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# VOLTERRA INTEGRAL INCLUSIONS VIA HENSTOCK-KURZWEIL-PETTIS INTEGRAL

Bianca Satco

*Université de Bretagne Occidentale, UFR Sciences et Techniques*
*Laboratoire de Mathématiques CNRS–UMR 6205*
*6 Avenue Victor Le Gorgeu, CS 93837, 29283 BREST Cedex 3, FRANCE*

**e-mail:** bianca.satco@univ-brest.fr

**Abstract**

In this paper, we prove the existence of continuous solutions of a Volterra integral inclusion involving the Henstock-Kurzweil-Pettis integral. Since this kind of integral is more general than the Bochner, Pettis and Henstock integrals, our result extends many of the results previously obtained in the single-valued setting or in the set-valued case.

**Keywords:** Volterra integral inclusion, Henstock-Kurzweil integral, Henstock-Kurzweil-Pettis integral, set-valued integral.

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## 1. Introduction

On the real line, the Henstock-Kurzweil integral extends the classical Lebesgue integral. It has the advantage to integrate highly oscillating derivatives. This property, together with its natural definition, makes the Henstock-Kurzweil integral a very useful tool in studying differential or integral inclusions.

In Banach spaces and, even more generally, in locally convex topological vector spaces, many authors studied the set-valued integration. Thus, there are quite a number of papers treating set-valued integrals of Aumann type by means of Bochner or Pettis integrable selections or by using the support functional of multifunction (e.g. [2, 20, 25, 13, 11]). Various results on differential and integral inclusions were obtained via these set-valued integrals (see [1, 7, 27, 3, 24] and the references therein).

In the present work, we will consider the Henstock-Kurzweil-Pettis integral, which is a Pettis-type integral defined by using, for the canonical bi-linear form, the Henstock-Kurzweil integral instead of the Lebesgue one.

We obtain an existence result of continuous solutions of the Volterra integral inclusion

$$(*)\qquad x(t) \in (\mathrm{HKP}) \int_0^t k(t,s)F(s,x(s))ds,$$

where $X$ is a separable Banach space, $F : [0,1] \times X \to \mathcal{P}_0(X)$ and $k : [0,1] \times [0,1] \to \mathbb{R}$.

To establish the result, we apply a set-valued variant of Mönch's fixed point theorem ([22]). To this end, we impose on the multifunction $F$ a condition involving a measure of weak noncompactness, a Henstock-Kurzweil-Pettis integrability condition, as well as some conditions of uniform integrability appropriate to the Henstock-Kurzweil integral setting. Also, on the real function $k$ some bounded variation assumptions are made.

Our result extends those obtained for the set-valued case in [24] under Bochner integrability assumptions and in [7] in Pettis setting. Moreover, it generalizes the single-valued case results obtained in [9] under the Henstock-Kurzweil-Pettis-integrable assumption, and in [14] under the Henstock-integrable assumption.

## 2. Notations and Preliminary Facts

We begin by introducing the Henstock-Kurzweil integral, a concept that extends the classical Lebesgue integral on the real line. A gauge $\delta$ on the unit interval $[0,1]$ provided with the $\sigma$-algebra $\Sigma$ of Lebesgue measurable sets and with the Lebesgue measure $\mu$ is a positive function; a partition of $[0,1]$ (that is a finite family $(I_i,t_i)_{i=1}^n$ of nonoverlapping intervals covering $[0,1]$ with the tags $t_i \in I_i$) is said to be $\delta$-fine if for each $i \in \{1,\dots,n\}$, $I_i \subset \,]t_i - \delta(t_i), t_i + \delta(t_i)[$.

A function $f : [0,1] \to \mathbb{R}$ is called Henstock-Kurzweil (shortly, HK-) integrable if there exists a real, denoted by (HK) $\int_0^1 f(t)dt$, satisfying that, for every $\varepsilon > 0$, one can find a gauge $\delta_\varepsilon$ such that, for any $\delta_\varepsilon$-fine partition $\mathcal{P} = (I_i,t_i)_{i=1}^n$ of $[0,1]$, $\left|\sum_{i=1}^n f(t_i)\mu(I_i) - (\mathrm{HK})\int_0^1 f(t)dt\right| < \varepsilon$.

Let us recall some useful results:

**Theorem 1** (Theorem 9.12 in [17]). *Any* HK-*integrable function* $f : [0,1] \to \mathbb{R}$ *is measurable and its primitive* (HK) $\int_0^{\cdot} f(t)dt$ *is continuous.*

The following notions were used in [17] in order to obtain convergence results:

**Definition 2.**

(i) A function $F : [0,1] \to \mathbb{R}$ is absolutely continuous in the restricted sense (shortly, $AC_*$) on $E \subset [0,1]$ if, for any $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that, whenever $\{[c_i,d_i], 1 \le i \le N\}$ is a finite collection of non-overlapping intervals that have endpoints in $E$ and satisfy $\sum_{i=1}^N (d_i - c_i) < \eta_\varepsilon$, one has $\sum_{i=1}^N osc(F,[c_i,d_i]) < \varepsilon$;

(ii) $F : [0,1] \to \mathbb{R}$ is said to be generalized absolutely continuous in the restricted sense (shortly, $ACG_*$) if it is continuous and the unit interval can be written as a countable union of sets on each of which $F$ is $AC_*$;

(iii) A family of real functions is uniformly $ACG_*$ if one can write the unit interval as a countable union of sets on each of which the family is uniformly $AC_*$ (i.e. the above mentioned $\eta_\varepsilon$ is the same for all elements of the family).

