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MATHEMATICAL TREATMENT FOR THERMOELASTIC PLATE WITH A CURVILINEAR HOLE IN S-PLANE

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Abstract

The Cauchy integral method has been applied to derive exact and closed expressions for Goursat's functions for the first and second fundamental problems for an infinite thermoelastic plate weakened by a hole having arbitrary shape.

The plate considered is conformally mapped to the area of the right half-plane. Many previous discussions of various authors can be considered as special cases of this work. The shape of the hole being an ellipse, a crescent, a triangle, or a cut having the shape of a circular arc are included as special cases.

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1. Introduction

The boundary value problems for isotropic homogeneous perforated infinite plates have been discussed by several authors [1, 2, 3, 4, 5, 6, 7].

It is known from [10] that the first and second fundamental problems in the plane theory of elasticity are equivalent to finding two analytic functions

$\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z = x+iy$, satisfying the boundary condition

$$k\phi_1(t) - t\,\overline{\phi_1'} - \overline{\psi_1}(t) = f(t), \tag{1.1}$$

where $k = -1$, $f(t)$ is a given function of stresses, for the first fundamental problem; while $k = \chi = \frac{\lambda+3\mu}{\lambda+\mu} \succ 1$, $f = 2\mu g(t)$ is a given function of the displacement for the second fundamental problem.

λ and μ are called the constants of Lame; χ is called Muskhelishvili's constant and t denoting the affix of a point on the boundary L.

Muskhelishvili solved in [10] the problem of a stretched infinite plate weakened by an elliptic hole using the mapping function $z = c(\zeta + m\zeta^{-1})$. This transformation conformally maps the infinite domain bounded internally by an ellipse onto the domain outside the unit circle $|\xi| = 1$ in the ξ-plane.

In [8] El-Sirafy used the complex variable methods and rational mapping functions to obtain the Goursat functions for a stretched infinite plate weakened by an inner curvilinear hole using the transformation

$$\frac{z}{c} = \frac{(s+1)^2 + m(s-1)^2}{(s-1)(s+1)^2 - n(s-1)^2}\,(c > 0,\ s = \sigma + i\tau;\ |n| < 1). \tag{1.2}$$

This transformation maps the perforated infinite plate onto the area of the right half-plane, $Re\,(s) \geq 0$.

The authors in [4, 5] considered the case of stretched infinite plates weakened by hypotrochoidal holes with four or five round corners, the Goursat functions $\phi(z)$ and $\psi(z)$ are obtained in a closed form.

Abdou and Khar Eldin [1] obtained the two Goursat's functions for the stretched infinite plate weakened by a hole whose edge is free from stresses, using the two rational mapping functions

$$\frac{z}{c} = w(s) = \frac{(s+1)^3 + m(s-1)^3}{(s-1)(s+1)^2 - n(s+1)(s-1)^2}$$

and

$$\frac{z}{c} = w(s) = \frac{(s+1)^3 + m(s-1)^3}{(s-1)[(s+1)^2 - n(s-1)^2]^2}\,(|n|,\ c > 0,\ s = \sigma + i\tau). \tag{1.3}$$

Here, m, n are real parameters subject to the conditions that $w(s)$ does not vanish on the right half-plane (i.e., $Re\ s \geq 0$) and w (∞) is bounded.

In this paper, the Cauchy integral method and the rational mapping function

$$\frac{z}{c} = w(s) = \frac{(s+1)^3 + m(s+1)(s-1)^2}{(s-1)(s+1)^2 - n(s+1)(s-1)^2}, \tag{1.4}$$

$$(c > 0,\ |n| < 1;\ s = \sigma + i\tau)$$

where $m, n,$ are real parameters subject to the conditions that $w\ (\infty)$ is bounded and $w(s)$ does not vanish on the right half-plane (i.e., $Rc\ s \geq 0$), are used to obtain exact and closed expressions for Goursat functions for the first and second fundamental problems of an infinite plate weakened by a curvilinear hole conformally mapped on the domain onto the right half-plane by (1.4).

If we take in (1.4) $s = \frac{\xi+1}{\xi-1}$, we have the transformation mapping

$$\frac{z}{c} = w(\xi) = \frac{\xi + m\xi^{-1}}{1 - n\xi^{-1}}.$$

In terms of $z = cw(\xi)$, $c > 0$, $w(\xi)$ does not vanish or becomes infinite for $|\xi| > 1$, the infinite region outside a closed contour conformally mapped outside the unit circle γ (see [2, 10]).

Some applications of the first and second fundamental problems on these domains are investigated, the interesting cases of an infinite plate weakened by an elliptic hole, a crescent like hole or a cut having the shape of a circular arc, and the hypotrochoidal hole with three rounded corners are considered as special cases, and the functions $\phi(z)$ and $\psi(z)$ are obtained in a closed form.

