi^2_{g_1,\delta}}{\sum_{i=2}^{s+1} a_i \chi^2_{g_i}}, \tag{3.5}$$

when g_1 is even. Since

$$F^+(z|1, a_2^s, g_1, g_2^s, \delta) = e^{-\delta/2} \sum_{\ell=0}^{+\infty} \frac{(\frac{\delta}{2})^\ell}{\ell!} F^+(z|1, a_2^s, g_1 + 2\ell, g_2^s), \tag{3.6}$$

to obtain the exact expression of this distribution it is necessary to obtain the exact expression of $F^+(z|1, a_2^s, g_1 + 2\ell, g_2^s)$. So, with $g_1 = 2m$ we have

$$\begin{aligned} F^+(z|1, a_2^s, 2m, g_2^s) &= pr(X_1 \leq z \sum_{i=2}^{s+1} X_i) \\ &= \int_0^{+\infty} ... \int_0^{+\infty} \left(\int_0^{z \sum_{i=2}^{s+1} x_i} f_1(x_1) dx_1 \right) \prod_{i=2}^{s+1} f_i(x_i) \prod_{i=2}^{s+1} dx_i. \end{aligned} \tag{3.7}$$

Since

$$\begin{aligned} \int_0^{z \sum_{i=2}^{s+1} x_i} f_1(x_1) dx_1 &= \int_0^{z \sum_{i=2}^{s+1} x_i} \frac{x_1^{m-1}}{(m-1)! 2^m} e^{-\frac{x_1}{2}} dx_1 \\ &= \frac{1}{(m-1)! 2^m} \int_0^{z \sum_{i=2}^{s+1} x_i} x_1^{m-1} e^{-\frac{x_1}{2}} dx_1 \\ &= \sum_{k=0}^{1} (-1)^k e^{-\frac{k}{2} z \sum_{i=2}^{s+1} x_i} \sum_{j=0}^{k(m-1)} \frac{\left(z \sum_{i=2}^{s+1} x_i\right)^j}{2^j j!} \end{aligned} \tag{3.8}$$

and

$$\left(\sum_{i=2}^{s+1} x_i\right)^j = \sum_{\left(\sum_{i=2}^{s+1} h_i = j\right)} \frac{j!}{\prod_{i=2}^{s+1} h_i!} \prod_{i=2}^{s+1} x_i^{h_i}, \tag{3.9}$$

we get

$F^+(z|1, a_2^s, 2m, g_2^s)$

$$= \int_0^{+\infty} \dots \int_0^{+\infty} \left(\sum_{k=0}^{1} (-1)^k e^{-\frac{k}{2} z \sum_{i=2}^{s+1} x_i} \sum_{j=0}^{k(m-1)} \frac{(z \sum_{i=2}^{s+1} x_i)^j}{2^j j!} \right) \prod_{i=2}^{s+1} f_i(x_i) \prod_{i=2}^{s+1} dx_i$$

$$= \sum_{k=0}^{1} \sum_{j=0}^{k(m-1)} \frac{(-1)^k}{2^j} \int_0^{+\infty} \dots \int_0^{+\infty} \frac{e^{-\frac{k}{2} z \sum_{i=2}^{s+1} x_i} \left(z \sum_{i=2}^{s+1} x_i\right)^j}{j!} \prod_{i=2}^{s+1} f_i(x_i) \prod_{i=2}^{s+1} dx_i$$

$$= \sum_{k=0}^{1} \sum_{j=0}^{k(m-1)} \frac{(-1)^k z^j}{2^j j!} \int_0^{+\infty} \dots \int_0^{+\infty} e^{-\frac{k}{2} z \sum_{i=2}^{s+1} x_i} \left(\sum_{i=2}^{s+1} x_i\right)^j \prod_{i=2}^{s+1} f_i(x_i) \prod_{i=2}^{s+1} dx_i$$

$$= \sum_{k=0}^{1} \sum_{j=0}^{k(m-1)} \frac{(-1)^k z^j}{2^j j!} \int_0^{+\infty} \dots \int_0^{+\infty} e^{-\frac{k}{2} z \sum_{i=2}^{s+1} x_i} \sum_{\left(\sum_{i=2}^{s+1} h_i = j\right)} \frac{j!}{\prod_{i=2}^{s+1} h_i!}$$

$$\times \prod_{i=2}^{s+1} x_i^{h_i} \prod_{i=2}^{s+1} f_i(x_i) \prod_{i=2}^{s+1} dx_i.$$