(iv) The collection $\mathcal{K}$ of real HK-integrable functions is said to be uniformly HK-integrable if, for any $\varepsilon > 0$, there is a gauge $\delta_\varepsilon$ such that, for any $\delta_\varepsilon$-fine partition $(I_i,t_i)_{i=1}^n$ and any $f \in \mathcal{K}$, one has $|\sum_{i=1}^n f(t_i)\mu(I_i) -$ (HK) $\int_0^1 f(t)dt| < \varepsilon$.

**Theorem 3** (Theorems 13.26 and 13.29 in [17]). *Let* $f_n : [0,1] \to \mathbb{R}$ *be a sequence of* HK-*integrable functions pointwise convergent on* $[0,1]$. *If the sequence* ((HK) $\int_0^{\cdot} f_n(t)dt)_n$ *is equicontinuous and uniformly* $ACG_*$*, then* $(f_n)_n$ *is uniformly* HK-*integrable.*

For more details on this integral, we refer to [17].

Through this paper, $X$ is a real separable Banach space with a closed unit ball $B$, $X^*$ (resp. $X^{**}$) denotes its topological dual (resp. bidual), $B^*$ the closed unit ball of $X^*$ and $\mathcal{P}_0(X)$ (resp. $\mathcal{P}_c(X)$, $\mathcal{P}_{fc}(X)$, $\mathcal{P}_{wkc}(X)$) stands for the family of nonempty (resp. convex, closed convex, weakly compact convex) subsets of $X$. We denote the support functional of $A \in \mathcal{P}_{wkc}(X)$ by $\sigma(\cdot, A)$, where $\sigma(x^*, A) = \sup\{\langle x^*, x\rangle, x \in A\}$, for all $x^* \in X^*$.

A well known extension of the Lebesgue integral to Banach-valued setting is the Pettis integral (see [23]). It can be generalized, by considering, on the real line, the Henstock-Kurzweil integral instead of the Lebesgue one, as in [9]:

**Definition 4.** A function $f : [0,1] \to X$ is said to be Henstock-Kurzweil-Pettis (shortly, HKP-) integrable if:

(1) $f$ is scalarly HK-integrable, i.e., for any $x^* \in X^*$, $\langle x^*, f(\cdot)\rangle$ is HK-integrable;

(2) for each $[a,b] \subset [0,1]$, there is $x_{[a,b]} \in X$ such that, for every $x^* \in X^*$, $\left\langle x^*, x_{[a,b]}\right\rangle = (\mathrm{HK}) \int_a^b \langle x^*, f(s)\rangle\, ds$. We denote $x_{[a,b]} = (\mathrm{HKP}) \int_a^b f(s)ds$.

If in (2) only $x_{[a,b]} \in X^{**}$ is required, then $f$ is said to be Henstock-Kurzweil-Dunford (shortly, HKD-) integrable.

**Remark 5.** It follows from Theorem 1 that if $f$ is HKP-integrable, then it is scalarly measurable (therefore, measurable, since $X$ is separable) and its primitive $(\mathrm{HKP}) \int_0^{\cdot} f(s)ds$ is weakly continuous.

On the space of all HKP-integrable $X$-valued functions we can define the Alexiewicz norm,

$$\|f\|_A = \sup_{[a,b]\subset[0,1]} \left\|(\mathrm{HKP}) \int_a^b f(s)ds\right\|.$$

The Henstock-Kurzweil integral was generalized to Banach spaces in the following straightforward manner (see [6]):

**Definition 6.** A function $f : [0,1] \to X$ is Henstock integrable if we can find $(\mathrm{H}) \int_0^1 f(s)ds \in X$ such that, for every $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ satisfying, for any $\delta_\varepsilon$-fine partition $(I_i, t_i)_{i=1}^n$ of $[0,1]$, that

$$\left\| \sum_{i=1}^n f(t_i)\mu(I_i) - (\mathrm{H}) \int_0^1 f(s)ds \right\| < \varepsilon.$$

Note that any Henstock-integrable function is HKP-integrable.

In the set-valued setting, the reader is referred to [13] for Pettis integral. We consider its extension introduced in [11]:

**Definition 7.** A $\mathcal{P}_{wkc}(X)$-valued function $\Gamma$ is said to be "HKP-integrable in $\mathcal{P}_{wkc}(X)$" (or, simply, HKP-integrable) if:

(1) it is scalarly HK-integrable, i.e., for any $x^* \in X^*$, $\sigma(x^*, \Gamma(\cdot))$ is HK-integrable;

(2) for every $[a, b] \subset [0, 1]$, there exists $I_a^b \in \mathcal{P}_{wkc}(X)$ such that, for each $x^* \in X^*$, $\sigma(x^*, I_a^b) = (\mathrm{HK}) \int_a^b \sigma(x^*, \Gamma(t))dt$. We denote $I_a^b$ by $(\mathrm{HKP}) \int_a^b \Gamma(t)dt$.