2. Basic equations

Consider a region of an elastic medium of an infinite plate denoted by S and bounded by a single contour L, with a curvilinear hole C where the origin lies inside the hole.

If $\widehat{xx}, \widehat{yy}, \widehat{xy}$ represent the components of stress, while u, v represent the components of displacement and in the absence of body forces, we have the formulae of Kolsov-Muskhelishvili [8] in the following form

$$\widehat{xx} + \widehat{yy} = 4Re\{\phi_1'(z)\} \tag{2.1}$$

$$y\widehat{y} - x\widehat{x} + 2ix\widehat{y} = 2[z\phi_1''(z) + \psi_1'(z)] \tag{2.2}$$

and

$$2\mu(u + iv) = k\phi_1(z) - \overline{z\phi_1'(z)} - \psi_1(z). \tag{2.3}$$

In terms of the conformal mapping function, it means that

$$z = cw(\xi),\ c > 0,\ w'(\xi) \neq 0 \text{ or } \infty \text{ for } |\xi| > 1.$$

The infinite region outside a closed contour is conformally mapped on the region outside the unit circle γ. The complex potentials $\phi(z)$ and $\psi(z)$ can be written in the form

$$\phi_1(t) = -\frac{X + iY}{2\pi(1+\chi)}\, ln\, t + \Gamma t + \phi_0(t), \tag{2.4}$$

$$\psi_1(t) = \frac{\chi(X - iY)}{2\pi(1+\chi)}\, ln\, t + \Gamma^* t + \psi_0(t), \tag{2.5}$$

where X, Y are the components of the resultant vector of all external forces acting on L; Γ, Γ^* are constants and $\phi_0(t)$, $\psi_0(t)$ are holomorphic functions at infinity. Using (2.4) and (2.5) in (2.3), we obtain

$$k\phi_0(t) - \overline{t\phi_0'(t)} - \overline{\psi_0(t)} = f(t), \tag{2.6}$$

where $k = -1,\ f(t) = -f^*(t)$ for the first displacement problem; while $k = \chi,\ f(t) = 2\mu g(t)$ for the second fundamental problem.

3. Method of solution

The expressions $\frac{\overline{w(i\tau)}}{w'(i\tau)}$ will be assumed in the form

$$\frac{\overline{w(i\tau)}}{w'(i\tau)} = \overline{\alpha(i\tau)} + \beta(i\tau), \tag{3.1}$$

where

$$\alpha(i\tau) = \frac{k^*}{a + i\tau}, \quad a = \frac{1+n}{1-n}$$

(3.2) $$k^* = 4na^2(n^3 + nm)J_0^{-1}, \quad J_0 = (1 - 2n^2 - mn^2)$$

and

$$\beta(s) = \frac{1}{s-a}\left[\frac{H(s)}{E(s)} + k^*\right],$$

where

$$H(s) = (1-n)(s^2-1)(s+a)^2[m(s-1)(s+1)^2 + (s-1)^3]$$

(3.3) $$E(s) = 2[-(s+1)^4 + 2n(s+1)^3(s-1) + m(s+1)^2(s-1)^2].$$

$\beta(s)$ is a regular function within the right half-plane except at infinity. The boundary condition (2.6) takes the form

(3.4) $$k\phi(i\tau) - \alpha(\tau)\overline{\phi'(i\tau)} - \overline{\psi(i\tau)} = f_*(\tau)$$

where

$$\phi(s) = \phi_0(w(s))$$

$$f_*(\tau) = f(w(i\tau)) - \gamma_0 + w(i\tau)(\overline{\Gamma} - k\Gamma) + \overline{w(i\tau)}\,\overline{\Gamma}^*$$
$$-\frac{X - iY}{2\pi(1+\chi)\overline{w'(i\tau)}}(w(i\tau) - \overline{w'(i\tau)})$$

$$\psi(s) = \psi_0(w(s)) + \beta(s)\phi'(s) + \overline{\gamma_0} - \frac{X + iY}{2\pi(1+\chi)}$$

(3.5) $$\gamma_0 = c\left(\frac{1+m}{1-n}\right)\left(\overline{\Gamma} - k\Gamma + \overline{\Gamma}^*\right)$$

and assume that $\phi(\infty) = \psi(\infty) = 0$.