(3.10)

Thus

$$F^{+}(z|1, a_2^s, 2m, g_2^s)$$

$$= \sum_{k=0}^{1} \sum_{j=0}^{k(m-1)} \sum_{\left(\sum_{i=2}^{s+1} h_i = j\right)} \frac{(-1)^k z^j}{2^j \prod_{i=2}^{s+1} h_i!} \prod_{i=2}^{s+1} \int_0^{+\infty} e^{-\frac{k}{2} z x_i} x_i^{h_i} f_i(x_i) dx_i$$

$$= \sum_{k=0}^{1} \sum_{j=0}^{k(m-1)} \sum_{\left(\sum_{i=2}^{s+1} h_i = j\right)} \frac{(-1)^k z^j}{2^j \prod_{i=2}^{s+1} h_i!} \prod_{i=2}^{s+1} \int_0^{+\infty} e^{-\frac{k}{2} z x_i} x_i^{h_i} \frac{x_i^{\frac{g_i}{2}-1} e^{-\frac{x_i}{2a_i}}}{\Gamma(\frac{g_i}{2})(2a_i)^{g_i/2}} dx_i$$

$$= \sum_{k=0}^{1} \sum_{j=0}^{k(m-1)} \sum_{\left(\sum_{i=2}^{s+1} h_i = j\right)} \frac{(-1)^k z^j}{2^j \prod_{i=2}^{s+1} h_i!} \prod_{i=2}^{s+1} \int_0^{+\infty} \frac{e^{-(\frac{kz}{2}+\frac{1}{2a_i}) x_i} x_i^{(h_i+\frac{g_i}{2}-1)}}{\Gamma(\frac{g_i}{2})(2a_i)^{g_i/2}} dx_i$$

$$= \sum_{k=0}^{1} \sum_{j=0}^{k(m-1)} \sum_{\left(\sum_{i=2}^{s+1} h_i = j\right)} \frac{(-1)^k z^j}{2^j \prod_{i=2}^{s+1} h_i!} \prod_{i=2}^{s+1} \frac{\Gamma(h_i+\frac{g_i}{2})}{\Gamma(\frac{g_i}{2})(2a_i)^{g_i/2}} \left(\frac{kz}{2}+\frac{1}{2a_i}\right)^{-(h_i+\frac{g_i}{2})}, \tag{3.11}$$

and

$$F^{+}(z|1, a_2^s, 2m, g_2^s, \delta)$$

$$= e^{-\delta/2} \sum_{\ell=0}^{+\infty} \frac{(\frac{\delta}{2})^\ell}{\ell!} F^{+}(z|1, a_2^s, 2m+2\ell, g_2^s) =$$

$$= e^{-\delta/2}\sum_{\ell=0}^{+\infty}\frac{(\frac{\delta}{2})^{\ell}}{\ell!}\sum_{k=0}^{1}\sum_{j=0}^{k(m+\ell-1)}\sum_{\left(\sum_{i=2}^{s+1}h_i=j\right)}\frac{(-1)^k z^j}{2^j\prod_{i=2}^{s+1}h_i!}$$

(3.12)

$$\times\prod_{i=2}^{s+1}\frac{\Gamma(h_i+\frac{g_i}{2})}{\Gamma(\frac{g_i}{2})(2a_i)^{g_i/2}(\frac{kz}{2}+\frac{1}{2a_i})^{(h_i+\frac{g_i}{2})}}.$$