The notation $S_\Gamma^{\mathrm{HKP}}$ stands for the family of HKP-integrable selections of $\Gamma$.

We will use the measure of weak noncompactness defined, for any $A \subset X$, by: $\beta(A) = \inf\{r > 0 : \text{there is a weakly compact} K_r \text{ such that } A \subset K_r + rB\}$. For its properties we refer the reader to [21]. Let us recall only the following equality in the space of $X$-valued functions continuous on $[0, 1]$:

**Theorem 8** (Theorem 2 in [21]). *Let $\mathcal{H}$ be a bounded and equicontinuous subset of $C([0, 1], X)$. Then, denoting by $\widetilde{\beta}$ the measure of weak noncompactness in the space $C([0, 1], X)$, one has that $\widetilde{\beta}(\mathcal{H}) = \sup_{t\in[0,1]} \beta(\mathcal{H}(t))$.*

It mainly relies on the following result, given in [12]:

**Proposition 9.** *Let $(f_n)_n$ be a bounded sequence of $C([0, 1], X)$. Then $(f_n)_n$ is convergent to $f \in C([0, 1], X)$ with respect to the weak topology of $C([0, 1], X)$ if and only if $(f_n(t))_n$ is weakly convergent to $f(t)$ for every $t \in [0, 1]$.*

In the sequel, we prove a set-valued version of Mönch's fixed point theorem, that we recall below:

**Theorem 10** ([22]). *Let $D$ be a closed, convex subset of a Banach space and $N : D \to D$ be continuous with the further property that for some $x_0 \in D$ one has: $C \subset D$ countable, $\overline{C} = \overline{conv}(\{x_0\} \cup N(C)) \Longrightarrow \overline{C}$ compact. Then $N$ has a fixed point.*

In our result, conditions with respect to the weak topology of a Banach space are imposed. A similar result considering the strong topology is Theorem 3.1 in [24]. Our theorem offers also a variant of the multi-valued fixed point result in [18], where the upper semicontinuity was required. We follow the same method of proof as in [24]. For the convenience of the reader, we give it in whole.

**Theorem 11.** *Let $D$ be a closed convex subset of a separable Banach space $X$ and $G : D \to \mathcal{P}_c(D)$ satisfying the following conditions:*

(i) Graph$(G)$ *is sequentially weakly closed;*

(ii) *for some $x_0 \in D$, every $M \subset D$ satisfying that $M = conv(\{x_0\} \cup G(M))$ is relatively weakly compact. Then $G$ has a fixed point in $D$.*

***Proof.*** Take $M_0 = \{x_0\}$ and, for all $n \geq 1$, $M_n = conv(\{x_0\} \cup G(M_{n-1}))$. It is a nondecreasing sequence of convex sets contained in $D$, so $M = \bigcup_{n=0}^{\infty} M_n \subset D$ is convex. Moreover, $M = conv(\{x_0\} \cup G(M))$.

Using hypothesis (ii) one gets that $M$ is relatively weakly compact, whence its weak closure $K = \overline{M}^w$ is weakly compact and convex.

It also satisfies the inclusion $K \subset G^-(K)$, where $G^-(K) = \{x \in D : G(x) \cap K \neq \emptyset\}$. Indeed, since the weak topology associated to a separable Banach space is metrizable on any weakly compact set, we are able to find, for a fixed $x \in K$, a sequence $(x_n)_n \subset M$ weakly convergent to $x$. Any sequence $y_n \in G(x_n) \subset K$ has (by Eberlein-Smulian's Theorem) a subsequence, not relabelled, weakly convergent to an element $y \in K$.
As $(x_n, y_n) \stackrel{X_w \times X_w}{\longrightarrow} (x, y)$, by hypothesis (i), $y \in G(x)$ and so, $G(x) \cap K \neq \emptyset$.

Finally, consider $\widehat{G} : K \to \mathcal{P}_{fc}(K)$, $\widehat{G}(x) = G(x) \cap K$. It has nonempty, closed convex values, a weakly sequentially closed graph (therefore, weakly closed, since the weak topology is metrizable on $K$) and $\widehat{G}(K)$ is relatively weakly compact; thus $\widehat{G}$ is upper semicontinuous with respect to the weak topology on $X$. We can apply Kakutani-Ky Fan's Theorem in order to obtain a fixed point of $\widehat{G}$, that is, obviously, a fixed point of $G$, too. ■

## 3. Existence result for Volterra integral inclusions via Henstock-Kurzweil-Pettis integral

In what follows, consider the Volterra integral inclusion

$$(*) \qquad x(t) \in (\text{HKP}) \int_0^t k(t,s) F(s, x(s)) ds,$$

where $F : [0,1] \times X \to \mathcal{P}_{wkc}(X)$ and $k : [0,1] \times [0,1] \to \mathbb{R}$.