Multiplying both sides of (2.4) by $\frac{1}{2\pi(s-i\tau)}$ and integrating with respect to τ from $-\infty$ to ∞, we have

(3.6) $$k\phi(s) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha(i\tau)\overline{\phi'(i\tau)}}{s - i\tau}d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_*(\tau)}{s-i\tau}d\tau$$

and by using (3.1) in (3.6), we obtain:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha(\tau)\overline{\phi'(i\tau)}}{s-i\tau}d\tau=\frac{ck^*b}{s+a}\,, \tag{3.7}$$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_*(i\tau)d\tau}{s-i\tau}&=A_1(s)-\frac{2c\overline{\Gamma}^*}{1+s}+\frac{2c(k^*\Gamma-\overline{\Gamma})}{(1-n)^2}\left[\frac{m+n^2}{(s+a)}\right]\\&+\frac{(1+n)(n^3+mn)(X-iY)}{1+\chi(1-n)(1+mn^2)(s+a)}\end{aligned} \tag{3.8}$$

and

$$A_1(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(w(i\tau))}{s-i\tau}d\tau\,, \tag{3.9}$$

where b is a complex constant to be determined. Substituting (3.7–3.9) in (3.6), we get:

$$\begin{aligned}k\phi(s)=\frac{ck^*b}{s+a}+A_1(s)+\frac{\gamma_1(1+n)(X-iY)}{\pi\gamma_2(1+\chi)(1-n)(s+a)}-\frac{2c\overline{\Gamma}^*}{1+s}\\+\frac{2c(k\Gamma-\overline{\Gamma})}{(1-n)^2}\left(\frac{m+n^2}{a+s}\right)\end{aligned} \tag{3.10}$$

where $\gamma_1=(n^3+nm)$; $\gamma_2=1+mn^2$.

Differentiating (3.10) and inserting $\phi(i\tau)$ in 3.7, the complex constant b can be determined in the form:

$$\begin{aligned}b=\frac{2a^2}{c(16a^4k^2-k^{*2})}\Bigg\{&\left[8a^2k\overline{A_1'(a)}-2k^*A_1'(0)\right]\\&+c(1-n)^2(4a^2k\Gamma^*-k^*\overline{\Gamma}^*)-c\left[(4a^2k^2+k^*)\overline{\Gamma}-(4a^2+K^*)k\Gamma\right]\\&\left(\frac{m+n^2}{(1+n)^2}\right)-\frac{(1+n)\gamma_1}{c\gamma_2(1-n)(1+k)}\left[\frac{x}{4a^2k+k^*}+\frac{iY}{4a^2k-k^*}\right]\Bigg\}.\end{aligned} \tag{3.11}$$

Inserting (3.11) in (3.10) the function $\phi(s)$ becomes

$$k\phi(s) = A_1(s) + \frac{k\gamma_1 J_0 a(XJ_1 - iYJ_2)}{\pi\gamma_2(1+\chi)(s+a)} + \frac{2c\overline{\Gamma}^*}{(1+s)} + \frac{2(m+n^2)h_1}{s+a} + \frac{2n\gamma_1 J_1 J_2}{(1-n)^2(s+a)}[h_2+h_3+h_4] \tag{3.12}$$

where

$$J_1 = (kJ_0 + n\gamma_1)^{-1}, \quad J_2 = (J_0 k - n\gamma_1)^{-1},$$

$$\begin{aligned} h_1 &= c\frac{k\Gamma - \overline{\Gamma}}{(1-n)^2}, \\ h_2 &= 2(1+n)^2\left[kJ_0\overline{A_1'(a)} - n\gamma_1 A_1'(a)\right], \\ h_3 &= c(1-n^2)^2[kJ_0\Gamma^* - n\gamma_1\overline{\Gamma}^*], \\ h_4 &= c(m+n^2)[(J_0+n\gamma_1)k\Gamma - (k^2J_0+n\gamma_1)\overline{\Gamma}]\,. \end{aligned} \tag{3.13}$$

From the boundary condition (2.4), we can determine $\psi(s)$ in the form

$$\begin{aligned} \psi(s) \;=\; & A_2(s) + \frac{kB_1(S)}{k^*} + \frac{2c(\Gamma - k\overline{\Gamma})}{1+S} - \frac{cK(1-n)^2(s+a+2)\overline{\Gamma}^*}{2k^*(1+s)^2} \\ & + \left(\frac{s+3a}{(s+a)^2}\right)\left\{\frac{2n\gamma_1 h_1(m+n^2)}{kJ_0} + \frac{n(1+n)\gamma_1^2(J_1X - iJ_2Y)}{\pi\gamma_2(1+\chi)(1-n)}\right. \\ & \left. + \frac{2n^2\gamma_1^2 J_1 J_2}{(1-n)^2 J_0 k}(h_2+h_3+h_4)\right\} \\ & + \frac{2c(m+n^2)\Gamma^*}{(s+a)(1-n)^2} \end{aligned} \tag{3.14}$$

where

$$B_1(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\overline{A_1(i\tau)}d\tau}{(i\tau - a)(s - i\tau)} \tag{3.15}$$

and

$$A_2(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\overline{fw(i\tau)}}{s - i\tau}d\tau\,.$$