Up to now we have considered the indicator variables $J_{1,i}$, $i = 1, ..., r$, and $J_{2,j}$, $j = 1, ..., s$, to have Poisson distributions with fixed parameters. We now assume these parameters to be random variables $L_{1,i}$, $i = 1, ..., r$ and $L_{2,j}$, $j = 1, ..., s$. With $\lambda_{L_1^r,L_2^s}(t_1^r, t_2^s)$ the joint moment generating function for these variables and

$$\lambda_{L_1^r,L_2^s}^{<\ell_1^r,\ell_2^s>}(t_1^r, t_2^s) = \frac{\partial^{\ell_{1,1}+...+\ell_{1,r}+\ell_{2,1}+...+\ell_{2,s}}\lambda(t_1^r, t_2^s)}{\prod_{i=1}^{r}\partial t_{1,i}^{\ell_{1,i}}\prod_{j=1}^{s}\partial t_{2,j}^{\ell_{2,j}}}. \tag{3.13}$$

Desconditioning

$$F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, l_1^r, l_2^s)$$

$$= \sum_{\ell_{1,1}=0}^{+\infty}...\sum_{\ell_{2,s}=0}^{+\infty}c(\ell_1^r, \ell_2^s, l_1^r, l_2^s)F^+(z|a_1^r, a_2^s, g_1^r + 2\ell_1^r, g_2^s + 2\ell_2^s) \tag{3.14}$$

in order to the random parameters vectors L_1^r and L_2^s, we get

$$\begin{aligned}
&F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r, L_2^s}) \\
&= \sum_{\ell_{1,1}=0}^{+\infty} \cdots \sum_{\ell_{2,s}=0}^{+\infty} \frac{\lambda_{L_1^r, L_2^s}^{<\ell_1^r, \ell_2^s>}\Big(-\frac{1}{2}1^r, -\frac{1}{2}1^s\Big)}{\prod_{i=1}^{r} \ell_{1,i}! 2^{\ell_{1,i}} \prod_{j=1}^{s} \ell_{2,j}! 2^{\ell_{2,j}}} \\
&F^{+}(z|a_1^r, a_2^s, g_1^r + 2\ell_1^r, g_2^s + 2\ell_2^s).
\end{aligned} \tag{3.15}$$

If $pr(L_2^s = 0^s) = 1$ $[pr(L_1^r = 0^r) = 1]$, $F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r, L_2^s})$ reduces to $F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r})$ $[F^{+}(z|a_1^r,\ a_2^s, g_1^r, g_2^s, \lambda_{L_2^s})]$ with

$$\begin{cases}
F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r}) \\
= \sum_{\ell_{1,1}=0}^{\infty} \cdots \sum_{\ell_{1,r}=0}^{\infty} \frac{\lambda_{L_1^r}^{<\ell_1^r>}(-\frac{1}{2}1^r)}{\prod_{i=1}^{r} \ell_{1,i}! 2^{\ell_{1,i}}} F^{+}(z|a_1^r, a_2^s, g_1^r + 2\ell_1^r, g_2^s) \\
F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_2^s}) \\
= \sum_{\ell_{2,1}=0}^{\infty} \cdots \sum_{\ell_{2,s}=0}^{\infty} \frac{\lambda_{L_2^s}^{<\ell_2^s>}(-\frac{1}{2}1^s)}{\prod_{j=1}^{s} \ell_{2,j}! 2^{\ell_{2,j}}} F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s + 2\ell_2^s)
\end{cases}. \tag{3.16}$$

Let q_i^r $[q_j^s]$ be the vector with all the r $[s]$ components null, except the i-th $[j$-th$]$ which is 1, then $\overline{L}_i^r = (1^r - q_i^r)L_1^r$ $[\overline{L}_j^s = (1^s - q_j^s)L_2^s]$ will have all the components equal to the ones of L_1^r $[L_2^s]$ to exception of i-th $[j$-th$]$

that is null, being the vector of the random non-centrality parameters when $pr(L_{1,i} = 0) = 1$, $i = 1, ..., r$ $[pr(L_{2,j} = 0) = 1$, $j = 1, ..., s]$. From (3.3) and (3.4) it is easy to get

$$\begin{cases} F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{\overline{L}_i^r, L_2^s}) > F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r, L_2^s});\ i = 1, ..., r \\ F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r, L_2^s}) > F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r, \overline{L}_j^s});\ j = 1, ..., s \end{cases} \cdot \tag{3.17}$$

As a rule, when one of the components of L_1^r $[L_2^s]$ is null, with probability 1, the values of $F^{+}(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r, L_2^s})$ increase [decrease].

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