In order to establish an existence result of continuous solutions, we will make use of the following auxiliary lemmas:

**Lemma 12.** *For any sequence* $(\overline{y}_n)_n$ *of measurable selections of a* $\mathcal{P}_{wkc}(X)$*-valued measurable multifunction* $\Gamma$*, there exists a sequence* $z_n \in \operatorname{conv}\{\overline{y}_m, m \geq n\}$ *weakly a.e. convergent to a measurable* $\overline{y}$*.*

***Proof.*** Let $(x_p^*)_{p\in\mathbb{N}}$ be a Mackey-dense sequence in the unit ball $B^*$. For every $p \in \mathbb{N}$, consider the countable measurable partition $(E_m^p)_m$ of $[0,1]$ defined by $E_m^p = \{t \in [0,1] : m-1 < \max(|\sigma(x_p^*, \Gamma(t))|, |\sigma(-x_p^*, \Gamma(t))|) \leq m\}$. For each $m \in \mathbb{N}$, the sequence $(|\langle x_p^*, \overline{y}_n\rangle|)_n \subset L^1([0,1])$ is uniformly integrable (therefore, relatively weakly compact) on $E_m^p$ whence, by an appropriate diagonal process, we can find a sequence of convex combinations, denoted by $(z_n)_n$, such that, for every $p \in \mathbb{N}$, $\langle x_p^*, z_n(t)\rangle \to \phi_p(t)$ a.e., where $\phi_p$ is a measurable function which is Bochner integrable on every $E_m^p$. On the other hand, for a.e. $t \in [0,1]$, $(z_n(t))_n \subset \Gamma(t)$ that is weakly compact, so one can find $\overline{y}(t) \in \Gamma(t)$ and a subsequence of $(z_n(t))_n$ weakly convergent to $\overline{y}(t)$. It follows that $\phi_p(t) = \langle x_p^*, \overline{y}(t)\rangle$ a.e. on $[0,1]$. Therefore, $\langle x_p^*, z_n(t)\rangle \to \langle x_p^*, \overline{y}(t)\rangle, \forall p \in \mathbb{N}, \forall t \in [0,1]\setminus N$, where $N \subset [0,1]$ is of null measure. Using again the fact that $(z_n(t))_n \subset \Gamma(t)$, it follows, by the choice of the sequence $(x_p^*)_p$, that $(z_n)_n$ weakly converges to $\overline{y}$ a.e. Thus $\overline{y}$ is a.e. the weak limit of a sequence of measurable functions, so it is measurable because $X$ is a Banach separable space. ∎

It is known that the HK-integrability is preserved under multiplication by real functions of bounded variation. Using this fact and an integration by parts result for HK-integral given in [17] we can prove:

**Lemma 13.** *Let* $f : [0,1] \to X$ *be* HKP*-integrable and* $g$ *be a real function of a bounded variation on* $[0,1]$*. Then* $gf$ *is* HKP*-integrable.*

***Proof.*** Applying Theorem 12.21 in [17] one deduces that $gf$ is scalarly HK-integrable (so, by Theorem 3 in [15], it is HKD-integrable) and, for any $x^* \in X^*$ and each $[a,b] \subset [0,1]$, (HK) $\int_a^b g(s)\langle x^*, f(s)\rangle ds = g(b)$(HK) $\int_a^b \langle x^*, f(s)\rangle ds - \int_a^b ((\text{HK}) \int_a^s \langle x^*, f(\tau)\rangle d\tau) dg(s)$, the latter being a Riemann-Stieltjes integral. By the definition of HKP-integral, it follows that

$$\begin{aligned}(1)\qquad &(\text{HK}) \int_a^b \langle x^*, g(s)f(s)\rangle ds \\ &= g(b)\langle x^*, (\text{HKP}) \int_a^b f(s)ds\rangle - \int_a^b \langle x^*, (\text{HKP}) \int_a^s f(\tau)d\tau\rangle dg(s).\end{aligned}$$

This implies that

$$(\text{HKD})\int_a^b g(s)f(s)ds = g(b)(\text{HKP})\int_a^b f(s)ds - \int_a^b \left((\text{HKP})\int_a^s f(\tau)d\tau\right) dg(s),$$