4. Thermoelastic potential function

For two dimensional problems for a thermoelastic infinite plate, we have $\sigma_{zx} = \sigma_{zy} = 0$, where σ_{zx} and σ_{zy} are the components of stress in the fixed direction z. Also from the equilibrium equation:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0\,. \end{aligned} \tag{4.1}$$

From Fourier's law, q, the rate of heat flow per unit area is proportional to the temperature gradient in the direction of the normal n to the element

$$q = -k\frac{\partial \theta}{\partial n} \tag{4.2}$$

where k is the thermal conductivity of the material, θ is the temperature difference and the minus sign is due to the fact that the heat flows in the direction of decreasing temperature.

In the stationary case of a thermoelastic infinite plate, we have the boundary value problem:

$$\nabla^2 \theta = 0 \tag{4.3}$$

with the boundary condition: $\frac{\partial \theta}{\partial n} = 0$ at $r = r_0$, (on the boundary) where $z = x + iy$, $x = r\,\cos\,\rho$ and $y = r\,\sin\,\rho$.

The solution of (4.3) from [14] is given by

$$\theta = q\left(Im\,z + \frac{r_0^2 \sin^2 \rho}{Im\,z}\right). \tag{4.4}$$

At the origin $\theta = O(\frac{1}{z})$, and has the greatest value as $z \longrightarrow 0$. Also when $z \longrightarrow \infty$, the increasing temperature will take the value $\theta = qy$. Neglecting the variation of the strain and stress with respect to the thickness of the plate, the thermoelastic potential function Φ takes the form:

$$\nabla^2 \Phi = (1 + \upsilon)\alpha\theta \tag{4.5}$$

(see [11]), where α is the coefficient of thermal expansion and υ is the Poisson's ratio. In the case of a uniform heat flow using (4.4) in (4.5), we get:

$$\Phi = \frac{(1+\upsilon)\alpha q r_0^2}{4}\,(Im\, z + \ln z\overline{z})\,. \tag{4.6}$$

5. Stress components in the presence of heat

From [10, 11], the stress components satisfy the following equation:

$$\sigma_{xx} + \sigma_{yy} = 4G\left[\phi'(z) + \overline{\phi'(z)} - \lambda\theta\right] \tag{5.1}$$

and

$$\begin{aligned} \sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} &= 2G\left[\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} + 2i\frac{\partial^2\Phi}{\partial x\partial y}\right] \\ &\quad - 4G\left[z\overline{\phi''(z)} + \overline{\psi'(z)}\right], \end{aligned} \tag{5.2}$$

where G is the shear modulus and $\lambda = \frac{1+\upsilon}{2}\alpha$ is the heat transfer coefficient. The partial derivatives of Φ with respect to x and y take the form:

$$\begin{aligned} \frac{\partial^2\Phi}{\partial x^2} &= \frac{(1+\upsilon)\alpha q r_0^2}{2}\;\frac{y^3 - x^2 y}{(x^2+y^2)^2} \\ \frac{\partial^2\Phi}{\partial y^2} &= \frac{(1+\upsilon)\alpha q r_0^2}{2}\;\frac{3x^2 y + y^3}{(x^2+y^2)^2} \\ \frac{\partial^2\Phi}{\partial x\partial y} &= \frac{(1+\upsilon)\alpha q r_0^2}{2}\;\frac{x^3 - x y^2}{(x^2+y^2)^2}\,. \end{aligned} \tag{5.3}$$

Solving equations (5.1) and (5.2) together we get the stress components as

$$\begin{aligned} \sigma_{xx} &= -2G\left\{\gamma_1^* - Re\left[2\phi'(z) - M(z,\overline{z})\right]\right\}, \\ \sigma_{yy} &= -2G\left\{\gamma_2^* - Re\left[2\phi'(z) + M(z,\overline{z})\right]\right\}, \quad \text{and} \\ \sigma_{xy} &= G\left\{\gamma_1 - 4GImM(z,\overline{z})\right\} \end{aligned}$$

where

$$\gamma_1^* = \frac{\partial^2 \Phi}{\partial y^2}, \quad \gamma_2^* = \frac{\partial^2 \Phi}{\partial x^2}, \quad \gamma_1 = \frac{\partial^2 \Phi}{\partial y \partial x}$$

and $M(z, \overline{z}) = z\overline{\phi''(z)} + \overline{\psi'(z)}$. So the stress components in the presence of heat are completely determined.

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