the latter integral being of Riemann-Stieltjes-type in the sense that there exists the element $I = \int_a^b((\text{HKP})\int_a^s f(\tau)d\tau)dg(s) \in X$ with the following property: for any weak neighbourhood $U$ of the origin, there exists $\delta_U > 0$ such that $S(\mathcal{P}', g) - I \in U$, for any partition $\mathcal{P}' = ((c_i, c_{i+1}), t_i)_{i=1}^N$ satisfying $|||\mathcal{P}'||| = \max_{i=1}^N |c_{i+1} - c_i| < \delta_U$, where $S(\mathcal{P}', g) = \sum_{i=1}^N (\text{HKP}) \int_a^{t_i} f(\tau) d\tau(g(c_{i+1}) - g(c_i))$. Indeed, the net $S(\mathcal{P}', g)_{\mathcal{P}'}$, where the set of partitions is considered ordered by $|||\cdot|||$, is weakly Cauchy in $X$. On the other hand, by Remark 5, there exists a weakly compact $Y \subset X$ such that $\{(\text{HKP}) \int_a^t f(s)ds, t \in [a, b]\} \subset Y$, whence, for any partition $\mathcal{P}$, $S(\mathcal{P}, g) \in V(g)\overline{conv}(Y \cup \{0\})$, $V(g)$ being the total variation of $g$ on $[a, b]$. Consequently, our weakly Cauchy net $S(\mathcal{P}', g)_{\mathcal{P}'}$ is contained in a weakly compact (so, weakly complete) subset of $X$ and then it has a weak limit, $I \in X$.

Finally, $(\text{HKD}) \int_a^b g(s)f(s)ds \in X$, and so $gf$ is HKP-integrable. ■

**Corollary 14.** *Let $\Gamma : [0, 1] \to \mathcal{P}_{wkc}(X)$ be* HKP*-integrable and $g$ be a real function of bounded variation on $[0, 1]$. Then $g\Gamma$ is* HKP*-integrable.*

***Proof.*** By Theorem 1 in [11], there exist a HKP-integrable function $\gamma$ and a $\mathcal{P}_{wkc}(X)$-valued Pettis integrable multifunction $G$ such that $\Gamma(t) = \gamma(t) + G(t)$, for every $t \in [0, 1]$, whence $g(t)\Gamma(t) = g(t)\gamma(t) + g(t)G(t)$. As $g$ is measurable and bounded on $[0, 1]$, by the characterisation of Pettis integrable multifunctions (Theorem 5.4 in [13]) it follows that $gG$ is a Pettis integrable multifunction. By the preceding lemma, $g\gamma$ is HKP-integrable, therefore $g\Gamma$ is HKP-integrable. ■

We also obtain the following

**Lemma 15.** *Let $\mathcal{F}$ be a $\|\cdot\|_A$-bounded family of* HKP*-integrable functions on $[0, 1]$ and let $k : [0, 1] \times [0, 1] \to \mathbb{R}$ be such that for every $t \in [0, 1]$, $k(t, \cdot) \in BV([0, 1])$ and the function $t \mapsto k(t, \cdot)$ is continuous from $[0, 1]$ to the space $BV([0, 1])$ provided with the norm $\|f\|_{BV} = |f(0)| + V(f)$, where $V(f)$ denotes the total variation of $f$. If $\{(\text{HKP}) \int_0^{\cdot} y(s)ds, y \in \mathcal{F}\}$ is strongly equicontinuous, then $\{(\text{HKP}) \int_0^{\cdot} k(\cdot, s)y(s)ds, y \in \mathcal{F}\}$ has the same feature.*

***Proof.*** Let us remark that, as $k$ is continuous on a compact set, it is bounded, so $\sup_{t\in[0,1]} \|k(t, \cdot)\|_{BV} < \infty$. Fix $c \in [0, 1]$ and $\varepsilon > 0$. There is

$\delta_{\varepsilon,c}>0$ such that, for any $t$ with $|t-c|<\delta_{\varepsilon,c}$, we have

$$\|k(t,\cdot)-k(c,\cdot)\|_{BV}<\frac{\varepsilon}{4\sup\limits_{y\in\mathcal{F}}\|y\|_A}$$

and, for every

$$y\in\mathcal{F},\left\|(\text{HKP})\int_c^t y(s)ds\right\|<\frac{\varepsilon}{4\sup\limits_{t\in[0,1]}\|k(t,\cdot)\|_{BV}}.$$

Then

$$\begin{aligned}
&\left\|(\text{HKP})\int_0^t k(t,s)y(s)ds-(\text{HKP})\int_0^c k(c,s)y(s)ds\right\|\\
&\le\left\|(\text{HKP})\int_0^c (k(t,s){-}k(c,s))\,y(s)ds\right\|+\left\|(\text{HKP})\int_c^t k(t,s)y(s)ds\right\|\\
&=\sup_{x^*\in B^*}\left|(\text{HK})\int_0^c (k(t,s)-k(c,s))\,\langle x^*,y(s)\rangle\,ds\right|\\
&+\sup_{x^*\in B^*}\left|(\text{HK})\int_c^t k(t,s)\,\langle x^*,y(s)\rangle\,ds\right|
\end{aligned}$$

whence, applying the integration by parts result, we obtain that

$$\begin{aligned}
&\left\|(\text{HKP})\int_0^t k(t,s)y(s)ds-(\text{HKP})\int_0^c k(c,s)y(s)ds\right\|\\
&\le\sup_{x^*\in B^*}\left|[k(t,c)-k(c,c)]\,(\text{HK})\int_0^c\langle x^*,y(s)\rangle\,ds\right|\\
&+\sup_{x^*\in B^*}\left|\int_0^c\left((\text{HK})\int_0^s\langle x^*,y(\tau)\rangle\,d\tau\right)d\,(k(t,s)-k(c,s))\right|\\
&+\sup_{x^*\in B^*}\left|k(t,t)(\text{HK})\int_c^t\langle x^*,y(s)\rangle\,ds+\int_c^t\left((\text{HK})\int_c^s\langle x^*,y(\tau)\rangle\,d\tau\right)dk(t,s)\right|\\
&\le|k(t,c)-k(c,c)|\,\|y\|_A+\|y\|_A\|k(t,\cdot)-k(c,\cdot)\|_{BV}\\
&+|k(t,t)|\left\|(\text{HKP})\int_c^t y(s)ds\right\|+\sup_{s\in[c,t]}\left\|(\text{HKP})\int_c^s y(\tau)d\tau\right\|\,\|k(t,\cdot)\|_{BV}\\
&\le 2\|y\|_A\|k(t,\cdot)-k(c,\cdot)\|_{BV}+2\|k(t,\cdot)\|_{BV}\sup_{s\in[c,t]}\left\|(\text{HKP})\int_c^s y(\tau)d\tau\right\|<\varepsilon.
\end{aligned}$$

**Lemma 16.** *Let $S$ be a subset of a Banach space and $M' > 0$. Then the measure $\beta$ of weak noncompactness satisfies that $\beta([-M', M']S) \leq M'\beta(S)$.*

***Proof.*** Let $r > 0$ be such that there exists a weakly compact $K_r$ satisfying the inclusion $S \subset K_r + rB$. Then $mS \subset mK_r + mrB$, for every $m \in [-M', M']$, whence $[-M', M']S \subset [-M', M']K_r + rM'B$. Now, by Krein-Smulian's Theorem, $[-M', M']K_r = M'\overline{conv}(\{0\} \cup K_r) \bigcup M'\overline{conv}(\{0\} \cup (-K_r))$ is weakly compact, so the conclusion follows by passing to the infimum over $r$. ∎

**Theorem 17.** *Let $F : [0,1] \times X \to \mathcal{P}_{wkc}(X)$ and $k : [0,1] \times [0,1] \to \mathbb{R}$ satisfy the following conditions:*

- (H1) *there exists a positive constant $c$ such that, for every bounded $A \subset X$, $\beta(F([0,1] \times A)) \leq c\beta(A)$;*
- (H2) *for every $x \in C([0,1], X)$, the multifunction $F(\cdot, x(\cdot))$ is* HKP*-integrable;*
- (H3) *the family $\{(\text{HKP}) \int_0^{\cdot} y(s)ds, y \in \mathcal{F}\}$ is strongly equicontinuous, where we denoted by $\mathcal{F} = \{y \in S^{\text{HKP}}_{F(\cdot,x(\cdot))}, x \in C([0,1], X)\}$;*
- (H4) *for each $x^* \in X^*$, $\{(\text{HK}) \int_0^{\cdot} \langle x^*, y(s)\rangle ds, y \in \mathcal{F}\}$ is uniformly $ACG_*$;*
- (H5) *for any $t \in [0,1]$, $F(t, \cdot)$ is upper semicontinuous from $X_w$ to $X_w$;*
- (H6) *for every $t \in [0,1]$, $k(t, \cdot) \in BV([0,1])$ and the function $t \mapsto k(t, \cdot)$ is continuous with respect to the norm $\|\cdot\|_{BV}$.*

*Then there exists $\alpha \in ]0,1]$ satisfying that the integral inclusion*

$$(*) \qquad x(t) \in (\text{HKP}) \int_0^t k(t,s)F(s,x(s))ds$$

*has a continuous solution on $[0, \alpha]$.*

***Proof.*** The method of proof is inspired by that of the main theorem in [9].

By (H3), $\{(\text{HKP}) \int_0^{\cdot} y(s)ds, y \in \mathcal{F}\}$ is strongly equicontinuous in 0, whence, for a fixed $M > 0$, there exists $\alpha > 0$ such that, for any $t \in [0, \alpha]$ and any $y \in \mathcal{F}$, $\|(\text{HKP}) \int_0^t y(s)ds\| \leq M$. Moreover, as $k$ is continuous on a compact set, it is bounded, so $\overline{M} = \sup_{t\in[0,1]} \|k(t, \cdot)\|_{BV} < \infty$. One can suppose, without any loss of generality, that $\alpha\overline{M}c < 1$.

Consider the set

$$\mathcal{K} = \{x \in C([0, \alpha], X) : \sup_{t\in[0,\alpha]} \|x(t)\| \leq 2M\overline{M}\}.$$

It is closed and convex. The set-valued function $\Xi : \mathcal{K} \to \mathcal{P}_0(\mathcal{K})$ defined by

$$\Xi(x) = \left\{ y \in \mathcal{K} : y(t) {=} (\text{HKP}) \int_0^t k(t,s)\overline{y}(s)ds, \forall t {\in} [0,\alpha], \text{ where } \overline{y} \in S^{\text{HKP}}_{F(\cdot,x(\cdot))} \right\}$$

has nonempty and convex values. Indeed, as $F(\cdot, x(\cdot))$ is measurable, it has at least one measurable selection and, by Theorem 1 in [11], this selection is HKP-integrable.

Let us now show that Graph($\Xi$) is sequentially weakly closed. Consider $(x_n, y_n)_n \subset$ Graph($\Xi$) convergent with respect to the weak topology of $\mathcal{K}$ to $(x, y)$. For every $n \in \mathbb{N}$, we can find $\overline{y}_n \in S^{\text{HKP}}_{F(\cdot,x_n(\cdot))}$ such that, for each $t \in [0, \alpha]$, $y_n(t) = (\text{HKP}) \int_0^t k(t,s)\overline{y}_n(s)ds$. Since $x_n \to x$ with respect to the weak topology of $\mathcal{K}$, by Proposition 9, for any $s \in [0, \alpha]$, $x_n(s) \xrightarrow{w} x(s)$, consequently the set $\{x_n(s), n \in \mathbb{N}\}$ is relatively weakly compact. Hypothesis 1) gets that $\beta(F(\{s\} \times \{x_n(s), n \in \mathbb{N}\})) \le c\beta(\{x_n(s), n \in \mathbb{N}\}) = 0$, which means that $\Gamma(s) = \bigcup_{n\in\mathbb{N}} F(s, x_n(s))$ is relatively weakly compact. Using Lemma 12, we are able to find a measurable function $\overline{y}$ and a sequence $z_n \in conv\{\overline{y}_m, m \ge n\}$ such that $(z_n)_n$ be weakly a.e. convergent to $\overline{y}$. On the other hand, from (H5) it follows that, for any weak neighbourhood $V$ of the origin, there exists $n_{s,V} \in \mathbb{N}$ such that, for every $n \ge n_{s,V}$, $F(s, x_n(s)) \subset F(s, x(s)) + V$. Obviously, the preceding $\overline{y}$ is a measurable selection of the $\mathcal{P}_{wkc}(X)$-valued HKP-integrable multifunction $F(\cdot, x(\cdot))$, so it is HKP-integrable.

By (H3) and (H4), for any $x^* \in X^*$, the sequence $((\text{HK}) \int_0^{\cdot} \langle x^*, z_n(s)\rangle ds)_n$ is equicontinuous and uniformly $ACG_*$. Consequently, it satisfies the hypothesis of Theorem 3, so $(\langle x^*, z_n(\cdot)\rangle)_n$ is uniformly HK-integrable. Applying the passage to the limit Theorem 13.16 in [17] one deduces that, for any $t \in [0, \alpha]$, (HKP) $\int_0^t z_n(s)ds \xrightarrow{w}$ (HKP) $\int_0^t \overline{y}(s)ds$. Using again (H3), the sequence $((\text{HKP}) \int_0^{\cdot} z_n(s)ds)_n$ is $C([0, \alpha], X)$-bounded and then, by Proposition 9 it follows that $((\text{HKP}) \int_0^{\cdot} z_n(s)ds)_n$ converges to (HKP) $\int_0^{\cdot} \overline{y}(s)ds$ with respect to the weak topology of $C([0, \alpha], X)$. One deduces that there exists a sequence $u_n \in conv\{z_m, m \ge n\}$ such that $((\text{HKP}) \int_0^{\cdot} u_n(s)ds)_n$ converges uniformly to (HKP) $\int_0^{\cdot} \overline{y}(s)ds$. That is to say that, for every $\varepsilon > 0$, there exists $n_\varepsilon > 0$ such that, for every $n \ge n_\varepsilon$ and every $t \in [0, \alpha]$, one has $\|(\text{HKP}) \int_0^t u_n(s)ds - (\text{HKP}) \int_0^t \overline{y}(s)ds\| < \varepsilon$. From the integration by parts, it follows, after computations similar to those in the proof of Lemma 15, that

$$\left\|(\mathrm{HKP})\int_0^t k(t,s)u_n(s)ds-(\mathrm{HKP})\int_0^t k(t,s)\overline{y}(s)ds\right\|$$

$$=\sup_{x^*\in B^*}\left|(\mathrm{HK})\int_0^t k(t,s)\left\langle x^*,u_n(s)-\overline{y}(s)\right\rangle ds\right|$$

$$\le\sup_{x^*\in B^*}\left|k(t,t)(\mathrm{HK})\int_0^t\left\langle x^*,u_n(s)-\overline{y}(s)\right\rangle ds\right|$$

$$+\sup_{x^*\in B^*}\left|\int_0^t\left((\mathrm{HK})\int_0^s\left\langle x^*,u_n(\tau)-\overline{y}(\tau)\right\rangle d\tau\right)dk(t,s)\right|\le 2\overline{M}\varepsilon,$$

for every $n\ge n_\varepsilon$ and every $t\in[0,\alpha]$. In other words, $((\mathrm{HKP})\int_0^{\cdot}k(\cdot,s)$ $u_n(s)ds)_n$ uniformly converges to $(\mathrm{HKP})\int_0^{\cdot}k(\cdot,s)\overline{y}(s)ds$. At the same time, we have that $\int_0^t k(t,s)u_n(s)ds\overset{w}{\to}y(t)$, whence $y(t)=\int_0^t k(t,s)\overline{y}(s)ds,\forall t$. Thus, Graph($\Xi$) is sequentially weakly closed.

Finally, consider an arbitrary $\mathcal{M}\subset\mathcal{K}$ with $\mathcal{M}=conv(\{x_0\}\cup\Xi(\mathcal{M}))$ and prove that $\mathcal{M}$ is relatively weakly compact. The strong equicontinuity is satisfied by (H3) and Lemma 15. It suffices to show that, for every $t\in[0,\alpha]$, $\mathcal{M}(t)$ is relatively weakly compact, that is, $\beta(\mathcal{M}(t))=0$ and then, applying Theorem 8, it will follow that $\mathcal{M}$ is relatively weakly compact.

Using a mean result for Henstock-Kurzweil-Pettis set-valued integral, we have

$$\beta\left(\Xi(\mathcal{M})(t)\right)=\beta\left(\bigcup_{x\in\mathcal{M}}\Xi(x)(t)\right)\le\beta\left(\bigcup_{x\in\mathcal{M}}(\mathrm{HKP})\int_0^t k(t,s)F(s,x(s))ds\right)$$

$$\le t\beta\left(\bigcup_{x\in\mathcal{M}}\overline{conv}\left(k\left(t\times[0,t]\right)F\left([0,t]\times x([0,t])\right)\right)\right)$$

$$\le t\beta\left(k\left(t\times[0,t]\right)F\left([0,t]\times\mathcal{M}([0,t])\right)\right)$$

$$\le t\beta\left([-\overline{M},\overline{M}]F\left([0,t]\times\mathcal{M}([0,t])\right)\right).$$

Applying now Lemma 16 and using (H1), we obtain that

$$\beta(\mathcal{M}(t))=\beta\left(\Xi(\mathcal{M})(t)\right)\le t\overline{M}\beta\left(F([0,t]\times\mathcal{M}([0,t]))\right)$$

$$\le\alpha\overline{M}c\beta(\mathcal{M}([0,t]))\le\alpha\overline{M}c\beta(\mathcal{M}([0,\alpha])),$$

whence $\beta(\mathcal{M}([0,\alpha])) \leq \alpha\overline{M}c\beta(\mathcal{M}([0,\alpha]))$. Since $\alpha\overline{M}c < 1$, it follows that $\beta(\mathcal{M}([0,\alpha]))=0$ and, consequently, $\beta(\mathcal{M}(t))=0$.

The assumptions of Theorem 11 are satisfied, therefore $\Xi$ has a fixed point that is, obviously, a continuous solution to our integral inclusion $(*)$. ■

In the particular case $k(t,s) = 1, \forall t, s \in [0,1]$, we obtain the following

**Corollary 18.** *Let $F : [0,1] \times X \rightarrow \mathcal{P}_{wkc}(X)$ satisfy the assumptions* (H1)–(H5) *in* Theorem 17. *Then there exists $\alpha \in ]0,1]$ such that the integral inclusion* $(**)$ $x(t) \in$ (HKP) $\int_0^t F(s,x(s))ds$ *has a continuous solution on* $[0,\alpha]$.

**Remark 19.** Our main theorem extends the existence result of solutions of Volterra inclusions obtained in [24] under Bochner integrability assumptions, as well as the result proved in [14] in the single-valued Henstock integrability setting, under contraction assumptions, using the successive approximations method.

The preceding corollary is a generalization of the existence result proved in [9] (Theorem 5) in the HKP-integrable single-valued case.

The Henstock-Kurzweil-Pettis integral is continuous with respect to the weak topology. In Theorem 17, the condition (H3) is stronger. However, our results can be applied in the following particular cases, where the strong continuity holds:

(1) the case of Pettis integrable functions; we obtain then a generalization of the existence Theorem VI-7 in [7], where a weakly compact convex Pettis integrable multifunction $\Gamma(\cdot)$ containing $F(\cdot,x)$, for any $x \in X$ was considered;

(2) the case of Henstock integrable functions. In this setting an existence result for the integral inclusion $(**)$ was obtained, as a particular case of Theorem 4.1 in [26], under the assumption that $F$ has at least one selection continuous with respect to the second variable; it is well known that under upper semicontinuity assumption (which is the case here), this is not always satisfied.

Finally, let us remark that, recently, some interesting properties of Henstock-type integrals were proved (see, for example, [5], where HKP-integral was introduced in Riesz spaces, as well as [8], where the author presents a very interesting discussion about different types of solutions of differential equations, depending on the vector integral considered). Also, new results were obtained on Pettis-type integrals (e.g. [4] and [16] that contains an application of the relation between decomposability and uniform integrability to differential inclusions).

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